SOLUTIONS

TO

MATHEMATICAL PROBLEM PAPERS

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BY THE

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PREFACE

THIS volume contains complete solutions to all the questions in the Second Edition of my *Mathematical Problem Papers*.

In order to make the book as helpful and interesting as possible to private students of Mathematics, as well as to busy teachers, I have employed a great variety of methods, and have endeavoured not to sacrifice clearness and soundness to mere brevity.

No results are assumed, other than may be found in the standard text-books on the various branches of Mathematics involved. I have not considered it necessary to make much use of the methods of the Calculus, except in questions on that subject. Teachers who prefer to use these methods more freely will easily be able to modify many of the solutions accordingly.

I have to thank my former colleague, Mr H. E. Lee, B.A., and my former pupil, Mr R. J. Fulford, B.Sc., for their kindness in reading the proof-sheets, and verifying a large number of the solutions. As I have also myself carefully revised the solutions while in the Press, I hope they will be found free from any serious errors.

Some of the earlier solutions have appeared in the *Mathematical Gazette*, but the issues containing them are no longer in print. They are reproduced by the courtesy of the Editor.

E. M. RADFORD.

ST JOHN'S COLLEGE,
BATTERSEA,
February, 1915.

PART I

I.

1. Let P be the given point and A the intersection of the given straight lines. Take points B and C, one on each line, such that P lies in the angle BAC, and $AB = AC =$ half the given perimeter. Describe a circle touching the lines at B and C, and from P draw tangents to this circle. Then evidently either of these tangents fulfils the required condition, provided P lies between A and the circle. There are also, in any case, two other solutions, obtained by drawing the inner tangents from P to the similarly described circles in the angles adjacent to that containing P.

2. Let S, S' be the centres of the two circles, a and $\frac{1}{2}a$ their radii, P the centre of the variable circle and r its radius. Then $SP = a - r$, $S'P = \frac{1}{2}a + r$, $\therefore$ $SP + S'P = \frac{3}{2}a$. Hence the locus of P is an ellipse with S, S' as foci, and whose major axis is $\frac{3}{2}a$. Also, since $SS' = \frac{1}{2}a$, it easily follows that the eccentricity is $\frac{1}{3}$, the semi-minor axis $\frac{a}{\sqrt{2}}$, and the latus-rectum $\frac{4}{3}a$.

3. Evidently $b - c$, $c - a$, $a - b$ are factors, and the remaining factor must be a symmetrical homogeneous function of a, b, c of the third degree. We may therefore assume that the given expression is equal to

$$\Pi (b - c) \,.\, [p \,.\, \Sigma a^3 + q \,.\, \Sigma a^2 b + r \,.\, abc],$$

where p, q, r are numerical constants to be determined.

Putting $c = 0$, we get

$$a^5 b - b^5 a = b(-a)(a - b) \,.\, [p(a^3 + b^3) + q(a^2 b + ab^2)].$$

But since $a^5 b - b^5 a = ab(a - b)(a + b)(a^2 + b^2)$, it follows that $p = q = -1$. To find r, put $a = 2$, $b = 1$, $c = -1$. We then find $4p + 2q - r = -6$, whence $r = 0$.

Hence the remaining factor is

$$-(\Sigma a^3 + \Sigma a^2 b) = -(\Sigma a)(\Sigma a^2).$$

4. We have

$$(\tfrac{3}{4})^{\frac{4}{5}} = (1 - \tfrac{1}{4})^{\frac{4}{5}}$$
$$= 1 + \tfrac{4}{5}(-\tfrac{1}{4}) + \frac{\frac{4}{5}(\frac{4}{5}-1)}{2!}(-\tfrac{1}{4})^2 + \dots$$
$$= 1 - u_1 - u_2 - u_3 - \dots \text{ say.}$$

Then

$$\begin{aligned} u_1 &= \cdot 200000 \\ u_2 = \tfrac{1}{10}\cdot\tfrac{1}{4}u_1 &= \cdot 005000 \\ u_3 = \tfrac{2}{5}\cdot\tfrac{1}{4}u_2 &= \cdot 000500 \\ u_4 = \tfrac{11}{20}\cdot\tfrac{1}{4}u_3 &= \cdot 000069 \\ u_5 = \tfrac{16}{25}\cdot\tfrac{1}{4}u_4 &= \cdot 000011 \\ u_6 = \tfrac{7}{10}\cdot\tfrac{1}{4}u_5 &= \underline{\cdot 000002} \\ &\ \cdot 205582. \end{aligned}$$

Hence

$$(\tfrac{3}{4})^{\frac{4}{5}} = 1 - \cdot 205582\dots$$
$$= \cdot 7944 \text{ correct to four decimal places.}$$

5. The left-hand side

$$= \tfrac{1}{2}(4 - \cos 24^\circ - \cos 42^\circ - \cos 78^\circ - \cos 96^\circ)$$
$$= \tfrac{1}{2}(4 - 2\cos 60^\circ \cos 36^\circ - 2\cos 60^\circ \cos 18^\circ)$$
$$= \tfrac{1}{2}(4 - \cos 18^\circ - \cos 36^\circ)$$
$$= 1 + \sin^2 9^\circ + \sin^2 18^\circ.$$

6. Here

$$\Sigma PA_1 = 2R\,.\sum_{r=0}^{r=n-1} \sin\left(a + \frac{r\pi}{n}\right)$$
$$= 2R\,\frac{\sin\left(a + \frac{\overline{n-1}\pi}{2n}\right)\sin\left(n\,.\,\frac{\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$$
$$= 2R\,.\,\frac{\cos\left(a - \frac{\pi}{2n}\right)}{\sin\frac{\pi}{2n}}$$
$$= 2R\left(\cos a \cot\frac{\pi}{2n} + \sin a\right).$$

7. The equation to any conic circumscribing this triangle must be of the form

$$(y+kx)(y-kx)+(x\cos\alpha+y\sin\alpha-p)(lx+my)=0\ldots\text{(i)}.$$

Using the conditions for a circle, we find

$$l\cos\alpha-k^2=m\sin\alpha+1 \text{ and } l\sin\alpha+m\cos\alpha=0,$$

whence

$$\frac{l}{\cos\alpha}=\frac{m}{-\sin\alpha}=1+k^2.$$

Substituting in (i) and simplifying, we obtain the equation given.

8. If P be the point m, and P' the point m', the equations to PP' and $P'Q$ are respectively

$$2x-(m+m')\,y+2amm'=0$$

and

$$2x-(m'-m)\,y-2amm'=0,$$

and if the first of these passes through $(-a, 0)$, the second passes through $(a, 0)$.

9. Let the true weight of the first article be Q, that of the second R, and the apparent and true weights of the third article X and W respectively. Let a and b be the lengths of the arms, S and S' the weights of the pans, w the weight of the instrument, x the distance of the centre of gravity from the fulcrum. Then we have the equations

$$(Q+S)\,a+w\,.\,x=(Q_1+S')\,b,$$
$$(Q_2+S)\,a+w\,.\,x=(Q+S')\,b,$$
$$(W+S)\,a+w\,.\,x=(X+S')\,b,$$
$$(X+S)\,a+w\,.\,x=(W+S')\,b,$$

whence

$$(W-Q_2)\,a=(X-Q)\,b,$$
$$(X-Q)\,a=(W-Q_1)\,b,$$
$$\therefore\ \frac{W-Q_2}{W-Q_1}=\frac{b^2}{a^2}=\frac{W-R_2}{W-R_1} \text{ similarly.}$$

This gives the required value of W.

1—2

10. Let $ABCD$ be the square, A the corner of the square in contact with the wall, B the corner to which the string is attached. Draw BN perpendicular to the wall, and let the direction of the string meet the horizontal through A in O. Then O must be vertically below the centre of the square. Let the distances of B, C and D from the wall be x, y, z respectively. Then $AO = 2x$.

Also $\frac{y}{2} = \frac{x+z}{2}$ = distance of centre from wall $= 2x$, whence

$$\frac{x}{1} = \frac{y}{4} = \frac{z}{3},$$

and evidently $AN = z$, so that the angle required is

$$\cot^{-1}\frac{z}{x} = \cot^{-1} 3.$$

11. When the wedge is at rest, let T be the tension of the string, R the reaction between the plane and the mass m'. Then, if f is the acceleration,

$$T - m'g \sin \alpha = m'f, \quad mg - T = mf,$$

whence $$T = \frac{mm'(1 + \sin \alpha)}{m + m'} \cdot g.$$

Also $R = m'g \cos \alpha$, and the horizontal force on the wedge is

$$R \sin \alpha - T \cos \alpha = \frac{m' \cos \alpha \, (m' \sin \alpha - m)}{m + m'} \cdot g,$$

and to keep the wedge from moving, a force equal and opposite to this is required.

12. If α is the angle of projection, and β the angle of the plane, the vertical altitude above the plane after time t is

$$u \sin \alpha \,.\, t - \tfrac{1}{2}gt^2 - (u \cos \alpha \,.\, t) \tan \beta = ut \,.\, \frac{\sin(\alpha - \beta)}{\cos \beta} - \tfrac{1}{2}gt^2.$$

Now, whatever λ may be,

$$ut \,.\, \lambda - \tfrac{1}{2}gt^2 = \tfrac{1}{2} \,.\, \frac{u^2\lambda^2}{g} - \tfrac{1}{2}g\left(\frac{u\lambda}{g} - t\right)^2,$$

and therefore its greatest value is $\frac{1}{2} \,.\, \frac{u^2\lambda^2}{g}$.

Hence the greatest value of the above vertical altitude is

$$\frac{1}{2}\cdot\frac{u^2\sin^2(\alpha-\beta)}{g\cos^2\beta}=\frac{1}{4}\cdot\frac{u^2}{g\cos^2\beta}[1-\cos 2(\alpha-\beta)].$$

But for the maximum range $2\alpha-\beta=\frac{\pi}{2}$. Hence the above is

$$\frac{1}{4}\cdot\frac{u^2}{g\cos^2\beta}(1-\sin\beta),$$

i.e. $\frac{1}{4}$ of the maximum range.

II.

1. Produce AD to E, making DE either $=2AD$ or $\frac{1}{2}AD$. Draw EC parallel to AB, meeting AC in C. Produce CD to meet AB in B. Then BC is the line required, since

$$BD : DC = AD : DE.$$

2. Let $ABCD$ be the parallelogram; E, F, G, H the middle points of the sides. Then E and G are the middle points of parallel chords, therefore EG passes through the centre of the conic, and similarly for FH. Hence the centre of the conic is the intersection of EG, FH, which is also the intersection of the diagonals.

3. By the ordinary rule of partial fractions,

$$\frac{x^2}{(x-a)(x-b)(x-c)}\equiv\Sigma\frac{a^2}{(a-b)(a-c)}\cdot\frac{1}{x-a}.$$

Putting $x=\frac{1}{2}(a+b+c)$, the given identity easily follows.

4. If $$\Sigma a=p,\quad \Sigma ab=q,\quad \Sigma abc=r,$$

then $$\Sigma a^3=p^3-3pq+3r.$$

Hence Σa^3-3r is divisible by p, the quotient being

$$p^2-3q=\Sigma a^2-\Sigma ab.$$

5. (i) Since in any triangle $\frac{a}{r} = \cot \frac{1}{2} B + \cot \frac{1}{2} C$ with similar formulae, we have

$$\left(\frac{b_1}{r_1} - \cot \tfrac{1}{2} C\right)\left(\frac{b_2}{r_2} - \cot \tfrac{1}{2} C\right) = \cot \tfrac{1}{2} A_1 \,.\, \cot \tfrac{1}{2} A_2$$
$$= 1, \text{ since } \tfrac{1}{2} A_1 + \tfrac{1}{2} A_2 = 90°.$$

(ii) Using the same formulae,

$$\frac{a-c}{r_2} = \cot \tfrac{1}{2} C - \cot \tfrac{1}{2} A_2 = \cot \tfrac{1}{2} C - \tan \tfrac{1}{2} A_1$$
$$= \frac{\cos \frac{1}{2}(C + A_1)}{\sin \frac{1}{2} C \cos \frac{1}{2} A_1} = \frac{\sin \frac{1}{2} B_1}{\sin \frac{1}{2} C \cos \frac{1}{2} A_1}.$$

Also $$\frac{a}{r_1} = \cot \tfrac{1}{2} B_1 + \cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(B_1 + C)}{\sin \frac{1}{2} B_1 \sin \frac{1}{2} C}$$
$$= \frac{\cos \frac{1}{2} A_1}{\sin \frac{1}{2} B_1 \sin \frac{1}{2} C}.$$

Multiplying these, the result follows.

6. Since $1 - \Sigma \cos^2 A - 2 \cos A \cos B \cos C \equiv 0$,

and since

$$1 - \Sigma \cos^2 \theta \sin^2 \phi = 1 - \Sigma \sin^2 \theta + \Sigma \sin^2 \theta \sin^2 \phi$$
$$= (1 - \sin^2 \theta)(1 - \sin^2 \phi)(1 - \sin^2 \psi) + \sin^2 \theta \sin^2 \phi \sin^2 \psi$$
$$= \Pi \cos^2 \theta + \Pi \sin^2 \theta;$$

it follows that

$$\Pi \cos^2 \theta + \Pi \sin^2 \theta - 2\Pi \sin \theta \cos \theta = 0,$$
$$\text{i.e. } (\Pi \cos \theta - \Pi \sin \theta)^2 = 0,$$

or $$\Pi \tan \theta = 1.$$

7. The polar of (X, Y) is $xX + yY = a^2$. Transferring to (h, k) as origin, this becomes

$$xX + yY = a^2 - hX - kY \quad \text{..................(i)},$$

while the equation to the circle is now

$$x^2 + y^2 + 2hx + 2ky + h^2 + k^2 - a^2 = 0 \text{............(ii)}.$$

Forming the equation to the lines joining the origin to the intersections of (i) and (ii) by the ordinary rule, and writing down the condition that these are perpendicular, we obtain the required locus of (X, Y).

8. For the intersections of

$$\frac{x-x'}{\cos\theta}=\frac{y-y'}{\sin\theta}=r \text{ with } y^2=4ax,$$

we have $$(r\sin\theta+y')^2=4a(r\cos\theta+x').$$

If (x', y') is the middle point of the chord, the values of r given by this equation will be equal and opposite,

$$\therefore\ y'\sin\theta=2a\cos\theta,$$

i.e. the equation to the chord is

$$y'(y-y')=2a(x-x').$$

If this passes through (h, k) we have

$$y'(k-y')=2a(h-x'),$$

so that the locus of (x', y') is

$$y(k-y)=2a(h-x).$$

9. If ϕ be the eccentric angle of P, we have

$$SP=a-e\,.\,a\cos\phi,\quad SQ=a+e\,.\,a\sin\phi,$$

$$\therefore\ (SP-SQ)^2=a^2e^2(\sin\phi+\cos\phi)^2.$$

Also $$PQ^2=a^2(\cos\phi+\sin\phi)^2+b^2(\sin\phi-\cos\phi)^2,$$

$$\therefore\ PQ^2-(SP-SQ)^2=b^2(\cos\phi+\sin\phi)^2+b^2(\sin\phi-\cos\phi)^2$$
$$=2b^2.$$

10. Let A be the upper end of the rod, G its middle point, N the point of contact with the circle (centre O), α the angle AON, θ the angle the rod makes with the vertical. Then if ON and the direction of the string meet in K, GK is vertical.

Hence $$\frac{AN}{GN}=\frac{\tan AKN}{\tan GKN}=\frac{\cot 2\alpha}{\cot\theta}.$$

But $\tan\alpha=\dfrac{na}{b}$, $\therefore\ \tan\theta=\dfrac{n}{1-n}\,.\,\tan 2\alpha=\dfrac{n}{1-n}\cdot\dfrac{2nab}{b^2-a^2n^2}.$

11. Let f_1, f_2 be the accelerations of m along and perpendicular to the face of the wedge; f_3, f_4 those of m'; f the horizontal acceleration of the wedge; R and R' the pressures.

Then $\quad mf_2 = mg\cos\alpha - R, \qquad m'f_4 = m'g\cos\beta - R',$

$$f_2 = f\sin\alpha, \qquad f_4 = -f\sin\beta.$$

$$Mf = R\sin\alpha - R'\sin\beta$$
$$= m(g\cos\alpha - f\sin\alpha)\sin\alpha - m'(g\cos\beta + f\sin\beta)\sin\beta,$$

giving the required value of f.

12. With the usual notation we have

$$[M'][L']^2[T']^{-2} = \tfrac{1}{2} \,.\, 2240m \,.\, (\tfrac{22}{15}v)^2[M][L]^2[T]^{-2},$$
$$[M'][L']^2[T']^{-3} = 32 \,.\, 550h \,.\, [M][L]^2[T]^{-3},$$
$$[M'][L'][T']^{-2} = 32 \,.\, 2240n \,.\, [M][L][T]^{-2}.$$

Multiplying the first equation by the square of the third, and dividing by the square of the second, we get the required value for $[M']$.

III.

1. Since $TDBC$ and $UEBC$ are cyclic,

$$\therefore\ T\hat{B}U = T\hat{B}C + U\hat{B}C = A\hat{D}C + U\hat{E}C$$
$$= 360° - 2D\hat{A}B - (180° - D\hat{A}B)$$
$$= 180° - B\hat{A}U, \text{ since } D\hat{A}B = B\hat{A}U.$$
$$\therefore\ A\hat{B}T + A\hat{B}U + B\hat{A}U = 180°,$$
$$\therefore\ A\hat{B}T = A\hat{U}B \text{ and } B\hat{A}T = B\hat{A}U,$$
$$\therefore\ A\hat{T}B = A\hat{B}U,$$

and the triangles ATB, ABU are similar. Hence the result.

2. If PM, QN are the ordinates, we have

$$\frac{MG^2}{GN^2} = \frac{PM^2}{QN^2} = \frac{AM}{AN},$$

$$\therefore \frac{(AG-AM)^2}{(AN-AG)^2}=\frac{2AM.AG}{2AN.AG}=\frac{AG^2+AM^2}{AN^2+AG^2},$$

$$\therefore \frac{AG^2+AM^2}{AM}=\frac{AN^2+AG^2}{AN}=\frac{AN^2-AM^2}{AN-AM}$$

$$=AN+AM.$$

$$\therefore \ AG^2=AM.AN,$$

whence the property in question immediately follows.

3. The term involving x^{n-2} is

$$\frac{-\frac{11}{3}(-\frac{11}{3}-1)\dots(-\frac{11}{3}-\overline{n-2}+1)}{(n-2)!}(-\tfrac{3}{4}x)^{n-2}$$

$$=(-1)^{n-2}\frac{11.14\dots(3n+2)}{(n-2)!}.(-1)^{n-2}(\tfrac{1}{4}x)^{n-2}$$

$$=\frac{11.14\dots(3n+2)}{4^{n-2}.(n-2)!}.x^{n-2}.$$

The given form is obtained by multiplying numerator and denominator by $2.5.8$.

4. Let N be the number. Then by Fermat's Theorem,

$$N^6=7m+1,$$

$$\therefore N^{42}=(7m+1)^7$$

$$=M(49)+1, \text{ by the Binomial Theorem.}$$

5. Let L, M, N be the points of contact. Then if Δ' be the area of LMN, we have

$$\Delta'=\tfrac{1}{2}r^2.\Sigma\sin A=2r^2.\Pi\cos\frac{A}{2},$$

and $$\Delta=2R^2.\Pi\sin A.$$

Also $\quad r=4R.\Pi\sin\dfrac{A}{2}, \quad$ whence $\quad \dfrac{\Delta'}{\Delta}=\dfrac{r}{2R}.$

6. If $$f(x)=\cot x-4x,$$

we have $$f(0)=\infty, \quad f\left(\frac{\pi}{6}\right)=\sqrt{3}-\frac{2\pi}{3},$$

which is negative.

Hence the root lies between 0 and $\frac{\pi}{6}$. From the graphs $y=\cot x$ and $y=4x$, its value is approximately $\cdot 48$.

7. If the tangent $x\cos\theta + y\sin\theta = a$ coincides with the polar $xx' + yy' = b(x + x')$, we have

$$\frac{x'-b}{\cos\theta} = \frac{y'}{\sin\theta} = \frac{bx'}{a},$$

whence $$(x'-b)^2 + y'^2 = \left(\frac{bx'}{a}\right)^2,$$

giving the required locus for (x', y').

8. The intersection of the tangents at m and m' is the point $amm', a(m+m')$, and the intersection of the normals is

$$a(m^2 + mm' + m'^2) + 2a, \qquad -amm'(m+m').$$

Now $amm' = c$. Hence, putting $m + m' = \lambda$, the latter point is

$$x = a\left(\lambda^2 - \frac{c}{a}\right) + 2a, \quad y = -c\lambda.$$

Eliminating λ, the result follows.

9. The abscissae of the intersections of

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \text{ and } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are determined by the equation

$$\frac{x^2}{a^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) - \frac{2xx'}{a^2} + 1 - \frac{y'^2}{b^2} = 0.$$

If x_1, x_2 are the roots of the equation, and if (x', y') lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$, we have $x_1 + x_2 = \frac{1}{2}x'$.

Hence if $(\bar{x}, \bar{y})$ be the centroid,

$$\bar{x} = \tfrac{1}{3}(x_1 + x_2 + x') = \tfrac{1}{2}x'.$$

Similarly $\bar{y} = \frac{1}{2}y'$, so that

$$\frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{b^2} = \tfrac{1}{4}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) = 1.$$

10. With the usual notation

$$W\sin\beta = T + \mu R = T + \mu W\cos\beta$$

and $$T = W\sin\gamma + \mu R' = W\sin\gamma + \mu W\cos\gamma,$$

$$\therefore\ W(\sin\beta - \sin\gamma) = \mu W(\cos\beta + \cos\gamma),$$

whence $\mu = \tan\frac{1}{2}(\beta - \gamma)$.

11. Let T, T' be the tensions of the strings round the axle and wheel respectively, and f, f' the accelerations of the weights. Then

$$T - 3g = 3f, \quad g - T' = f', \quad f = \tfrac{1}{4}f', \quad T = 4T',$$

whence
$$3g + 3f = 4g - 4f' = 4g - 16f,$$
$$\therefore f = \tfrac{1}{19}g.$$

12. Taking the point of projection as origin, and θ as the angle of projection, the equation to the path is

$$y = x\tan\theta - \tfrac{1}{2}g \cdot \frac{x^2}{V^2\cos^2\theta},$$

and to the line of the plane

$$y = x\tan\alpha + c\sec\alpha.$$

Eliminating y, we get

$$\tfrac{1}{2}g \cdot \frac{x^2}{V^2\cos^2\theta} + x(\tan\alpha - \tan\theta) + c\sec\alpha = 0.$$

Hence, if the intersections are real,

$$(\tan\alpha - \tan\theta)^2 > \frac{2g}{V^2\cos^2\theta} \cdot c\sec\alpha,$$

$$\text{i.e. } V^2 > 2cg \cdot \frac{\cos\alpha}{\sin^2(\alpha - \theta)},$$

which is impossible unless

$$V^2 > 2cg\cos\alpha.$$

IV.

1. Let Q be the middle point of AB, and draw PN perpendicular to AB. Call AB unity, and let $AP = k$. Then

$$PQ = \tfrac{1}{2} + (1 - k) = \tfrac{3}{2} - k.$$

Hence
$$QN^2 - AN^2 = (\tfrac{3}{2} - k)^2 - k^2 = \tfrac{9}{4} - 3k$$

and
$$QN + AN = \tfrac{1}{2}.$$
$$\therefore QN - AN = \tfrac{9}{2} - 6k,$$
$$\therefore AN = 3k - 2.$$

But $AN = \tfrac{1}{2}AP = \tfrac{1}{2}k$,

$$\therefore 3k - 2 = \tfrac{1}{2}k, \text{ whence } k = \tfrac{4}{5}.$$

2. Let P, P' be the two points, and let PP' meet the directrix in Z. Then SZ is the external bisector of $P\hat{S}P'$. Hence, if PP' is divided internally in Y in the ratio in which Z divides it externally, the locus of S is the circle on YZ as diameter.

3. Here $$a^2+b^2=\tfrac{1}{2}(u+v), \quad ab=\tfrac{1}{2}(u-v),$$
$$\therefore\ a^4+b^4=(a^2+b^2)^2-2a^2b^2$$
$$=\tfrac{1}{4}(u+v)^2-\tfrac{1}{2}(u-v)^2.$$
Also $$a^6+b^6=(a^2+b^2)\left[(a^2+b^2)^2-3a^2b^2\right]$$
$$=\tfrac{1}{4}(u+v)(4uv-u^2-v^2).$$

4. Since
$$\frac{x+2}{x^2+x+1}=\frac{(2+x)(1-x)}{1-x^3}=(2-x-x^2)(1-x^3)^{-1}$$
the result follows from the Binomial Theorem.

5. The given expression
$$=2\cos\frac{8\pi}{15}\cos\frac{6\pi}{15}+2\cos\frac{6\pi}{15}\cos\frac{2\pi}{15}$$
$$=2\cos\frac{2\pi}{5}\left(\cos\frac{8\pi}{15}+\cos\frac{2\pi}{15}\right)$$
$$=2\cos\frac{2\pi}{5}\,.\,2\cos\frac{\pi}{5}\cos\frac{\pi}{3}$$
$$=2\cos\frac{2\pi}{5}\cos\frac{\pi}{5}=2\sin\frac{\pi}{10}\,.\,\cos\frac{\pi}{5}$$
$$=2\,.\,\frac{\sqrt{5}-1}{4}\,.\,\frac{\sqrt{5}+1}{4}=\tfrac{1}{2}.$$

6. If the first circle touches IE in N, we have
$$r=IE=IN+NE=\rho_1\tan\frac{A}{2}+\rho_1,$$
$$\therefore\ \frac{\rho_1}{r-\rho_1}=\cot\frac{A}{2}.$$
But since $$\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}, \qquad \therefore\ \Sigma\cot\frac{A}{2}=\Pi\cot\frac{A}{2}.$$
Hence the result.

7. Let the straight line be $lx+my=k$. Then P is $\left(\frac{k}{l},\ 0\right)$ and any line through P is $y=\lambda\left(x-\frac{k}{l}\right)$. If this is perpendicular to $x+py=0$, we have

$$\lambda-p=(\lambda p-1)\cos\omega,\ \text{ i.e. } \lambda(1-p\cos\omega)=p-\cos\omega.$$

Hence the line is

$$(1-p\cos\omega)\,y=(p-\cos\omega)\left(x-\frac{k}{l}\right).$$

Similarly the line through Q is

$$(1-q\cos\omega)\,x=(q-\cos\omega)\left(y-\frac{k}{m}\right).$$

Eliminating k, the locus of the intersection is

$$(q-\cos\omega)\left[(1-p\cos\omega)\,l+(p-\cos\omega)\,m\right]y$$
$$=(p-\cos\omega)\left[(1-q\cos\omega)\,m+(q-\cos\omega)\,l\right]x.$$

8. Let the parabola be $y^2=4ax$. Then the equations to t_1 and t_2 may be taken in the forms

$$x\cos\alpha_1+y\sin\alpha_1+a\sin\alpha_1\tan\alpha_1=0,$$
$$x\cos\alpha_2+y\sin\alpha_2+a\sin\alpha_2\tan\alpha_2=0.$$

The equation to h is found by adding these, and is thus

$$x\cos\frac{\alpha_1+\alpha_2}{2}+y\sin\frac{\alpha_1+\alpha_2}{2}+\frac{a(\sin\alpha_1\tan\alpha_1+\sin\alpha_2\tan\alpha_2)}{2\cos\frac{\alpha_1-\alpha_2}{2}}=0,$$

whence also the equation to t is

$$x\cos\frac{\alpha_1+\alpha_2}{2}+y\sin\frac{\alpha_1+\alpha_2}{2}+a\sin\frac{\alpha_1+\alpha_2}{2}\tan\frac{\alpha_1+\alpha_2}{2}=0.$$

The perpendiculars from $(a, 0)$ on h and t are

$$a\left(\cos\frac{\alpha_1+\alpha_2}{2}+\frac{\sin\alpha_1\tan\alpha_1+\sin\alpha_2\tan\alpha_2}{2\cos\frac{\alpha_1-\alpha_2}{2}}\right)\ \text{ and }\ a\sec\frac{\alpha_1+\alpha_2}{2},$$

and the product of these is

$$a^2\left(1+\frac{\sin\alpha_1\tan\alpha_1+\sin\alpha_2\tan\alpha_2}{\cos\alpha_1+\cos\alpha_2}\right)=a^2\cdot\frac{\sec\alpha_1+\sec\alpha_2}{\cos\alpha_1+\cos\alpha_2}$$

$$=a^2\sec\alpha_1\sec\alpha_2,$$

which proves the result, since the perpendiculars from $(a, 0)$ on t_1 and t_2 are $a\sec\alpha_1$ and $a\sec\alpha_2$.

9. If the points $(a\cos\phi, b\sin\phi)$, $(-a\sin\phi, b\cos\phi)$ both lie on the second ellipse, we have

$$\frac{a^2\cos^2\phi}{\alpha^2}+\frac{b^2\sin^2\phi}{\beta^2}=1, \qquad \frac{a^2\sin^2\phi}{\alpha^2}+\frac{b^2\cos^2\phi}{\beta^2}=1,$$

whence, adding, $$\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}=2.$$

10. In general, the centre of gravity of a quadrilateral coincides with that of four equal particles at the angular points and a fifth equal negative particle at the intersection of the diagonals. Hence in this case the centre of gravity of the four equal particles at the angular points must be at the intersection of the diagonals, and this can only be when the diagonals bisect each other, i.e. when the quadrilateral is a parallelogram.

11. Let f, f' be the downwards and upwards accelerations in space of the masses 5 lbs. and 3 lbs., and T the tension of the string. Then

$$5g-T=5f, \qquad T-3g=3f'.$$

Also, since the accelerations of the two masses relative to the lift are the same, we have

$$f+2=f'-2,$$

$$\text{i.e. } f'-f=4.$$

Hence $$\left(\frac{T}{3}-g\right)-\left(g-\frac{T}{5}\right)=4,$$

$$\text{i.e. } \frac{8T}{15}=2g+4=2g+\tfrac{1}{8}g=\tfrac{17}{8}g,$$

$$\therefore\ T=\tfrac{255}{64}g,$$

i.e. the tension is $3\frac{63}{64}$ lbs. wt.

Also, substituting for T in the first two equations, we find $f = \frac{13}{2}$ and $f' = \frac{21}{2}$ ft./(sec.)^2.

12. We have

$$\frac{v^2 \sin^2 \alpha}{2g} = h, \qquad \frac{v^2 \sin \alpha \cos \alpha}{g} = a, \qquad v^2 = 2gc,$$

whence the possible values of h are the roots of

$$\frac{h}{c}\left(1 - \frac{h}{c}\right) = \frac{a^2}{4c^2},$$

or

$$h^2 - ch + \frac{a^2}{4} = 0.$$

Hence, since b is the difference of the roots,

$$b^2 = c^2 - a^2.$$

V.

1. Since the quadrilaterals $AA'B'B$ and $\alpha A'B'\beta$ are both cyclic,

$$\therefore \; A'\hat{A}B = 180° - A'\hat{B'}B = A'\hat{\alpha}\beta.$$

Therefore $\alpha\beta$ is parallel to AB, and similarly for the other sides.

2. Let $ABCD$ be the quadrilateral, and let each of the tangents from A subtend an angle α at the focus, each of those from B an angle β, and so on. Then, since AC and BD are straight lines, $\alpha + \beta = \gamma + \delta$. But

$$2(\alpha + \beta + \gamma + \delta) = 360° \,; \quad \therefore \; \alpha + \beta = 90°,$$

i.e. the diagonals are at right angles. Further, since

$$2\alpha + 2\beta = 180°,$$

each of the lines joining the points of contact of opposite sides is a focal chord. Therefore the extremities of the third diagonal are on the directrix.

3. The series $1-a_1+a_2-\dots$ to $(n+2)$ terms is the absolute term in the product of $(1+x)^{2n}$, and the series

$$1-\frac{1}{x}+\frac{1}{x^2}-\dots+(-1)^{n+1}\frac{1}{x^{n+1}}=\frac{1-(-1)^{n+2}\dfrac{1}{x^{n+2}}}{1+\dfrac{1}{x}},$$

and is therefore

$$\text{the coefficient of } x^{n+1} \text{ in } (1+x)^{2n-1}\left[x^{n+2}-(-1)^{n+2}\right]$$

$$= \dots\dots\dots\dots\dots\dots\dots\dots\dots (-1)^{n+1}(1+x)^{2n-1}$$

$$=(-1)^{n+1}\frac{(2n-1)!}{(n+1)!\,(n-2)!}.$$

4. Let x be the distance of the cutting plane from the centre. Then, regarding the spherical cap cut off as the difference between a spherical sector and a cone, we have

$$\tfrac{1}{3}\,.\,2\pi(1-x)-\tfrac{1}{3}x\pi(1-x^2)=\tfrac{1}{3}\pi,$$

since the volume of the cap is $\frac{1}{4}$ that of the sphere.

This equation at once reduces to $x^3-3x+1=0$, and the root between 0 and 1, determined from the graphs $y=x^3$ and $y=3x-1$, is about ·35.

5. Since $\qquad A=\pi+\beta-\gamma, \quad C=\pi+\alpha-\beta;$

$$\therefore \quad B=-\pi+\gamma-\alpha.$$

Hence the identity to be proved is

$$\Sigma\sin(\beta-\gamma)\cos 2\alpha+[\Sigma\sin(\beta-\gamma)]\,[\Sigma\cos(\beta+\gamma)]=0.$$

Now

$$2\sin(\beta-\gamma)\cos(\beta+\gamma)=\sin 2\beta-\sin 2\gamma,$$

$$2\sin(\beta-\gamma)\cos(\gamma+\alpha)=\sin(\alpha+\beta)-\sin(2\gamma+\alpha-\beta),$$

$$2\sin(\beta-\gamma)\cos(\alpha+\beta)=\sin(2\beta+\alpha-\gamma)-\sin(\alpha+\gamma).$$

Hence the second term of the identity is clearly equal to

$$\tfrac{1}{2}\left[\Sigma\sin(2\alpha+\gamma-\beta)-\Sigma\sin(2\alpha+\beta-\gamma)\right],$$

while the first term is

$$\tfrac{1}{2}\left[\Sigma\sin(2\alpha+\beta-\gamma)-\Sigma\sin(2\alpha+\gamma-\beta)\right].$$

6. If $A'B'C'$ is the triangle formed,

$$A'B'^2 = \tfrac{1}{4}\sec^2\theta\,[a^2 + b^2 - 2ab\cos(C - 2\theta)]$$
$$= \tfrac{1}{4}\sec^2\theta\,[c^2 - 4ab\sin\theta\sin(C - \theta)].$$

Hence, if the triangles are similar,

$$\frac{c^2 - 4ab\sin\theta\sin(C-\theta)}{b^2 - 4ca\sin\theta\sin(B-\theta)} = \frac{c^2}{b^2} = \frac{b\sin(C-\theta)}{c\sin(B-\theta)},$$

$$\therefore\quad \sin^3 B\sin(C-\theta) = \sin^3 C\sin(B-\theta),$$

whence $$\tan\theta = \frac{\tan B\tan C(\sin^2 B - \sin^2 C)}{\sin^2 B\tan B - \sin^2 C\tan C}.$$

The denominator is

$$\tan B - \tan C - \tfrac{1}{2}(\sin 2B - \sin 2C).$$

Hence, dividing through by $\sin(B - C)$, we have

$$\tan\theta = \frac{\tan B\tan C\sin A}{\dfrac{1}{\cos B\cos C} + \cos A} = \frac{\sin A\sin B\sin C}{1 + \cos A\cos B\cos C}.$$

7. The lines being $y - m_1x = 0$, $y - m_2x = 0$, the co-ordinates of M are given by

$$\frac{x}{1} = \frac{y}{m_1} = \frac{x' + m_1y'}{1 + m_1^2},$$

and similarly for N.

Hence, since the remaining angular point is the origin, the area of the triangle is

$$\tfrac{1}{2}(m_1 - m_2)\frac{(x' + m_1y')(x' + m_2y')}{(1 + m_1^2)(1 + m_2^2)},$$

which, remembering that $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1m_2 = \dfrac{a}{b}$, reduces to the given expression.

8. The condition that the chord m, m' of $y^2 = 4ax$ should coincide with the normal at m is

$$\frac{m}{2} = -\frac{1}{m + m}.$$

Since $m = \tan\alpha$, $m' = \tan\alpha'$, this becomes

$$\sin\alpha\sin(\alpha + \alpha') = -2\cos^2\alpha\cos\alpha',$$

i.e. $$\cos\alpha' - \cos(2\alpha + \alpha') = -2(1 + \cos 2\alpha)\cos\alpha',$$

i.e. $$3\cos\alpha' + \cos(2\alpha - \alpha') = 0.$$

9. Any tangent to the hyperbola is of the form

$$m^2x - 2mc + y = 0 \text{......................(i)},$$

and this will be a tangent to the ellipse if

$$a^2m^4 + b^2 = 4c^2m^2 \text{......................(ii)}.$$

We have therefore to eliminate m between (i) and (ii).

Now from (i)

$$(m^2x + y)^2 = 4c^2m^2,$$

$$\text{i.e. } m^4x^2 + (2xy - 4c^2)\, m^2 + y^2 = 0 \text{ ....(iii)}.$$

From (ii) and (iii)

$$\frac{m^4}{-4c^2y^2 - b^2(2xy - 4c^2)} = \frac{m^2}{b^2x^2 - a^2y^2} = \frac{1}{a^2(2xy - 4c^2) + 4c^2x^2},$$

whence the required equation follows.

10. Let A, N be the points of contact of the ruler with the wall and rail respectively, and draw NO, AO perpendicular to the axis of the ruler and the wall. Then O must be vertically beneath the centre of gravity of the ruler. Hence

$$AO = l \sin\theta + a\cos\theta,$$

$$ON = OA\cos\theta - 2a = l\sin\theta\cos\theta + a\cos^2\theta - 2a.$$

$$\therefore \quad b = OA - ON\cos\theta$$

$$= l\sin\theta + a\cos\theta - (l\sin\theta\cos\theta + a\cos^2\theta - 2a)\cos\theta$$

$$= l\sin^3\theta + a\cos\theta\sin^2\theta + 2a\cos\theta.$$

11. The second ball reaches the ground in time $\sqrt{\frac{h}{g}}$, and rebounds with velocity $e\,.\sqrt{gh}$. Let t be the time after the rebound at which the balls pass each other. The upward distance described by the second ball is $e\sqrt{gh}\,.\,t - \frac{1}{2}gt^2$, and the downward distance described by the first is $\frac{1}{2}g\left(\sqrt{\frac{h}{g}} + t\right)^2$. The sum of these is h, whence we obtain

$$t = \frac{1}{2(1+e)}\,.\sqrt{\frac{h}{g}},$$

and therefore

$$e\sqrt{gh}\,.\,t-\tfrac{1}{2}gt^2=\frac{e}{2\,(1+e)}\,.\,h-\frac{1}{8}\cdot\frac{1}{(1+e)^2}\,.\,h$$
$$=\frac{4e^2+4e-1}{8\,(1+e)^2}\,.\,h.$$

12. Let f_1, f_2 be the accelerations of the masses perpendicular to the respective faces of the wedge, f the horizontal acceleration of the wedge, R and R' the pressures. Then

$$m_1g\,.\,\frac{1}{\sqrt{2}}-R=m_1f_1,$$

$$m_2g\,.\,\frac{1}{\sqrt{2}}-R'=m_2f_2,\qquad f_1=f\cdot\frac{1}{\sqrt{2}},$$

$$(R-R')\,.\,\frac{1}{\sqrt{2}}=Mf\ \text{(i)},\qquad f_2=-f\cdot\frac{1}{\sqrt{2}}.$$

From these

$$R=m_1(g-f)\cdot\frac{1}{\sqrt{2}},\qquad R'=m_2(g+f)\cdot\frac{1}{\sqrt{2}},$$

whence (i) gives the required value for f.

VI.

1. Since $B\hat{E}D=B$, and $C\hat{E}D=C$, therefore $B\hat{E}C=180°-A$, i.e. E is on the circle ABC. Produce AD to meet this circle in F. Then $B\hat{F}D=C=C\hat{E}D$. Hence evidently F must be the reflection of E in the line through D perpendicular to BC, i.e. $DE=DF$. But $AD\,.\,DF=BD\,.\,DC$. Therefore $AD\,.\,DE=DC^2$.

2. Let U be the intersection, Y the foot of the perpendicular from S on the tangent. Then the triangle SYU is given in species; one vertex (S) is fixed, and another vertex (Y) describes a circle, viz. the auxiliary circle. Hence the locus of the third vertex U is also a circle.

3. We have

$$(\Sigma x)^3 - \Sigma x^3 \equiv 3 \,.\, \Pi\,(y+z).$$

But the expression $(\Sigma x)^{2n+1} - \Sigma x^{2n+1}$ vanishes identically when $y = -z$, i.e. $(y+z)$ is a factor, and similarly for $(z+x)$ and $(x+y)$. Hence the result.

4. If $m < n$, we have by the ordinary rule

$$\frac{x^m}{(x-a_1)(x-a_2)\ldots(x-a_n)} = \sum_{r=1}^{r=n} \frac{a_r{}^m}{(a_r-a_1)\ldots(a_r-a_n)} \cdot \frac{1}{x-a_r}.$$

Putting $x = 0$ in this identity, the first result follows.

If $m = n-1$, the above identity may be written

$$\prod_{r=1}^{r=n} \cdot \left(1 - \frac{a_r}{x}\right)^{-1} = \sum_{r=1}^{r=n} \cdot \frac{a_r{}^{n-1}}{(a_r-a_1)\ldots(a_r-a_n)} \cdot \left(1 - \frac{a_r}{x}\right)^{-1}.$$

Expanding each side, and equating the absolute terms and the coefficients of $\frac{1}{x}$, the other two results follow.

5. Using the graph $y = \cos x$ and the circle $x^2 + y^2 = 2$, (x and y being, of course, on the same scale), we obtain the approximate value $x = 1{\cdot}4$ at the intersection. To obtain a closer approximation, assume $x = 1{\cdot}4 + \xi$, retaining only the first power of ξ, and using the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. This leads to the closer value $x = 1{\cdot}405$.

6. Let $a = \frac{\pi}{14}$, and call the given product x. Then

$$\begin{aligned} 4x \cos a &= 2 \sin 2a \sin 3a \sin 5a \\ &= (\cos 3a - \cos 7a) \sin 3a \\ &= \tfrac{1}{2} \sin 6a \\ &= \tfrac{1}{2} \cos a, \qquad \text{since } 7a = \frac{\pi}{2}, \end{aligned}$$

$$\therefore \quad x = \tfrac{1}{8}.$$

7. The line $y = 3x + c$ meets the given conic where

$$2x^2 - x(3x+c) + 3(3x+c)^2 = 1,$$

$$\text{i.e. } 26x^2 + 17cx + 3c^2 - 1 = 0.$$

Hence, if $(\bar{x}, \bar{y})$ be the middle point of the intercept,

$$\bar{x} = \tfrac{1}{2}(\text{sum of roots}) = -\tfrac{17}{52}c.$$

Also $$\bar{y} = 3\bar{x} + c = \tfrac{1}{52}c, \text{ i.e. } \bar{x} = -17\bar{y}.$$

8. The chord joining m, m' on $y^2 = 4ax$ is

$$2x - (m + m')\,y + 2amm' = 0,$$

and the tangent at μ to $y^2 = 4a'x$ is

$$x - \mu y + a'\mu^2 = 0.$$

If these coincide, we must have $\mu = \dfrac{m + m'}{2}$.

But $\mu = \cot\theta_1$, $m = \cot\theta_2$, $m' = \cot\theta_3$. Hence the result.

9. If P is (x', y'), the tangent at P is $\dfrac{yy'}{b^2} - \dfrac{xx'}{a^2} = 1$, and the equation to CQ, CR is

$$y^2 = c^2\left(\frac{yy'}{b^2} - \frac{xx'}{a^2}\right)^2.$$

Hence the equation to the bisectors of the angle QCR is

$$\frac{c^2x'y'}{a^2b^2}(x^2 - y^2) + \left(\frac{c^2x'^2}{a^4} - \frac{c^2y'^2}{b^4} + 1\right)xy = 0,$$

and these will coincide with $(xy' - yx')(xx' + yy') = 0$, provided

$$\frac{a^2b^2}{c^2}\left(\frac{c^2x'^2}{a^4} - \frac{c^2y'^2}{b^4} + 1\right) = y'^2 - x'^2,$$

i.e. if $c^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) = 1$, remembering that $\dfrac{y'^2}{b^2} - \dfrac{x'^2}{a^2} = 1$.

10. Let T be the tension of the string, S the pressure on either peg, and let ϕ be the angle which each of the lower portions of the string makes with the horizontal. Then evidently

$$\cos\phi = \frac{c}{l - c}, \qquad \sin\phi = \frac{\sqrt{l(l - 2c)}}{l - c}.$$

Also $\quad 2T\sin\phi = W, \qquad \therefore\quad T = \dfrac{W}{2}\cdot\dfrac{l - c}{\sqrt{l(l - 2c)}},$

and $\quad S^2 = 2T^2(1 + \cos\phi) = 2T^2\cdot\dfrac{l}{l - c}.$

$$\therefore\quad S = \sqrt{2}\cdot\frac{W}{2}\cdot\frac{l - c}{\sqrt{l(l - 2c)}}\cdot\sqrt{\frac{l}{l - c}}$$

$$= W\cdot\sqrt{\frac{l - c}{2(l - 2c)}}.$$

11. Let l be the length of the wire, r the radius of the quadrant. Then

$$2r + \frac{\pi}{2} r = l, \quad \text{i.e. } r = \frac{2}{4+\pi} l.$$

The weights of the straight and curved portions are in the ratio $2 : \frac{\pi}{2}$, and the distances of their centres of gravity from the centre along the middle radius are respectively $\frac{r}{2\sqrt{2}}$ and $\frac{2\sqrt{2}}{\pi} r$. Hence the distance of the centre of gravity of the whole from the centre is

$$\frac{\frac{r}{\sqrt{2}} + \sqrt{2}r}{2 + \frac{\pi}{2}} = \frac{3\sqrt{2}r}{4+\pi} = \frac{6\sqrt{2}}{(4+\pi)^2} l.$$

12. The horizontal distance of the point of intersection of the paths satisfies the equation

$$x \tan \alpha - \tfrac{1}{2} g \cdot \frac{x^2}{u^2 \cos^2 \alpha} = x \tan \alpha' - \tfrac{1}{2} g \cdot \frac{x^2}{u^2 \cos^2 \alpha'},$$

$$\text{i.e.} \quad x (\tan \alpha - \tan \alpha') = \tfrac{1}{2} g \cdot \frac{x^2}{u^2} (\tan^2 \alpha - \tan^2 \alpha'),$$

whence
$$\tan \alpha + \tan \alpha' = \frac{2u^2}{gx}.$$

But
$$t = \frac{x}{u \cos \alpha'}, \quad T = \frac{u \sin \alpha}{g}; \qquad \therefore \quad Tt = \frac{x}{g} \tan \alpha.$$

Hence
$$Tt + T't' = \frac{x}{g} (\tan \alpha + \tan \alpha') = \frac{2u^2}{g^2}.$$

VII.

1. Since $PM'A$ is a transversal of triangle $BC'B'$,

$$\therefore \frac{B'M'}{M'C'} \cdot \frac{C'P}{PB} \cdot \frac{BA}{AB'} = -1,$$

$$\text{i.e.} \quad \frac{B'M'}{M'C'} \cdot \frac{C'P}{PB} \cdot \frac{BC}{CM} = -1 \quad \text{..................(i).}$$

Also, since CPB' is a transversal of triangle $AC'B$,

$$\therefore \frac{C'P}{PB} \cdot \frac{BB'}{B'A} \cdot \frac{AC}{CC'} = -1,$$

$$\text{i.e.} \quad \frac{C'P}{PB} \cdot \frac{BM}{MC} \cdot \frac{BC}{CM} = -1 \quad \text{..................(ii).}$$

From (i) and (ii), it follows that

$$\frac{B'M'}{M'C'} = \frac{BM}{MC}.$$

2. If TQ' is the tangent to the circle, then $S\hat{T}Q' = S\hat{Q}T$. But if TP is the other tangent to the parabola, the triangles SQT, STP are similar.

Hence $\qquad S\hat{Q}T = S\hat{T}P, \qquad \therefore S\hat{T}Q' = S\hat{T}P,$

i.e. TP and TQ' coincide.

3. (i) The relation in the roots is

$$2\alpha'\beta' + 2\alpha\beta - (\alpha + \beta)(\alpha' + \beta') = 0,$$

$$\text{i.e. } (\alpha' - \alpha)(\beta' - \beta) = (\beta - \alpha')(\beta' - \alpha),$$

so that the roots, measured as lengths from a common origin, determine a harmonic range.

(ii) The relation in the rods is

$$[2\alpha'\beta' + 2\alpha\beta - (\alpha+\beta)(\alpha'+\beta')]^2 - (\alpha-\beta)^2(\alpha'-\beta')^2 = 0,$$

i.e. $(2\alpha'\beta' + 2\alpha\beta - 2\alpha'\beta - 2\alpha\beta')(2\alpha'\beta' + 2\alpha\beta - 2\alpha\alpha' - 2\beta\beta') = 0,$

i.e. $(\alpha'-\alpha)(\beta'-\beta)(\alpha'-\beta)(\beta'-\alpha) = 0.$

Hence the equations must have a common root.

4. (i) $\Sigma n(n+1)(n+3) = \Sigma n(n+1)(n+2) + \Sigma n(n+1)$

$$= \tfrac{1}{4}n(n+1)(n+2)(n+3) + \tfrac{1}{3}n(n+1)(n+2)$$

$$= \tfrac{1}{12}n(n+1)(n+2)(3n+13).$$

(ii) The nth term is $n + 3^{n-1}$,

$$\therefore\ S_n = \frac{n(n+1)}{2} + \tfrac{1}{2}(3^n - 1) = \tfrac{1}{2}(n^2 + n - 1 + 3^n).$$

(iii) In the identity

$$a_1(1-a_2) + a_1a_2(1-a_3) + \dots + a_1a_2\dots a_{n-1}(1-a_n)$$

$$\equiv a_1 - a_1a_2\dots a_n$$

put $\quad a_1 = a, \quad a_2 = \dfrac{a+1}{b}, \quad a_3 = \dfrac{a+2}{b+1}, \ \dots\ a_n = \dfrac{a+n-1}{b+n-2}.$

Then since $b > a+1$, these quantities (except a_1) are all proper fractions. Hence, denoting the sum to infinity of the given series by S, we have

$$(b-a-1)(S-1) = a,$$

$$\therefore\ S = \frac{b-1}{b-a-1}.$$

5. Since $\quad \Sigma x^2(y-z) = -\Pi(y-z),$

$$\therefore\ \Sigma(2x^2-1)(y-z) + 2\,.\,\Pi(y-z) = 0.$$

Putting $x = \cos 2A$, etc., the result follows.

6. We have

$$b \cos B + c \cos C = R(\sin 2B + \sin 2C)$$
$$= 2R \sin A \cos(B - C)$$
$$= a \cos(B - C),$$
$$\therefore\ a \cos(A + 2B) + b \cos B + c \cos C = 0,$$

since $$A + 2B = 180° + (B - C).$$

Writing down the two similar equations and eliminating a, b, c the result follows.

7. Let the first pair of lines be

(i) $y = mx$, (ii) $y = m'x$,

and the second pair

(iii) $y = \mu(x - a)$, (iv) $y = \mu'(x - a)$.

The intersection of (i) and (iii) is

$$\frac{y}{m} = \frac{x}{1} = \frac{\mu a}{\mu - m}$$ and so for (ii) and (iv).

Hence the middle point of the diagonal (i, iii), (ii, iv) is

$$x = \tfrac{1}{2}\left(\frac{\mu a}{\mu - m} + \frac{\mu' a}{\mu' - m'}\right),$$
$$y = \tfrac{1}{2}\left(\frac{m\mu a}{\mu - m} + \frac{m'\mu' a}{\mu' - m'}\right).$$

From these

$$2x - a = \frac{\mu\mu' - mm'}{(\mu - m)(\mu' - m')} \cdot a,$$
$$2y = \frac{(m + m')\mu\mu' - (\mu + \mu')mm'}{(\mu - m)(\mu' - m')} \cdot a.$$

Hence $$\frac{2x - a}{2y} = \frac{\dfrac{a'}{b'} - \dfrac{a}{b}}{-\dfrac{2h}{b} \cdot \dfrac{a'}{b'} + \dfrac{2h'}{b'} \cdot \dfrac{a}{b}} = \frac{a'b - ab'}{2(ah' - a'h)},$$

i.e. $(ah' - a'h)(2x - a) + (ab' - a'b)y = 0.$

Similarly for the middle point of the other diagonal.

8. The equation to the chord of which (x', y') is the middle point is

$$y'(y - y') = 2a(x - x'). \qquad \text{(See II. 8.)}$$

But $y' = mx' + c$ and therefore this equation may be written

$$y'(y - y') = 2a\left(x - \frac{y' - c}{m}\right),$$

or
$$y'^2 - y'\left(y + \frac{2a}{m}\right) + 2a\left(x + \frac{c}{m}\right) = 0,$$

and the envelope is as given.

9. The chord of the ellipse having (x', y') for its middle point is

$$\frac{x'(x - x')}{a^2} + \frac{y'(y - y')}{b^2} = 0,$$

$$\text{i.e. } \frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2}.$$

But $lx + my = n$ touches $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, if

$$a^2l^2 - b^2m^2 = n^2.$$

Hence in this case

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)^2,$$

giving the required locus for (x', y').

10. If the beam makes an angle θ with the horizontal, and the string an angle ϕ with the vertical, we have, taking moments about A,

$$w \,.\, l \sin\phi = W \,.\, \frac{a}{2}\cos\theta \quad \text{..................(i).}$$

Also
$$\frac{l}{\cos(\theta - \phi)} = \frac{a}{\sin\phi} = \frac{l - a\sin\theta}{\cos\theta\cos\phi}.$$

Hence, from (i)

$$w^2l^2a^2 = W^2 \,.\, \frac{a^2}{4}\cos^2\theta\left[a^2 + \frac{(l - a\sin\theta)^2}{\cos^2\theta}\right],$$

$$\text{i.e. } \frac{4w^2l^2}{W^2} = l^2 - 2al\sin\theta + a^2,$$

giving the required value of $\sin\theta$.

11. With the usual notation (see V. 12), we have

$$mg \sin a - \mu R = mf_1,$$

$$mg \cos a - R = mf_2, \qquad f_2 = f \sin a,$$

$$R \sin a - \mu R \cos a = Mf.$$

From these

$$\frac{R \sin (a - \epsilon)}{\cos \epsilon} = Mf, \qquad R = mg \cos a - mf \sin a,$$

$$\therefore \left[M + m \sin a . \frac{\sin (a - \epsilon)}{\cos \epsilon}\right] f = mg \cos a . \frac{\sin (a - \epsilon)}{\cos \epsilon},$$

given the value of f.

12. Let θ be the angle of projection and T the time of flight in either case, R the given range. Then

$$u \cos \theta . T = R \cos a,$$

$$u \sin \theta . T - \tfrac{1}{2} g T^2 = R \sin a,$$

whence, eliminating θ, we get

$$\tfrac{1}{4} g^2 T^4 - (u^2 - gR \sin a) T^2 + R^2 = 0.$$

The roots of the quadratic in T^2 are t^2 and t'^2;

$$\therefore t^2 + t'^2 = \frac{4 (u^2 - gR \sin a)}{g^2},$$

$$t^2 t'^2 = \frac{4R^2}{g^2}, \quad \text{i.e. } tt' = \frac{2R}{g},$$

$$\therefore t^2 + t'^2 + 2tt' \sin a = \frac{4u^2}{g^2}.$$

VIII.

1. Let O, O_1 be the centres of the smaller circle and one of the larger ones, B_1 the point of contact, A the point from which the tangents are drawn, AP the tangent, and let AB_1 meet the larger circle again in L.

Then $$O_1\hat{L}B_1 = O_1\hat{B}_1L = O\hat{A}B_1,$$

$$\therefore\ O_1L \text{ is parallel to } OA,$$

$$\therefore\ AP^2 : AB_1^2 = AB_1 \,.\, AL : AB_1^2 = AL : AB_1 = OO_1 : OB_1.$$

Hence, evidently, if AQ, AR be the tangents to the other circles, B_2, B_3 the points of contact with the smaller circle, we shall have

$$AP : AB_1 = AQ : AB_2 = AR : AB_3.$$

But triangle $B_1B_2B_3$ is evidently equilateral, and therefore by Ptolemy's Theorem, supposing A to lie between B_2 and B_3,

$$AB_1 = AB_2 + AB_3,$$

$$\therefore\ AP = AQ + AR.$$

2. We have

$$S\hat{R}P = 180^\circ - S\hat{Q}P, \qquad S\hat{R}'Q = 180^\circ - S\hat{P}'Q.$$

But the triangles SPQ, SQP' are similar,

$$\therefore\ S\hat{Q}P = S\hat{P}'Q, \qquad \therefore\ S\hat{R}P = S\hat{R}'Q,$$

i.e. PR and QR' are parallel.

3. From the data

$$(1-ax)(1-bx)(1-cx)(1-dx) = 1 - px^3 + qx^4.$$

Taking logarithms and expanding,

$$\sum_{n=1} \frac{a^n + b^n + c^n + d^n}{n} \,.\, x^n = (px^3 - qx^4) + \tfrac{1}{2}(px^3 - qx^4)^2 + \ldots.$$

Equating coefficients of x^4 and x^8, we have

$$\frac{\Sigma a^4}{4} = -q, \qquad \frac{\Sigma a^8}{8} = \tfrac{1}{2}q^2,$$

$$\therefore\ \Sigma a^8 = \tfrac{1}{4}(\Sigma a^4)^2.$$

4. By the ordinary rule

$$\frac{(t-x)(t-y)(t-z)}{(t-a)(t-b)(t-c)(t-d)} = \Sigma \frac{(a-x)(a-y)(a-z)}{(a-b)(a-c)(a-d)} \cdot \frac{1}{t-a}.$$

Putting $t=0$ in this identity, we see that the given expression is equal to $xyz/abcd$.

5. If $\tan \alpha = l$, etc. the given condition implies

$$\tan(\alpha+\beta+\gamma) = \infty, \text{ i.e. } \alpha+\beta+\gamma = (2n+1)\frac{\pi}{2}.$$

$$\therefore \Sigma \sin 2\alpha = 4 \,.\, \Pi \cos \alpha,$$

$$\therefore \Sigma \frac{\cos \alpha}{\sin\beta \sin\gamma} = 2 \,.\, \Pi \cot \alpha.$$

$$\therefore \Sigma \frac{\cot\beta \cot\gamma}{\sec^2 \alpha} = 2 \,.\, \frac{\Pi \cot \alpha}{\Pi \sec \alpha},$$

and the given result follows.

6. We have

$$x^2 - 2xy \cos\gamma + y^2 = (x\cos\beta + y\cos\alpha)^2,$$

$$\text{i.e. } x^2 \sin^2\beta - 2xy(\cos\gamma + \cos\alpha\cos\beta) + y^2 \sin^2\alpha = 0.$$

If the solution is unique, the left side must be a perfect square,

$$\therefore \cos\gamma + \cos\alpha\cos\beta = \pm \sin\alpha \sin\beta,$$

or

$$\cos(\alpha \pm \beta) = -\cos\gamma,$$

$\therefore \alpha \pm \beta \pm \gamma$ is an odd multiple of π.

7. Let p and q be the perpendiculars from the origin on the two lines, θ the angle between the lines. Then the area of the parallelogram is $pq/\sin\theta$.

Now if the lines are

$$lx + my + n = 0, \qquad l'x + m'y + n' = 0,$$

then

$$pq = \frac{nn'}{\sqrt{l^2+m^2}\sqrt{l'^2+m'^2}},$$

$$\tan\theta = \frac{lm' - l'm}{ll' + mm'}, \qquad \therefore \sin\theta = \frac{lm' - l'm}{\sqrt{l^2+m^2}\sqrt{l'^2+m'^2}},$$

therefore the area is

$$\frac{nn'}{lm'-l'm}=\frac{c}{2\sqrt{h^2-ab}},$$

since
$$\frac{ll'}{a}=\frac{mm'}{b}=\frac{lm'+l'm}{2h}=\frac{nn'}{c}.$$

8. Writing the equation in the form

$$(x-y+\lambda)^2-(6+2\lambda)x-(10-2\lambda)y+9-\lambda^2=0,$$

the condition that the lines $x-y+\lambda=0$ and $(6+2\lambda)x+\ldots=0$ are at right angles is

$$(6+2\lambda)-(10-2\lambda)=0, \text{ whence } \lambda=1,$$

so that the axis is $x-y+1=0.$

Hence, taking

$$X=\frac{x+y-1}{\sqrt{2}}, \qquad Y=\frac{x-y+1}{\sqrt{2}},$$

the equation becomes $Y^2-4\sqrt{2}\,X=0,$

so that the latus-rectum is $4\sqrt{2}$.

9. If the equations to P_1P_2 and Q_1Q_2 are

$$x\cos\alpha+y\sin\alpha=p \text{ and } x\cos\beta+y\sin\beta=q,$$

the equation to the circle must be of the form

$$xy-c^2+\lambda(x\cos\alpha+y\sin\alpha-p)(x\cos\beta+y\sin\beta-q)=0$$

The conditions for a circle are

$$\cos\alpha\cos\beta=\sin\alpha\sin\beta, \text{ i.e. } \cos(\alpha+\beta)=0,$$

and
$$1+\lambda\sin(\alpha+\beta)=0, \text{ whence } \lambda=\pm 1.$$

Also, since the circle passes through the origin,

$$-c^2+\lambda pq=0.$$

Hence $pq=c^2$ numerically.

10. The stresses are:

$$\text{in } BC, \frac{10}{\sqrt{3}}\text{ lbs.}; \quad \text{in } AC, 10\text{ lbs.};$$

$$\text{in } AB, \frac{20}{\sqrt{3}}\text{ lbs.}; \quad \text{in } CD, \frac{50}{\sqrt{3}}\text{ lbs.}$$

11. The pull (P) of the engine

$$=\frac{550H}{\frac{22}{15}v} \text{ lbs. weight.}$$

$$=\tfrac{75}{448}\cdot\frac{H}{v} \text{ tons' weight.}$$

Now let T be the tension of the coupling, λM the resistance on mass M. Then

$$P-T=\lambda M, \qquad T=\lambda nM',$$

$$\therefore \frac{T}{P}=\frac{nM'}{M+nM'},$$

$$\therefore T=\tfrac{75}{448}\cdot\frac{HnM'}{(M+nM')v} \text{ tons' weight.}$$

12. If α is the inclination of the string to the vertical, then

$$\cos\alpha=\frac{g}{4\pi^2n^2l}.$$

When the length of the string is diminished by x, let the new number of revolutions be N. Then also

$$\cos\alpha=\frac{g}{4\pi^2N^2(l-x)}.$$

$$\therefore N^2(l-x)=n^2l,$$

$$\text{i.e. } \frac{N}{n}=\left(\frac{l}{l-x}\right)^{\frac{1}{2}}=\left(1-\frac{x}{l}\right)^{-\frac{1}{2}}=1+\tfrac{1}{2}\frac{x}{l} \text{ approx.},$$

$$\therefore N-n=\tfrac{1}{2}\frac{nx}{l}.$$

IX.

1. We evidently have

$$B\hat{C}K=B\hat{C}O=90^\circ-B=B\hat{A}K.$$

Hence K is on the circle ABC, and so for L and M.

2. Here

$$PG^2=SP^2+SG^2-2SG\,.\,SM=(SG+SN)^2,$$

PM being the ordinate of P.

$$\therefore\ SP^2 - SN^2 = 2SG\,.\,MN,$$

$$\text{i.e. } SP^2 - SP'^2 + P'N^2 = 2e\,.\,SP\,.\,MN$$
$$= 2SP\,(SP - SP'),$$
$$\therefore\ P'N^2 = (SP - SP')^2.$$

3. (i) Putting $x + y = p$, $xy = q$, the equations are

$$2a^3 + pq = a^3 + a^2p + aq, \text{ i.e. } (p-a)(q-a^2) = 0,$$

and
$$p^2 - 2q = 2a^2.$$

Hence, either

$$p = a, \quad q = -\frac{a^2}{2}, \text{ whence } x = \frac{1 \pm \sqrt{3}}{2}a, \quad y = \frac{1 \mp \sqrt{3}}{2}a$$

or
$$q = a^2, \quad p = 2a, \text{ whence } x = y = a.$$

(ii) Putting $y = mx$ and dividing, we get $x = 0$ or

$$\frac{1+2m}{2m-m^2} = \frac{4-m}{6-5m},$$

i.e. $m^3 + 4m^2 + m - 6 = 0$, whence $m = 1$, -2 or -3.

Hence, since $x = \dfrac{4-m}{1+2m}$, we get the table of roots

x	0	1	-2	$-\frac{7}{5}$
y	0	1	4	$\frac{21}{5}$

4. The expanded form of the determinant is

$$8 + 2\,.\,\Pi\left(a + \frac{1}{a}\right) - 2\,.\,\Sigma\left(a + \frac{1}{a}\right)^2$$

$$= 2\left(-2 + abc + \frac{1}{abc} + \Sigma\frac{bc}{a} + \Sigma\frac{a}{bc} - \Sigma a^2 - \Sigma\frac{1}{a^2}\right)$$

$$= 2\left(1 - \Sigma\frac{a}{bc} + \Sigma\frac{1}{a^2} - \frac{1}{abc}\right)(abc - 1)$$

$$= 2\,.\,\Pi\left(1 - \frac{a}{bc}\right).\,(abc - 1).$$

5. From the formula $b^2 = c^2 + a^2 - 2ca\cos B$, we have

$$b \,.\, \delta b = (c - a\cos B)\,\delta c + (a - c\cos B)\,\delta a + ca\sin B \,.\, \delta B,$$

i.e. $$ca\sin B \,.\, \delta B = b \,.\, \delta b - b\cos A \,.\, \delta c - b\cos C \,.\, \delta a.$$

Under the given conditions, the greatest possible value of the right-hand side is

$$\tfrac{1}{100}(b^2 - bc\cos A + ba\cos C) = \tfrac{1}{50}\,ab\cos C;$$

therefore the greatest value of δB is

$$\tfrac{1}{50} \cdot \frac{b\cos C}{c\sin B} = \tfrac{1}{50}\cot C.$$

Similarly for δC, and since $\delta A + \delta B + \delta C = 0$, we get from these the given maximum value for δA.

6. Putting $\cos\alpha + i\sin\alpha = a$, etc., we have

$$\cos\alpha - i\sin\alpha = \frac{1}{a}, \text{ etc.}$$

Hence the given equations involve both

$$\Sigma a = 0 \text{ and } \Sigma\frac{1}{a} = 0, \text{ i.e. } \Sigma bc = 0.$$

Hence $$\Sigma a^2 = 0, \text{ and } \therefore \Sigma\cos 2\alpha = 0;$$

$$\therefore \Sigma\cos^2\alpha = \Sigma\sin^2\alpha = \tfrac{1}{2}(\Sigma\cos^2\alpha + \Sigma\sin^2\alpha) = \tfrac{3}{2}.$$

7. If (X, Y) is the intersection of the lines, the equation of the bisectors is

$$\frac{(x-X)^2 - (y-Y)^2}{a-b} = \frac{(x-X)(y-Y)}{h},$$

and these meet $y = 0$, where

$$hx^2 - [2hX - (a-b)Y]\,x + h(X^2 - Y^2) - (a-b)XY = 0.$$

If x_1, x_2 are the roots of this equation,

$$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$$
$$= \frac{(a-b)^2 + 4h^2}{h^2} \,.\, Y^2 \text{ on reduction};$$

and the area of the triangle is

$$\tfrac{1}{2}(x_1 - x_2)\,Y = \frac{\sqrt{(a-b)^2 + 4h^2}}{2h}\,Y^2.$$

But $$Y = \frac{gh - af}{ab - h^2}, \quad \therefore\ Y^2 = \frac{ca - g^2}{ab - h^2},$$

using the condition for straight lines in the form

$$(gh - af)^2 = (ab - h^2)(ca - g^2).$$

8. The line joining the origin to the point m is $y = \dfrac{2}{m}x$. Hence if the lines joining the origin to the points m, m' are inclined at 45°, we have

$$\frac{2}{m} - \frac{2}{m'} = 1 + \frac{4}{mm'}, \text{ whence } m' = \frac{2m+4}{2-m},$$

and the chord m, m' now becomes

$$2(2-m)x - (4m - m^2 + 4)y + 2am(2m + 4) = 0,$$

$$\text{i.e. } m^2(y + 4a) - 2m(x + 2y - 4a) + 4(x - y) = 0,$$

and the envelope is

$$(x + 2y - 4a)^2 = 4(x - y)(y + 4a),$$

which reduces to the form given.

9. The centre is the point $\dfrac{x}{8} = \dfrac{y}{1} = -\tfrac{1}{10}$, and the line joining (x', y') to the centre is

$$-x(10y' + 1) + y(8 + 10x') + x' - 8y' = 0 \text{ ....(i).}$$

Also the polar of (x', y') is

$$x(17x' - 6y' + 13) + y(-6x' + 8y' - 4) + 13x' - 4y' + 2 = 0 \text{...(ii).}$$

If (x', y') is a point on either axis, the lines (i) and (ii) are perpendicular. Hence

$$-(17x' - 6y' + 13)(10y' + 1) + (-6x' + 8y' - 4)(10x' + 8) = 0,$$

giving the required equation.

10. Let the force along AB be λc. Then the sum of the moments about C is

$$\lambda c \sin CAB - \lambda b \sin CAD.$$

But $$\frac{b}{\sin CAB} = \frac{c}{\sin CAD} = 2R,$$

where R is the radius. Hence the sum of the moments about C is zero, i.e. the resultant acts along AC.

Again, if F be the resultant, we have, taking moments about B,

$$F \,.\, \sin BAC = \lambda \,.\, b \sin A,$$

$$\text{i.e. } F \,.\, \frac{b}{2R} = \lambda b \,.\, \frac{BD}{2R}, \quad \therefore \quad F = \lambda \,.\, BD.$$

11. Let T be the tension, and u the velocity, at the lowest point. Then

$$T - mg = \frac{mu^2}{l},$$

$$\text{i.e. } u^2 = (n-1)\, gl, \text{ since } T = nmg.$$

Also, if h be the vertical height through which the bob rises,

$$u^2 = 2gh,$$

$$\therefore \quad (n-1)\, l = 2h, \text{ i.e. } h = \frac{n-1}{2} \,.\, l,$$

and the angle of swing on each side of the vertical is

$$\cos^{-1}\left(\frac{l-h}{l}\right) = \cos^{-1}\left(\frac{3-n}{2}\right).$$

12. Let the angles of projection be $\alpha - \delta$, α, $\alpha + \delta$. Then

$$v_1^2 \sin 2(\alpha - \delta) = v_2^2 \sin 2\alpha = v_3^2 \sin 2(\alpha + \delta),$$

and $$v_1 \sin(\alpha - \delta) = 2v_3 \sin(\alpha + \delta).$$

From these

$$\frac{\sin 2(\alpha - \delta)}{\sin^2(\alpha - \delta)} = \tfrac{1}{4} \,.\, \frac{\sin 2(\alpha + \delta)}{\sin^2(\alpha + \delta)},$$

$$\text{i.e. } \frac{\tan(\alpha - \delta)}{\tan(\alpha + \delta)} = 4, \quad \therefore \quad \frac{\sin 2\alpha}{\sin 2\delta} = -\tfrac{5}{3}.$$

Now $$\frac{v_1^2 v_3^2}{v_2^4} = \frac{\sin^2 2\alpha}{\sin 2(\alpha - \delta) \sin 2(\alpha + \delta)} = \frac{\sin^2 2\alpha}{\sin^2 2\alpha - \sin^2 2\delta} = \tfrac{25}{16};$$

$$\therefore \quad v_1 v_3 = \tfrac{5}{4} v_2^2.$$

X.

1. By hypothesis, the triangles ABC, DEF are in perspective. Also if L, M, N be the middle points of the sides of DEF, the triangles DEF, LMN are in perspective, the centre of perspective being the centre of gravity of DEF. Hence the triangles ABC, LMN being both in perspective with the same triangle, viz. DEF, are in perspective with each other, whence the result.

2. Let SY be the perpendicular from the given focus on the given tangent. Then Y is on the auxiliary circle and therefore $CS : CY$, being the ratio of the eccentricity, is given. Hence S and Y being fixed points, the locus of C is a circle. Also

$$SS' : SC = 2 : 1,$$

and S is fixed, therefore the locus of S' is another circle.

3. Suppose the given expression equal to

$$(x+y+z)(ax^2+by^2+cz^2+pyz+qzx+rxy).$$

Then equating the remaining coefficients, we have

$$\left.\begin{array}{lll} c+p=d, & a+q=e, & b+r=f \\ b+p=d', & c+q=e', & a+r=f' \end{array}\right\} \quad \ldots\ldots\ldots\text{(i)},$$

$$p+q+r=g \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) follow $\qquad b-c+d-d'=0$, etc.

Also $\qquad p=d'-b, \qquad q=e'-c, \qquad r=f'-a.$

Hence from (ii) $\qquad d'+e'+f'=a+b+c+g.$

4. The turning points are $(-2 \pm \sqrt{7},\ -5 \pm 2\sqrt{7})$,

i.e. $(-4{\cdot}64,\ -10{\cdot}28)$ and $({\cdot}64,\ {\cdot}28)$,

and the asymptotes are

$$y=x-3 \text{ and } x+2=0.$$

5. Dividing through by $\cos^4 x \cos^4 y \cos^4 z$, and putting

$$\tan x = a, \text{ etc.},$$

the identity takes the algebraical form

$$8abc\,.\,\Pi(b-c) + 2\Sigma a\,(b-c)(1-bc)^2(1+ab)(1+ac)=0.$$

To verify this, we notice that

$$\Sigma\left[a(b-c)+a^2(b^2-c^2)+a^3bc(b-c)\right](1-2bc+b^2c^2)$$

produces nine sums, six of which vanish identically.

The others are

$$\Sigma a^3bc(b-c)=-abc\,.\,\Pi(b-c),$$

$$-2\,\Sigma a^2bc(b^2-c^2)=-2abc\,.\,\Pi(b-c),$$

and

$$\Sigma ab^2c^2(b-c)=-abc\,.\,\Pi(b-c).$$

Hence the whole sum is $-4abc\,.\,\Pi(b-c)$, and therefore the identity is true.

6. Let ρ be the radius of the circle in question. The centre of the circle is the orthocentre.

$$\therefore\ T_A{}^2=4R^2\cos^2 A-\rho^2,$$

$$\therefore\ \Sigma T_A{}^2\sin A\sin(B-C)=-\Sigma T_A{}^2(\cos^2 B-\cos^2 C)$$

$$=-4R^2.\Sigma\cos^2 A(\cos^2 B-\cos^2 C)+\rho^2.\Sigma(\cos^2 B-\cos^2 C)$$

$$=0.$$

7. The lines being $y=mx$, $y=m'x$, the required locus is

$$\frac{(y-mx)^2}{1+m^2}+\frac{(y-m'x)^2}{1+m'^2}=c^2,$$

i.e. $(2+m^2+m'^2)y^2-2(mm'+1)(m+m')xy$

$$+(m^2+m'^2+2m^2m'^2)x^2=c^2(1+m^2)(1+m'^2).$$

Putting $m+m'=-\dfrac{2h}{b}$, $mm'=\dfrac{a}{b}$, this easily reduces to the form given.

8. The co-ordinates of any point on $x^2+y^2=2bx$ may be written $(2b\cos^2\theta,\ 2b\sin\theta\cos\theta)$, and the polar of this point for $x^2+y^2=a^2$ is

$$x\,.\,2b\cos^2\theta+y\,.\,2b\sin\theta\cos\theta=a^2,$$

i.e. $2bx+2by\tan\theta=a^2(1+\tan^2\theta)$,

or

$$a^2\tan^2\theta-2by\tan\theta-(2bx-a^2)=0,$$

and the envelope is

$$b^2y^2+a^2(2bx-a^2)=0.$$

9. Let the equation to the other chord be $lx - my - k = 0$. Then the equation to the circle is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda(lx + my - 1)(lx - my - k) = 0,$$

the condition for a circle being

$$\frac{1}{a^2} + \lambda l^2 = \frac{1}{b^2} - \lambda m^2.$$

(The other condition has already been utilised in assuming the above form for the chord.)

Further, since the circle passes through the pole of

$$lx + my = 1, \text{ viz. } (a^2 l,\ b^2 m),$$

$$\therefore\ 1 + \lambda(a^2 l^2 - b^2 m^2 - k) = 0,$$

$$\text{i.e. } k = a^2 l^2 - b^2 m^2 + \frac{1}{\lambda} = a^2 l^2 - b^2 m^2 + \frac{l^2 + m^2}{\dfrac{1}{b^2} - \dfrac{1}{a^2}}$$

$$= \frac{a^4 l^2 + b^4 m^2}{a^2 - b^2}.$$

10. Let A and C be the points of contact of the rod with the wall and the wire, G its middle point. Draw AN making an angle λ with the horizontal downwards, and CN making an angle λ with the perpendicular to the rod away from A. Then N must be vertically below G.

$$\text{Now} \qquad \frac{AC}{AN} = \frac{\sin(\alpha - 2\lambda)}{\cos\lambda}, \quad \frac{AG}{AN} = \frac{\cos\lambda}{\sin\alpha},$$

$$\therefore\ \frac{AC}{AG} = \frac{\sin\alpha \sin(\alpha - 2\lambda)}{\cos^2\lambda}.$$

11. Let T be the tension of the upper string, T' of the lower, f the acceleration of M.

The accelerations of m and m' downwards are

$$g - \frac{T'}{m}, \text{ and } g - \frac{T'}{m'}.$$

Hence, since their accelerations relative to the pulley are the same,

$$2g - T'\left(\frac{1}{m} + \frac{1}{m'}\right) = 2f.$$

Also $T = Mf$, and $2T' + \mu g - T = \mu f$.

Eliminating T' and f from these equations, the result follows.

12. Let u be the velocity of projection, θ the elevation, t the time to reach the top of the plane. Then

$$V \cos \alpha = u \cos \theta = \frac{a \cos \alpha}{t}, \quad \therefore\ t = \frac{a}{V}.$$

Also $$V \sin \alpha = -(u \sin \theta - gt),$$

$$u \sin \theta \,.\, t - \tfrac{1}{2} gt^2 = a \sin \alpha.$$

From these $$(gt - V \sin \alpha)\, t - \tfrac{1}{2} gt^2 = a \sin \alpha.$$

Substituting for t, we get the required value for V^2.

Also $$u \sin \theta = g \,.\, \frac{a}{V} - V \sin \alpha = 3V \sin \alpha,$$

and the greatest height is

$$\frac{(u \sin \theta)^2}{2g} = \frac{9V^2 \sin^2 \alpha}{2g} = \tfrac{9}{8}(a \sin \alpha).$$

XI.

1. Let O_1 be the centre of the smaller circle. Then

$$EA^2 + EB^2 = 2EM^2 + 2AM^2.$$

But

$$EM^2 + EO^2 = 2EO_1^2 + 2O_1O^2$$
$$= 2(ET^2 + O_1T^2) + 2O_1O^2 = 2ET^2 + OM^2,$$

and $OE^2 - OM^2 = OA^2 - OM^2 = AM^2$, $\therefore\ EM^2 + AM^2 = 2ET^2$.

2. Let U be the intersection. Then since S, Y, Z, T are cyclic,

$$\therefore\ \angle SZY = STY = 180^\circ - S'TZ = ZTU.$$

Also $\quad TSY = UZT, \quad \therefore \; ZTU + UZT =$ a right angle.

Therefore TUZ is a right angle, i.e. the locus of U is the circle on YS' as diameter.

3. Let the roots of the first equation be α, β and those of the second α, β^2. Then

$$\alpha + \beta = -\frac{b}{a} \;\ldots\ldots\text{(i)}, \qquad \alpha + \beta^2 = -\frac{b'}{a'} \;\ldots\ldots\text{(iii)},$$

$$\alpha\beta = \frac{c}{a} \;\ldots\ldots\text{(ii)}, \qquad \alpha\beta^2 = \frac{c'}{a'} \;\ldots\ldots\text{(iv)}.$$

From (ii) and (iv) $\qquad \beta = \dfrac{ac'}{a'c}, \qquad \alpha = \dfrac{a'c^2}{a^2c'}.$

Substituting these values in (i) we obtain the given result.

4. (i) The series is the absolute term in the expansion of

$$(1+x)^a - \frac{2}{x}(1+x)^a + \frac{3}{x^2}(1+x)^a - \ldots,$$

which latter series may be supposed to be continued indefinitely, provided $|x| > 1$. Its sum is

$$(1+x)^a \left(1 + \frac{1}{x}\right)^{-2} = x^2 (1+x)^{a-2}.$$

Hence the absolute term is zero, provided $a \nless 2$. Hence the sum of the given series is zero, unless $a = 1$, in which case the sum is clearly -1.

(ii) This series is the absolute term in the expansion of

$$(1+x)^a \left(1 - \frac{1}{x}\right)^a,$$

i.e. the coefficient of x^a in $(-1)^a (1-x^2)^a$. If a is odd, this coefficient is zero; if a is even, it is

$$(-1)^{\frac{a}{2}} \cdot \frac{a!}{\left(\frac{a}{2}!\right)^2}.$$

5. Let PN be the tower, A and B the two points, N being on BA produced. Draw PL perpendicular to AB. Then, from the triangle PAB,

$$\frac{b}{\sin(\phi_1 - \phi_2)} = \frac{PA}{\sin\phi_2} = \frac{PL}{\sin\phi_1 \sin\phi_2} \quad \ldots\ldots\ldots\ldots\text{(i)}.$$

Now let PK be drawn perpendicular to the horizontal plane through the foot of the tower. Then

$$LK^2 = PL^2 - PK^2 = PN^2 - NL^2 - PK^2 = NK^2 - NL^2,$$

therefore KLN is a right angle,

$$\therefore \quad NL = \frac{NK}{\sqrt{2}} = \frac{a\cos\theta}{\sqrt{2}},$$

where θ is the angle the tower makes with the horizontal.

Hence $$\cos\theta = \sqrt{2}\,.\,\frac{NL}{a} = \sqrt{2}\left(1 - \frac{PL^2}{a^2}\right)^{\frac{1}{2}},$$

whence, using (i), we obtain the given expression.

6. If N be the centre of the nine-point circle, and T_A the tangent from A, then

$$T_A^2 = AN^2 - \tfrac{1}{4}R^2.$$

But $$AN^2 = \tfrac{1}{2}(AO^2 + AP^2 - 2PN^2),$$

O being the circumcentre, and P the orthocentre.

$$\therefore \quad AN^2 = \tfrac{1}{2}R^2 + 2R^2\cos^2 A - \tfrac{1}{4}R^2(1 - 8\cos A\cos B\cos C),$$

$$\therefore \quad T_A^2 = 2R^2\cos^2 A + 2R^2\cos A\cos B\cos C.$$

Also $$1 - \Sigma\cos^2 A - 2\cos A\cos B\cos C \equiv 0,$$

$$\therefore \quad \Sigma T_A^2 = 2R^2(1 + \cos A\cos B\cos C).$$

To prove the first part, we have at once

$$\begin{aligned}\Sigma a^2 &= 4R^2\,.\,\Sigma\sin^2 A\\ &= 4R^2(3 - \Sigma\cos^2 A)\\ &= 8R^2(1 + \cos A\cos B\cos C).\end{aligned}$$

7. Let $\tan(\alpha+\beta)=t_1$, $\tan(\alpha-\beta)=t_2$, $\tan x=t$. Then the equation is

$$\frac{t_1-t}{1+tt_1}\cdot\frac{t-t_2}{1+tt_2}\cdot\frac{t+t_2}{1-tt_2}=1,$$

i.e. $t^3(1-t_1t_2^2)-t^2(t_1+t_2^2)+t(t_1-t_2^2)+1+t_1t_2^2=0,$

$$\therefore\ \tan(x_1+x_2+x_3)=\frac{t_1+t_2^2+1+t_1t_2^2}{1-t_1t_2^2-t_1+t_2^2}=\frac{1+t_1}{1-t_1}$$

$$=\tan\left(\alpha+\beta+\frac{\pi}{4}\right).$$

8. Taking the circle in the form

$$x^2+y^2+2\lambda x+2\mu y+\nu=0,$$

the polar of $(0, c)$ is

$$yc+\lambda x+\mu(y+c)+\nu=0.$$

But this polar is $y=0$, $\therefore\ \lambda=0$ and $\mu c+\nu=0$.

Hence the circle may be written

$$x^2+y^2+2\mu y-\mu c=0 \text{ ....(i).}$$

The polar of $(a, 0)$ is

$$ax+\mu y-\mu c=0.$$

But this polar is $$cx-by+bc=0,$$

$$\therefore\ \frac{a}{c}=-\frac{\mu}{b},\ \text{ i.e. }\ \mu=-\frac{ab}{c}.$$

Substituting in (i) we obtain the form given.

9. Taking the equivalent polar form $\dfrac{2a}{r}=1+\cos\theta$, the tangents at $\beta\pm\alpha$ intersect where $\theta=\beta$, their equations being

$$\frac{2a}{r}=\cos\theta+\cos(\theta-\beta\mp\alpha),$$

and the value of r for the intersection is given by

$$\frac{2a}{r}=\cos\beta+\cos\alpha.$$

Hence the locus of the intersection is

$$\frac{2a}{r} = \cos\theta + \cos\alpha,$$

and remembering that $r\cos\theta = -x$, this is

$$(2a+x)^2 = (x^2+y^2)\cos^2\alpha,$$

which is equivalent to the form given. From the polar equation this conic is evidently a hyperbola of eccentricity $\sec\alpha$.

10. Let E be the middle point of BC, and let $AB = a$, $AF = x$. The areas of the whole and the part removed are in the ratio

$$a^2 : \tfrac{1}{4}a(a-x), \text{ i.e. } 4a : a-x,$$

and their centres of gravity are distant

$$\frac{a}{2}, \quad \frac{2a+x}{3},$$

respectively from AD. Hence the distance of the centre of gravity of the remainder from AD is

$$\bar{x} = \frac{2a^2 - \frac{1}{3}(2a+x)(a-x)}{3a+x}$$

$$= \frac{4a^2+ax+x^2}{3(3a+x)}.$$

For equilibrium, we must have $\bar{x} < x$,

$$\text{i.e. } 4a^2+ax+x^2 < 9ax+3x^2,$$

$$\text{i.e. } x^2+4ax-2a^2 > 0,$$

$$\text{i.e. } [x-(\sqrt{6}-2)a][x+(\sqrt{6}+2)a] > 0,$$

and the necessary condition is evidently $x > (\sqrt{6}-2)a$.

11. Let f be the acceleration of the man, f' that of the weight, both upwards, so that

$$f + f' = \tfrac{6}{7}g \quad \text{.........................(i).}$$

The equations of motion are

$$T - \tfrac{3}{2}mg = \tfrac{3}{2}mf', \qquad T - mg = mf,$$

$$\therefore \tfrac{1}{2}mg = mf - \tfrac{3}{2}mf', \text{ whence from (i) } f' = \tfrac{1}{7}g.$$

Also $T = \frac{3}{2}m(g+f') = \frac{12}{7}mg$, i.e. $1\frac{5}{7}$ times the man's weight.

12. If u is the velocity of projection, the velocity on reaching the summit is given by $V^2 = u^2 - 2gl \sin\theta$, and the equation to the subsequent path is

$$y = x \tan\theta - \tfrac{1}{2}g \cdot \frac{x^2}{V^2 \cos^2\theta}.$$

This has to pass through the point $(l \cos\theta, -l \sin\theta)$;

$$\therefore\ -l\sin\theta = l\sin\theta - \tfrac{1}{2}g \cdot \frac{l^2}{V^2},\ \text{ i.e. } V^2 = \tfrac{1}{4} \cdot \frac{gl}{\sin\theta}.$$

Hence $$u^2 = V^2 + 2gl\sin\theta = \tfrac{1}{4}gl\left(\frac{1}{\sin\theta} + 8\sin\theta\right).$$

XII.

1. $$B_1\hat{A}_1C_1 = B_1\hat{A}_1A + A\hat{A}_1C_1 = B_1\hat{B}A + A\hat{C}C_1,$$

$$\text{i.e. } A_1 = \tfrac{1}{2}(B + C),$$

and similarly $$B_1 = \tfrac{1}{2}(C + A).$$

Hence $$A_1 - B_1 = \tfrac{1}{2}(B - A).$$

Similarly $$A_2 - B_2 = \tfrac{1}{2}(B_1 - A_1) = \frac{1}{2^2}(A - B) \text{ and so on.}$$

Hence $A_n - B_n$ continually diminishes as n increases, and similarly for the other differences, i.e. $\triangle A_nB_nC_n$ tends to become equilateral.

2. Let $T_1T_2T_3$ be the triangle formed by the three fixed tangents, the sides touching the parabola at P, Q, R.

Then $S\hat{T}_1Q = S\hat{F}E$, both being equal to $S\hat{R}T_1$,

and $$S\hat{Q}T_1 = S\hat{E}F.$$

Hence the triangles SQT_1, SEF are similar,

$$\therefore\ SQ : QT_1 = SE : EF.$$

Similarly $$SQ : QT_3 = SE : ED,$$

$$\therefore\ EF : ED = QT_1 : QT_3, \text{ i.e. a constant ratio.}$$

Similarly for the other ratios.

3. If $\Sigma x = p$, $\Sigma xy = q$, $xyz = r$, the equations are

$$p = a, \qquad pq - 9r = c \ldots \text{(ii)},$$
$$2p^2 - 6q = b \ldots \text{(i)}, \qquad 2q^2 - 6pr = d \ldots \text{(iii)},$$

since $\Sigma xy^2 = pq - 3r$, $\Sigma x^2y^2 = q^2 - 2pr$.

From (i) and (ii),

$$q = \frac{2a^2 - b}{6}, \quad r = \frac{a(2a^2 - b) - 6c}{54},$$

whence from (iii),

$$\frac{(2a^2 - b)^2}{18} - \frac{a^2(2a^2 - b) - 6ac}{9} = d,$$

$$\text{i.e. } (2a^2 - b)^2 - 2a^2(2a^2 - b) + 12ac = 18d.$$

4. Taking the numbers $n - 2$, n, $n + 2$, $n + 4$, the result is

$$n(n^2 - 4)(n + 4) + 16 = (n^2 + 2n - 4)^2.$$

If $n = 2p - 1$, this is $(4p^2 - 5)^2$.

Suppose p ends in 0, 1, 2, 3 or 4,
then $4p^2 - 5$,, 5, 9, 1, 1 or 9,
$\therefore (4p^2 - 5)^2$,, 5, 1, 1, 1 or 1 as stated.

If $n = 2p$, the number is $16(p^2 + p - 1)^2$.

Now $p^2 + p - 1$ ends in 9, 1, 5, 1, 9,
$\therefore (p^2 + p - 1)^2$,, 1, 1, 5, 1, 1,
$\therefore 16(p^2 + p - 1)^2$,, 6, 6, 0, 6, 6.

Hence in four cases the last digit is 6, and in the other 0.

If p ends in 5, 6, 7, 8 or 9, we get the same sequences repeated.

5. Let the bisectors meet in O. Then

$$\frac{\triangle ODF}{\triangle CDF} = \frac{OF}{CF} = \frac{AF}{AC + AF} = \frac{c}{a + b + c}$$

$$\left(\text{since } \frac{AF}{AC} = \frac{BF}{BC} = \frac{c}{b + a}\right),$$

$$\frac{\triangle CDF}{\triangle CBF} = \frac{CD}{CB} = \frac{b}{b + c},$$

$$\frac{\triangle CBF}{\triangle ABC} = \frac{BF}{AB} = \frac{a}{a + b}.$$

Multiplying, we have

$$\frac{\triangle ODF}{\triangle ABC} = \frac{abc}{a+b+c} \cdot \frac{1}{(b+c)(a+b)}.$$

Hence

$$\frac{\triangle DEF}{\triangle ABC} = \frac{abc}{\Sigma a} \cdot \Sigma \frac{1}{(b+c)(a+b)} = \frac{abc}{\Sigma a} \cdot \frac{2\Sigma a}{\Pi(b+c)}$$

$$= \frac{2abc}{\Pi(b+c)} = \frac{2 \,.\, \Pi \sin A}{\Pi(\sin B + \sin C)}.$$

But $\qquad \sin B + \sin C = 2\cos\frac{A}{2}\cos\frac{B-C}{2}$, etc.

Hence the result.

6. If $\dfrac{\tan 7\theta}{\tan\theta} = 0$, and $\tan\theta = y$, we have

$$7 - 35y^2 + 21y^4 - y^6 = 0,$$

and the roots of this equation are $\therefore\ \tan\frac{r\pi}{7}$ $(r = 1, 2, \ldots, 6)$.

These are numerically equal in pairs. Hence putting $y^2 = x$, the roots of

$$x^3 - 21x^2 + 35x - 7 = 0$$

are $\tan^2\frac{r\pi}{7}$ $(r = 1, 2, 3)$.

Further $\qquad \sec^4\theta = 1 + 2\tan^2\theta + \tan^4\theta.$

$$\therefore \sum_{r=1,2,3} \sec^4\frac{r\pi}{7} = 3 + 2\Sigma\tan^2\frac{r\pi}{7} + \Sigma\tan^4\frac{r\pi}{7}$$

$$= 3 + (2 \times 21) + (21^2 - 2 \times 35)$$

$$= 416.$$

7. If the lines are $y = m_1 x$, $y = m_2 x$, the intersection of the first with $\dfrac{x}{\alpha} + \dfrac{y}{\beta} = 1$ is

$$\frac{x}{1} = \frac{y}{m_1} = \frac{1}{\dfrac{1}{\alpha} + \dfrac{m_1}{\beta}},$$

and the line through this point perpendicular to $y = m_2 x$ is

$$x + m_2 y = a\beta \frac{1 + m_1 m_2}{am_1 + \beta}.$$

Another perpendicular is $\alpha x - \beta y = 0$, and the intersection of these is

$$\frac{x}{\beta} = \frac{y}{\alpha} = \frac{\alpha\beta\,(1 + m_1 m_2)}{(am_1 + \beta)\,(am_2 + \beta)} = \frac{\alpha\beta\left(1 + \frac{a}{b}\right)}{\alpha^2 \frac{a}{b} + \alpha\beta\left(-\frac{2h}{b}\right) + \beta^2}$$

$$= \frac{(a + b)\,\alpha\beta}{a\alpha^2 - 2h\alpha\beta + b\beta^2},$$

and the distance from the origin is $(x^2 + y^2)^{\frac{1}{2}}$, as given.

8. The chord joining m, m' on $y^2 = 4ax$ is

$$2x - (m + m')\,y + 2amm' = 0,$$

and the parallel tangent to $y^2 = 4bx$ is

$$2x - (m + m')\,y + b\frac{(m + m')^2}{2} = 0.$$

If these coincide, we have $bp^2 = 4aq$, where $m + m' = p$, $mm' = q$.

The intersection of tangents is $x = aq$, $y = ap$ and therefore its locus is $by^2 = 4a^2 x$.

The intersection of normals is

$$x = a\,(p^2 - q) + 2a, \quad y = - apq,$$

$$\text{i.e. } x - 2a = aq\,\frac{4a - b}{b}, \quad y^2 = \frac{4a^3 q^3}{b},$$

$$\therefore \text{ the locus is } (4a - b)^3\,y^2 = 4b^2\,(x - 2a)^3.$$

9. The co-ordinates of the external point are $a \sec \alpha \cos \theta$, $b \sec \alpha \sin \theta$. The tangent at ϕ to the inner ellipse is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1.$$

If this tangent passes through the above point, then

$$\sec \alpha \cos \theta \cos \phi + \sec \alpha \sin \theta \sin \phi = 1,$$

$$\text{i.e. } \cos (\phi - \theta) = \cos \alpha,$$

$$\text{i.e. } \phi - \theta = \pm\,\alpha,$$

or

$$\phi = \theta \pm \alpha.$$

10. Let A be the point of contact with the ground, B with the cylinder. Let AN be the line of limiting friction, meeting the radius at B produced in N. Then, in limiting equilibrium, N is vertically above the centre of gravity, and we have

$$\frac{AG}{AN}=\frac{\sin\epsilon}{\cos 2\alpha}, \quad \frac{AB}{AN}=\sin(2\alpha+\epsilon), \quad AB=a\cot\alpha.$$

$$\therefore \frac{b}{a\cot\alpha}=\frac{\sin\epsilon}{\cos 2\alpha\sin(2\alpha+\epsilon)},$$

$$\text{i.e. } a\sin\epsilon=b\tan\alpha\cos 2\alpha\sin(2\alpha+\epsilon).$$

If $a\sin\epsilon$ exceeds this, the equilibrium is not limiting.

11. From O the centre of the circle draw a perpendicular OAN on the line, meeting the circle in A. Then the line of quickest descent is the line bisecting the angle between AN and the vertical. If this meets the given straight line in P, then

$$\angle PAN=\frac{\theta}{2}, \quad AN=l. \quad \therefore AP=l\sec\frac{\theta}{2}.$$

But, since AP makes an angle $\frac{\theta}{2}$ with the vertical, the time down AP is

$$\sqrt{\frac{2AP}{g\cos\frac{\theta}{2}}}=\sqrt{\frac{2l}{g}}\cdot\sec\frac{\theta}{2}.$$

12. Let α be the angle of the plane, θ the inclination of the direction of projection to the plane. Then the times in the successive trajectories are

$$\frac{2u\sin\theta}{g\cos\alpha}, \quad \frac{2eu\sin\theta}{g\cos\alpha}, \text{ etc.,}$$

$$\therefore \text{time to } r\text{th impact is } T=\frac{2u\sin\theta}{g\cos\alpha}\cdot\frac{1-e^r}{1-e}.$$

Since at the rth impact the particle strikes the plane at right angles, it afterwards begins to move down the plane, and if it returns to the point of projection, the time of descending is equal to the time of ascending, i.e. the time to the nth impact is $2T$.

$$\therefore 2\frac{1-e^r}{1-e}=\frac{1-e^n}{1-e}.$$

$$\text{i.e. } e^n-2e^r+1=0.$$

XIII.

1. Let O be the point of concurrence of the lines. Then since AC is a transversal of $\triangle P'OR'$, we have

$$\frac{P'Q}{QR'} \cdot \frac{R'C}{CO} \cdot \frac{OA}{AP'} = -1.$$

Similarly
$$\frac{Q'R}{RP'} \cdot \frac{P'A}{AO} \cdot \frac{OB}{BQ'} = -1,$$

and
$$\frac{R'P}{PQ'} \cdot \frac{Q'B}{BO} \cdot \frac{OC}{CR'} = -1.$$

Multiplying these, the result follows.

2. Let SL, SM, SN be the perpendiculars from the focus on the sides, and suppose that L falls between M and N. Then since S is on the circumcircle, L, M, N are collinear.

$$\therefore \triangle SLM + \triangle SLN = \triangle SMN.$$

Now let p, q be the perpendiculars from L, M on SN. Then p is also the perpendicular from L on SM, since the angles LSM, LSN are each equal to an angle of the triangle. Hence

$$p \,.\, SM + p \,.\, SN = q \,.\, SN.$$

But

$$p : q = SL : SM, \quad \therefore SL \,.\, SM + SL \,.\, SN = SM \,.\, SN,$$

$$\text{i.e.} \quad \frac{1}{SM} + \frac{1}{SN} = \frac{1}{SL}.$$

3. By the ordinary rule, the product of the determinants

$$\begin{vmatrix} b, & c, & a \\ c, & a, & b \\ a, & b, & c \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a, & b, & c \\ b, & c, & a \\ c, & a, & b \end{vmatrix}$$

is

$$\begin{vmatrix} B, & A, & B \\ B, & B, & A \\ A, & B, & B \end{vmatrix}$$

and each of these determinants is

$$3abc - a^3 - b^3 - c^3.$$

4. Since the number is of each of the forms $7x+4$ and $9y+3$, we have to solve the indeterminate equation

$$7x+4=9y+3$$

in positive integers. We have

$$7x-9y=-1=-1.(9.4-7.5),$$

$$\text{i.e. } 7(x-5)=9(y-4),$$

whence $$\frac{x-5}{9}=\frac{y-4}{7}=t, \text{ an integer.}$$

$\therefore$ $x=9t+5$ and the number is $63t+39$, or what is the same form $63t-24$. Also, since the number is even, t must be even. Hence, putting $t=2p$, the number is of the form $126p-24$.

5. The solutions are given by the intersections of the graphs

$$y=\cot x \text{ and } y=a+bx.$$

The latter, being a straight line, will cut the former once between any two of its successive asymptotes, i.e. there is a root of the equation between $n\pi$ and $(n+1)\pi$, where n is any integer. Hence the number of roots between α and β is

$$I\left(\frac{\beta}{\pi}\right)-I\left(\frac{\alpha}{\pi}\right)+p,$$

where $I(x)$ is the greatest integer in x, and p may be either 0, 1 or -1.

To find approximations to large roots, we note that when n is very great $x=n\pi$ will, as is evident from the graphs, be an approximate solution. Put $x=n\pi+y$, then the equation is

$$\cot y=a+bn\pi+by.$$

Now y being small, we have $\cot y=\frac{1}{y}$, neglecting y^2, etc. Hence

$$1=(a+bn\pi)y+by^2,$$

and, again neglecting y^2, this gives

$$y=(a+bn\pi)^{-1}.$$

6. Denoting $C\hat{A}B$ by α, we have

$$\frac{b}{CP_r} = \frac{\sin\left(\alpha + r \cdot \dfrac{\pi}{2n}\right)}{\sin \alpha};$$

$$\therefore\ b\Sigma \frac{1}{CP_r} = \frac{1}{\sin \alpha} \sum_{r=1}^{r=n-1} \sin\left(\alpha + r \cdot \frac{\pi}{2n}\right),$$

$$\text{i.e. } (b \sin \alpha)\,\Sigma \frac{1}{CP_r} = \frac{\sin\left(\alpha + \dfrac{\pi}{4}\right) \sin (n-1)\dfrac{\pi}{4n}}{\sin \dfrac{\pi}{4n}}$$

$$= \frac{1}{\sqrt{2}} (\sin \alpha + \cos \alpha) \frac{1}{\sqrt{2}} \left(\cot \frac{\pi}{4n} - 1\right);$$

$$\therefore\ \Sigma \frac{1}{CP_r} = \frac{1}{2b} (1 + \cot \alpha) \left(\cot \frac{\pi}{4n} - 1\right),$$

and $\cot \alpha = \dfrac{b}{a}$. Hence the result given.

7. The focal distance of the point (x', y') on $y^2 = 4ax$ is $x' + a$. Let the fixed time be $x = h$; then the circle will be of the form

$$x^2 + y^2 - 2hx + 2fy + c = 0.$$

Eliminating y between the two equations, we get

$$(x^2 + 4ax - 2hx + c)^2 = 16f^2 ax,$$

$$\text{i.e. } x^4 - (4h - 8a)\,x^3 + \dots = 0.$$

The sum of the roots of this equation is $4h - 8a$. Hence the sum of the focal distances of the four points of intersection is $4h - 4a$.

8. The tangents at α, β to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ meet at the point

$$a\frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}, \quad b\frac{\sin \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)},$$

and those at $\pi + \alpha$, β at the point

$$a\frac{\sin \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}(\alpha - \beta)}, \quad -b\frac{\cos \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}(\alpha - \beta)}.$$

Putting $\frac{1}{2}(\alpha+\beta)=\theta$, $\frac{1}{2}(\alpha-\beta)=\phi$, these points are

$$\left(\frac{a\cos\theta}{\cos\phi},\ \frac{b\sin\theta}{\cos\phi}\right) \text{ and } \left(\frac{a\sin\theta}{\sin\phi},\ -\frac{b\cos\theta}{\sin\phi}\right).$$

The first of these lies on the given conic.

$$\therefore\ Aa^2\cos^2\theta + 2Hab\sin\theta\cos\theta + Bb^2\sin^2\theta = \cos^2\phi.$$

Hence calling the second (x, y), it satisfies the equation

$$Aa^2 . \frac{y^2}{b^2} - 2Hab . \frac{xy}{ab} + Bb^2 . \frac{x^2}{a^2} = \cot^2\phi$$
$$= \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

as given.

9. The point O being $(\alpha', \beta', \gamma')$, a is $(0, \beta', \gamma')$. Hence the equation to bc is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & 0, & \gamma' \\ \alpha', & \beta', & 0 \end{vmatrix} = 0, \text{ i.e. } -\frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0,$$

and the intersection of this with $\alpha = 0$ lies on $\frac{\alpha}{\alpha'} + \frac{\beta}{\beta'} + \frac{\gamma}{\gamma'} = 0$, and so for the other two.

Also, since O is on the circumcircle, $\Sigma\beta'\gamma'\sin A = 0$. Hence the line last written passes through the fixed point

$$(\sin A,\ \sin B,\ \sin C).$$

10. Let O be the common vertex of the two tetrahedra, A and A' the centres of the bases. The volumes are in the ratio $1 : x^3$, and the distances of the centres of gravity from O are $\frac{3}{4}h$, $\frac{3}{4}xh$. Hence the distance of G, the C. of G. of the frustum, from O is

$$\bar{x} = \frac{\frac{3}{4}h - \frac{3}{4}x^4h}{1-x^3} = \frac{3}{4}h . \frac{(1+x)(1+x^2)}{1+x+x^2};$$

$$\therefore\ A'G = \bar{x} - xh = \frac{3-x-x^2-x^3}{4(1+x+x^2)} . h \quad \text{.........(i).}$$

Let AN be the perpendicular from A on the edge of the base in contact with the ground, and let ON cut the other edge in N'. Then, by the question, GN' must be vertical.

Now the edge of the larger tetrahedron is $\frac{\sqrt{3}}{\sqrt{2}} \,.\, h$,

$$\therefore AN = \frac{1}{3} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{\sqrt{2}} \cdot h = \frac{1}{2\sqrt{2}} h, \quad \therefore A'N' = \frac{1}{2\sqrt{2}} \cdot xh.$$

But $ON'G$ is a right angle;

$$\therefore A'G \,.\, xh = \left(\frac{1}{2\sqrt{2}} \,.\, xh\right)^2, \text{ i.e. } A'G = \tfrac{1}{8} xh \quad \ldots\ldots\text{(ii)}.$$

Equating the values of $A'G$ in (i) and (ii), we get the required equation.

11. Let t be the time one of the drops takes to reach the ground after leaving the rim. Then, since the drop has no initial vertical velocity, $\frac{1}{2} gt^2 = h$.

The horizontal distance described by the drop in time t, in the vertical plane perpendicular to the radius through its initial position, is

$$\omega a \,.\, t = \omega a \,.\, \sqrt{\frac{2h}{g}}\,.$$

Calling this distance x, the distance of the point at which the drop reaches the ground from the point at which the direction of the handle produced meets the ground is

$$(a^2 + x^2)^{\frac{1}{2}} = \left(a^2 + \omega^2 a^2 \,.\, \frac{2h}{g}\right)^{\frac{1}{2}} = a\left(1 + \frac{2\omega^2 h}{g}\right)^{\frac{1}{2}}.$$

12. The striking velocity of B is $\sqrt{2g \sin \alpha \,.\, b}$ and therefore its velocity after impact is $e\sqrt{2g \sin \alpha \,.\, b}$, so that after impact the distance it describes up the plane is

$$\frac{e^2 (2g \sin \alpha \,.\, b)}{2g \sin \alpha} = e^2 b.$$

Also the total time B is in motion is $(1+e)\sqrt{\dfrac{2b}{g\sin\alpha}}$, and in this time A describes a distance

$$\tfrac{1}{2}g\sin\alpha\,.\,(1+e)^2\frac{2b}{g\sin\alpha}=b(1+e)^2;$$

$$\therefore\ b(1+e)^2+e^2b=a+b,$$

$$\text{i.e. } b(2e+2e^2)=a.$$

XIV.

1. Since AL, BM, CN are concurrent,

$$\therefore \frac{BL}{LC}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=1.$$

Also, by the question

$$\frac{BL'}{L'C}=-\frac{BL}{LC},$$

$$\therefore \frac{BL'}{L'C}\cdot\frac{CM}{MA}\cdot\frac{AN}{NB}=-1,$$

i.e. the points M, N, L' are collinear.

2. We have

$$CN^2+PN^2=CA^2+CM^2.$$

But $$PN^2:CN^2-CA^2=CB^2:CA^2,$$

$$\therefore PN^2:PN^2+CN^2-CA^2=CB^2:CS^2,$$

$$\therefore PN:CM=CB:CS.$$

Again $$\left.\begin{aligned}PM'^2&=CP^2+CM'^2-2CM'\,.\,PN\\ P'M^2&=CP'^2+CM^2-2CM\,.\,P'N'\end{aligned}\right\}\cdots\cdots\cdots\cdots(\text{i}).$$

But $PN:CM=P'N':CM'$, $\therefore CM'\,.\,PN=CM\,.\,P'N'$.

Also $CP^2-CM^2=AM^2-CM^2=CA^2=CP'^2-CM'^2$, similarly.

Hence, from (i), we see that

$$PM'^2-P'M^2=0.$$

3. Since

$$(n+1)^5 - n^5 = 5n^2(n+1)^2 + 5n^2 + 5n + 1,$$

for all values of n, we have

$$(n+1)^5 - 1 = 5S + 5 \,.\, \Sigma n^2 + 5 \,.\, \Sigma n + n,$$

where S is the left-hand series.

$$\therefore\ 5S = (n+1)^5 - 1 - 5 \,.\, \frac{n(n+1)(2n+1)}{6} - 5 \,.\, \frac{n(n+1)}{2} - n$$

$$= (n+1)^5 - (n+1) - \frac{5n(n+1)}{6}(2n+4)$$

$$= n(n+1)\left[(n+2)(n^2+2n+2) - \tfrac{5}{3}(n+2)\right]$$

$$= n(n+1)(n+2)(n^2+2n+\tfrac{1}{3}).$$

4. (i) The given series is $n!$ times the coefficient of t^n in the expansion of

$$e^{xt} - n \,.\, e^{(x+y)t} + \frac{n(n-1)}{2!}e^{(x+2y)t} - \ldots,$$

i.e. of $e^{xt}(1-e^{yt})^n = (1 + xt + \ldots)(-yt - \ldots)^n$,

in which the coefficient of t^n is $(-y)^n$.

Hence the value of the given series is $(-1)^n \,.\, n! \,.\, y^n$.

(ii) The nth term is

$$\frac{4n-1}{n(n+2)} \,.\, 3^{n-1} = \left[\frac{9}{2(n+2)} - \frac{1}{2n}\right]3^{n-1}$$

$$= \tfrac{1}{2}\left(\frac{3^{n+1}}{n+2} - \frac{3^{n-1}}{n}\right).$$

Hence the sum of $(n+1)$ terms is

$$\tfrac{1}{2}\left(\frac{3^{n+2}}{n+3} + \frac{3^{n+1}}{n+2} - \tfrac{3}{2} - 1\right) = \tfrac{1}{2} \,.\, \frac{4n+9}{(n+2)(n+3)} \,.\, 3^{n+1} - \tfrac{5}{4}.$$

5. The identical relation between $\cos A$, $\cos B$, $\cos C$ is

$$1 - \Sigma \cos^2 A - 2\Pi \cos A = 0,$$

$$\text{i.e. } (\Sigma \sin^2 A - 2)^2 = 4\Pi (1 - \sin^2 A),$$

$$\text{i.e. } (y-2)^2 = 4\left[1 - y + \left(\frac{x^2-y}{2}\right)^2 - 2xz - z^2\right],$$

reducing to the given form.

6. If $\tan 5\theta = 1$ and $\tan\theta = x$, we have

$$5x - 10x^3 + x^5 = 1 - 10x^2 + 5x^4,$$

$$\text{i.e. } x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0,$$

or
$$(x-1)(x^4 - 4x^3 - 14x^2 - 4x + 1) = 0.$$

Also, under these conditions, $5\theta = n\pi + \dfrac{\pi}{4}$, i.e. $\theta = \dfrac{(4n+1)\pi}{20}$.

When $n = 1$, $\theta = \dfrac{\pi}{4}$, i.e. $x = 1$. Hence the roots of the equation in question are the tangents of $\dfrac{\pi}{20}, \dfrac{9\pi}{20}, \dfrac{13\pi}{20}, \dfrac{17\pi}{20}$, i.e. one root is $\tan 9°$, the others being

$$\tan 81°, \ \tan 117° \text{ and } \tan 153°.$$

7. Let the centre of one of the touching circles be (α, β) and its radius ρ. Then

$$(\alpha - a)^2 + \beta^2 = (r+\rho)^2 \ldots\text{(i)}, \qquad (\alpha - ma)^2 + \beta^2 = (mr + \rho)^2,$$

whence, subtracting,

$$2(a\alpha + r\rho) = (m+1)(a^2 - r^2);$$

$\therefore$ from (i)

$$\left.\begin{array}{l} \alpha^2 + \beta^2 - \rho^2 = m(a^2 - r^2) \\ \text{and similarly for the second circle} \\ \alpha'^2 + \beta'^2 - \rho'^2 = m(a^2 - r^2) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Hence the radical axis of the two circles passes through the origin. Hence, since the circles touch, the common tangent at the point of contact passes through the origin. But, from (ii), the square of the tangent from the origin to either circle is $m(a^2 - r^2)$. Hence, if (x, y) be the point of contact,

$$x^2 + y^2 = m(a^2 - r^2).$$

8. The centre is $(\frac{1}{5}, -\frac{18}{5})$ and the equation referred to parallel axes through the centre is $6x^2 + 24xy - y^2 = 30$. To make the axes of co-ordinates coincide with the principal axes, we must turn them through an angle given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{24}{7},$$

whence $\tan\theta = \frac{3}{4}$ or $-\frac{4}{3}$.

The equation in polar co-ordinates is

$$r^2 = \frac{30(1+\tan^2\theta)}{6 + 24\tan\theta - \tan^2\theta}.$$

This gives $r^2 = 2$ when $\tan\theta = \frac{3}{4}$, and $r^2 = -3$ when $\tan\theta = -\frac{4}{3}$. Hence the curve is a hyperbola with real semi-axis $\sqrt{2}$ inclined at an angle $\tan^{-1}\frac{3}{4}$ to the axis of x.

9. The point of intersection of the tangents at α, β to the parabola $\dfrac{l}{r} = 1 + \cos\theta$ is

$$\left(\frac{l}{2\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}},\ \frac{\alpha+\beta}{2}\right).$$

Hence it is easily seen that the equation to the circle through the intersections of tangents at α, β, γ is

$$r = \frac{l}{2\Pi\cos\dfrac{\alpha}{2}}\cos\left(\theta - \frac{\Sigma\alpha}{2}\right).$$

Hence at the intersection of the circle and parabola, we have

$$\left(kr - \frac{l-r}{r}\cos\lambda\right)^2 = \frac{2rl - l^2}{r^2}\sin^2\lambda,$$

where $$k = 2\Pi\cos\frac{\alpha}{2}\Big/ l,\quad \lambda = \frac{\Sigma\alpha}{2}.$$

This equation is a biquadratic in r, and the product of the roots is l^2/k^2.

Also $SP = \dfrac{l}{2\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}}$, etc., $\therefore SP \,.\, SQ \,.\, SR = \dfrac{l}{2k^2}$.

Hence the value of the expression in question is $2l$.

10. Suppose the rod containing w makes an angle α with the vertical. Then since the centre of gravity of the system must be vertically below the pivot, we have

$$w \,.\, l\cos\theta\sin\alpha + W \,.\, a\sin\alpha = w' \,.\, l\sin\theta\cos\alpha + W \,.\, a\cos\alpha,$$

or $$\sin\alpha\,(Wa + wl\cos\theta) = \cos\alpha\,(Wa + w'l\sin\theta) \quad \ldots\ldots\text{(i)}.$$

Also, for the equilibrium of the rings,

$$\frac{T}{\cos\alpha} = \frac{w}{\cos\theta}, \quad \frac{T}{\sin\alpha} = \frac{w'}{\sin\theta};$$

$$\therefore \frac{\sin\alpha}{w\sin\theta} = \frac{\cos\alpha}{w'\cos\theta}.$$

Combining this with (i), we obtain the given equation for θ.

11. Let P be the pull of the engine, and let steam be shut off after a length x of the journey. Then the acceleration during the first part is $f = \dfrac{P-F}{M}$, and the retardation during the second is $\dfrac{F}{M}$. The velocity when steam is shut off is $\sqrt{2fx}$.

$$\therefore 2fx = 2 \,.\, \frac{F}{M}(l-x).$$

Also the time occupied is $\sqrt{2fx}\left(\dfrac{1}{f} + \dfrac{M}{F}\right) = t.$

From these $$x = \frac{Fl}{Mf+F}, \quad f = \frac{2Fl}{Ft^2 - 2Ml},$$

$$P = Mf + F = \frac{F^2t^2}{Ft^2 - 2Ml},$$

and the greatest rate of working is $P \,.\, \sqrt{2fx}$.

12. If α is the angle of elevation, x the range on the horizontal plane, we have

$$-h = x\tan\alpha - \tfrac{1}{2}g \,.\, \frac{x^2}{V^2}(1 + \tan^2\alpha).$$

If this quadratic in $\tan\alpha$ has real roots,

$$x^2 > 4 \,.\, \tfrac{1}{2}g \,.\, \frac{x^2}{V^2}\left(\tfrac{1}{2}g \,.\, \frac{x^2}{V^2} - h\right),$$

whence
$$x^2 < \frac{V^2}{g^2}(V^2 + 2gh),$$
$$\text{i.e. } x < \frac{V^2}{g}\left(1 + \frac{2gh}{V^2}\right)^{\frac{1}{2}}.$$

For the battery on the plane, to find the greatest distance from the foot of the hill at which the other could be hit, we have only to change the sign of h in this result. Hence the distance over which the battery is unable to return fire is

$$\frac{V^2}{g}\left[\left(1 + \frac{2gh}{V^2}\right)^{\frac{1}{2}} - \left(1 - \frac{2gh}{V^2}\right)^{\frac{1}{2}}\right] \quad \text{...............(i)},$$

with the given values $\frac{2gh}{V^2} = \cdot 0192$, i.e. a small fraction. But, if k is small, $(1+k)^{\frac{1}{2}}$ is approximately equal to $1 + \frac{1}{2}k$. Hence the distance (i) is approximately

$$\frac{V^2}{g} \cdot \frac{2gh}{V^2} = 2h,$$

i.e. the distance required is about 600 ft.

XV.

1. If P be the point, the bisectors of $A\hat{P}D$ will be also those of $B\hat{P}C$. Hence, if these bisectors are PE, PF, the points E, F are harmonic conjugates for both A, D and B, C. To find such points draw the circles XAD, XBC, where X is any point not in AB. Let these intersect again in Y, and let XY cut AB in R (which will be outside the circles). Draw a tangent RZ to either circle, and with centre R and radius RZ draw a circle cutting AB in E, F. These are the points required, since evidently

$$RE^2 = RF^2 = RA \, . \, RD = RB \, . \, RC,$$

and the locus of P is the circle on EF as diameter.

2. If the diameter bisecting a chord meets the directrix in Z, then SZ is perpendicular to the chord. This may be proved as follows:—Let PP' be the chord, V its middle point. Draw SY perpendicular to the chord, meeting the directrix in Z. Join ZV meeting the axis at C. We have then to shew that C is the centre of the conic. Through V draw VN parallel to the axis meeting SZ in K and the directrix in N. Draw PM, $P'M'$ perpendicular to the directrix, and PL, $P'L'$ perpendicular to VN.

Then $SP^2 - SP'^2 = PY^2 - P'Y^2 = 4VP \,.\, VY = 4VK \,.\, VL$, since P, L, Y, K are cyclic.

Also $SP^2 - SP'^2 = e^2(PM^2 - P'M'^2) = e^2 \,.\, 4VN \,.\, VL$.

Hence $$VK = e^2 \,.\, VN, \text{ i.e. } CS = e^2 \,.\, CX,$$

$\therefore$ C is the centre.

The property in question immediately follows.

3. Denoting the nth term by u_n, we have

$$u_n = r^{n-1} + r^{n-2} + \dots + r + 1 = \frac{r^n - 1}{r - 1} \quad \dots\dots\dots\dots\text{(i)}.$$

Hence $$u_{n+1} - u_n = \frac{r^{n+1} - r^n}{r - 1} = r^n;$$

$$\therefore u_{n+2} - u_{n+1} = r(u_{n+1} - u_n).$$

The second result follows easily from the actual values of u_n, etc., given by (i).

4. We have

$$\frac{\Sigma n^3}{n} > \sqrt[n]{(n\,!)^3}, \text{ i.e. } \frac{n(n+1)^2}{4} > \sqrt[n]{(n\,!)^3}.$$

Also $$\frac{\Sigma n}{n} > \sqrt[n]{n\,!}, \text{ i.e. } \frac{n+1}{2} > \sqrt[n]{n\,!}.$$

Multiplying these, the result follows.

5. Let l be the length of each leg, x the side of the equilateral triangle. Then the projection of l on the ground is

$$\frac{2}{3} \cdot \frac{\sqrt{3}}{2} x, \text{ i.e. } \frac{x}{\sqrt{3}}, \quad \therefore \cos\alpha = \frac{x}{\sqrt{3}l}.$$

The angle between the legs is

$$2\sin^{-1}\left(\frac{x}{2l}\right) = 2\sin^{-1}\left(\frac{\sqrt{3}\cos\alpha}{2}\right)$$

$$= \cos^{-1}\left(1 - \frac{3\cos^2\alpha}{2}\right)$$

$$= \cos^{-1}\left[\tfrac{1}{4}(1 - 3\cos 2\alpha)\right], \text{ since } \cos^2\alpha = \tfrac{1}{2}(1 + \cos 2\alpha).$$

6. The roots of $\dfrac{\sin 7\theta}{\sin\theta} = 0$, considered as an equation in $\cos\theta$, are $\cos\dfrac{r\pi}{7}$ $(r = 1, 2 \ldots 6)$.

Putting $\sin 7\theta = \sin(3\theta + 4\theta)$, this equation is

$$(3 - 4\sin^2\theta)\cos 4\theta + \cos 3\theta \,.\, 4\cos\theta\cos 2\theta = 0,$$

and if $4\cos^2\theta = x$, this becomes

$$(x-1)\left(\frac{x^2}{2} - 2x + 1\right) + x(x-3)\left(\frac{x}{2} - 1\right) = 0,$$

or
$$x^3 - 5x^2 + 6x - 1 = 0,$$

and the roots are evidently

$$4\cos^2\frac{r\pi}{7} \quad (r = 1, 2, 3).$$

7. If P is the point (x', y'), the equation to the circle PQR must be of the form

$$y^2 - 4ax + \lambda(2ax - yy' + 2ax')(2ax + yy' + k) = 0,$$

with the condition

$$\lambda \,.\, 4a^2 = 1 - \lambda y'^2, \quad \text{i.e. } \lambda = 1/(4a^2 + y'^2).$$

Also, since the circle passes through (x', y'),

$$-1 + \lambda(2ax' + y'^2 + k) = 0,$$

whence
$$k = 4a^2 - 2ax'.$$

Substituting, and reducing, the equation to the circle is

$$a(x^2 + y^2) - (y'^2 + 2a^2)x + y'(x' - a)y + ax'(2a - x') = 0.$$

Hence, if ρ be its radius,

$$4a^2\rho^2 = \{y'^2 + (x' - a)^2\}(y'^2 + 4a^2).$$

Now the equation to PG is $xy' + 2ay = x'y' + 2ay'$ and this cuts the axis where $x = x' + 2a$, $\therefore\ PG^2 = 4a^2 + y'^2$,

$$\therefore\ 2a\rho = SP \,.\, PG.$$

8. If (x_1, y_1) is the pole of the normal chord at α, for the conic $\dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 1$, then the lines $\dfrac{xx_1}{A^2} + \dfrac{yy_1}{B^2} = 1$, and

$$ax \sin\alpha - by \cos\alpha = (a^2 - b^2) \sin\alpha \cos\alpha,$$

must coincide.

$$\therefore\ \frac{x_1}{A^2 a \sin\alpha} = \frac{y_1}{-B^2 b \cos\alpha} = \frac{1}{(a^2 - b^2)\sin\alpha\cos\alpha}.$$

Hence the line joining the poles in this case is

$$\begin{vmatrix} x, & y, & 1 \\ (a^2 + n^2b^2)\, a \sin\alpha, & -(b^2 + n^2a^2)\, b \cos\alpha, & (a^2 - b^2)\sin\alpha\cos\alpha \\ a^3 \sin\alpha, & -b^3\cos\alpha, & (a^2 - b^2)\sin\alpha\cos\alpha \end{vmatrix} = 0.$$

Subtracting the third row from the second, this is

$$\begin{vmatrix} x, & y, & 1 \\ b\sin\alpha, & -a\cos\alpha, & 0 \\ a^3\sin\alpha, & -b^3\cos\alpha, & (a^2 - b^2)\sin\alpha\cos\alpha \end{vmatrix} = 0,$$

or $$ax\cos\alpha + by\sin\alpha = a^2 + b^2,$$

which is a tangent to $a^2x^2 + b^2y^2 = (a^2 + b^2)^2$.

9. Taking the given line as the axis of x, and the point of contact as the origin, the equations are of the form

$$a(x^2 - y^2) + 2hxy = 2y \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

If (X, Y) is the centre, $aX + hY = 0$, $\quad hX - aY - 1 = 0$,

whence $$\frac{a}{Y} = \frac{h}{-X} = \frac{1}{-(X^2 + Y^2)}.$$

But since (i) passes through a fixed point, there exists a relation of the form $a\lambda + h\mu = 1$, and therefore we have

$$X^2 + Y^2 - \mu X + \lambda Y = 0,$$

shewing that the locus of (X, Y) is a circle.

10. Let O be the common extremity, R the pressure on either peg. Then resolving vertically for the system

$$R \sin \alpha = W.$$

Also taking moments about O for either rod, we have

$$T \,.\, 2a \cos \alpha + W \,.\, a \sin \alpha = R \,.\, c \operatorname{cosec} \alpha,$$

$$\therefore\ T \,.\, 2a \cos \alpha = -W \,.\, a \sin \alpha + W \,.\, c \operatorname{cosec}^2 \alpha.$$

$$\therefore\ T = \tfrac{1}{2} W \left(\frac{c}{a} \operatorname{cosec}^2 \alpha \sec \alpha - \tan \alpha\right).$$

11. If the velocity of projection be u, the maximum range on an inclined plane of angle θ is $u^2/g\,(1 + \sin\theta)$. Hence, if θ be the angle of elevation of the top of the wall, we must have

$$\frac{u^2}{g\,(1 + \sin\theta)} \not< \sqrt{h^2 + a^2}.$$

But $\quad \sin\theta = \dfrac{h}{\sqrt{h^2 + a^2}}, \quad \therefore\ u^2 \not< g\left(1 + \dfrac{h}{\sqrt{h^2 + a^2}}\right)\sqrt{h^2 + a^2}.$

12. Let u_1 be the velocity of rebound, v that of the wedge. Then, by Newton's Law,

$$-u_1 - v \sin \alpha = -u \quad \text{....................(i)}.$$

Also, since no kinetic energy is lost,

$$\therefore\ \tfrac{1}{2} M v^2 + \tfrac{1}{2} m u_1^2 = \tfrac{1}{2} m u^2 \quad \text{..................(ii)}.$$

From (i) and (ii),

$$(M + m \sin^2 \alpha)\, u_1^2 - 2 M u u_1 + (M - m \sin^2 \alpha)\, u^2 = 0,$$

$$\text{i.e.}\ \ (u_1 - u)\left[(M + m \sin^2 \alpha)\, u_1 - (M - m \sin^2 \alpha)\, u\right] = 0,$$

whence the result, since $u_1 \neq u$.

XVI.

1. In *any* tetrahedron, draw AH, BK perpendicular to the opposite faces. In general they will not intersect. Suppose, however, that they meet in I, and let the plane AIB meet CD in M, so that AKM, BHM are straight lines. Then AH, BK being both perpendicular to CD, it follows that CD is perpendicular to the plane AIB and therefore to AB. Conversely, if AB, CD are at right angles, then the perpendiculars from A and B on opposite faces intersect.

Further, $CA^2 - AD^2 = CM^2 - MD^2 = CB^2 - BD^2$,

$$\text{i.e. } CA^2 + BD^2 = AD^2 + BC^2.$$

Hence if AC, BD be also at right angles, we must have

$$AB^2 + CD^2 = AC^2 + BD^2 = AD^2 + BC^2,$$

whence BC, AD must also be at right angles, and the four perpendiculars from A, B, C, D on opposite faces must meet in the same point I.

Again the shortest distance between AB and CD is the perpendicular from M on AB, which also passes through I. So for the others.

2. Let P be the point of contact, C the centre. If CD be the semi-diameter conjugate to CP, then $CP^2 + CD^2 = a^2 + b^2$ and is therefore known. Hence CP being given, CD is given, i.e. we have given a pair of conjugate diameters in magnitude and position, whence the ellipse can be constructed.

3. We have

$$c(x + y + a - b) = (x + a)(y - b) = c(x + y) + ay - bx - ab,$$

$$\therefore\ bx - ay = -c(a - b) - ab.$$

So
$$b'x - a'y = -c(a' - b') - a'b'.$$

From these
$$x = c + \frac{aa'(b - b')}{ab' - a'b},$$

$$y = c + \frac{bb'(a - a')}{ab' - a'b}.$$

But $(x - c)(y - c) = c^2$: hence the result.

4. If u_n be either the numerator or denominator of the nth convergent,

$$u_n = 3u_{n-1} + 4u_{n-2}.$$

Assuming $u_n = A\lambda^n$, this gives $\lambda^2 - 3\lambda - 4 = 0$, i.e. $\lambda = 4$ or -1.

$$\therefore\ u_n = A \,.\, 4^n + B \,.\, (-1)^n.$$

Putting in initial values, we easily find

$$p_n = \tfrac{4}{5}\,[4^n - (-1)^n], \quad q_n = \tfrac{1}{5}\,[4^{n+1} + (-1)^n],$$

$$\therefore\ \frac{p_n}{q_n} = \frac{4\,[4^n + (-1)^{n-1}]}{4^{n+1} + (-1)^n}.$$

Hence the product of the first n convergents is

$$\frac{4^n\,(4+1)}{4^{n+1} + (-1)^n}.$$

5. Putting $\tan \frac{1}{2}x = t$, the equation becomes

$$\frac{1 - 6t^2 + t^4}{(1+t^2)^2} + a \,.\, \frac{1-t^2}{1+t^2} + b \,.\, \frac{2t}{1+t^2} + c = 0.$$

Clearing of fractions, it is at once evident that the only odd powers of t are those arising from the third term, viz. $2bt\,(1+t^2)$, i.e. the coefficients of t^3 and t are equal. Hence, if the roots of the equation are $t_1 = \tan \frac{1}{2}\alpha$, etc., we have

$$\Sigma t_1 = \Sigma t_1 t_2 t_3.$$

But in the formula for $\tan \frac{1}{2}(\alpha + \beta + \gamma + \delta)$, the numerator is $\Sigma t_1 - \Sigma t_1 t_2 t_3$. It therefore follows that

$$\tan \tfrac{1}{2}(\alpha + \beta + \gamma + \delta) = 0,$$

$$\text{i.e. } \tfrac{1}{2}(\alpha + \beta + \gamma + \delta) = n\pi.$$

6. The equation $\dfrac{\tan 13\theta}{\tan\theta} = 0$ is equivalent to

$$13 - 286 \tan^2\theta + \ldots + \tan^{12}\theta = 0,$$

and the roots are $\tan r\alpha$, $(r = 1, 2 \ldots 12)$.

Hence $\prod_{r=1}^{r=12} \tan r\alpha = 13$, and $\therefore$ since $\tan r\alpha = -\tan(13-r)\alpha$,

$$\prod_{r=1}^{r=6} \tan^2 r\alpha = 13.$$

On taking the square root, the positive sign must be taken, since the angles are all acute.

7. The intersection of tangents at m, m' is amm', $a(m+m')$. Hence

$$t_1^2 = (amm' - am^2)^2 + [a(m+m') - 2am]^2$$
$$= a^2(m-m')^2(m^2+1).$$

Also
$$SP^2 = (amm' - a)^2 + [a(m+m')]^2$$
$$= a^2(m^2+1)(m'^2+1).$$

Hence if $t_1 t_2 = 4a \,.\, SP$, we must have $(m-m')^2 = 4$.

$\therefore$ putting $m + m' = p$, the co-ordinates of P are

$$x = a \,.\, \frac{p^2-4}{4}, \quad y = ap,$$

and the locus is

$$y^2 = 4a(x+a).$$

8. Let the chord be $lx + my = 1$. Then, transferring the origin to the point α, the equation to the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right) = 0,$$

and that to the chord

$$lx + my + al\cos\alpha + bm\sin\alpha - 1 = 0.$$

Hence the equation to the lines joining the intersections to the origin is

$$(al\cos\alpha + bm\sin\alpha - 1)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$$
$$- 2(lx+my)\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right) = 0,$$

and the condition that these are at right angles is

$$(al\cos\alpha + bm\sin\alpha - 1)\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - 2\left(\frac{l\cos\alpha}{a} + \frac{m\sin\alpha}{b}\right) = 0$$

or
$$a(a^2-b^2)\,l\cos\alpha - b(a^2-b^2)\,m\sin\alpha = a^2+b^2,$$

which is the condition that the point in question lies on

$$lx + my = 1.$$

9. Let one of the conics be $\frac{l'}{r} = 1 + e'\cos(\theta - \gamma)$. The tangents at α are

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha),$$

and $$\frac{l'}{r} = e' \cos(\theta - \gamma) + \cos(\theta - \alpha).$$

If these coincide

$$\frac{e' \cos \gamma + \cos \alpha}{e + \cos \alpha} = \frac{e' \sin \gamma + \sin \alpha}{\sin \alpha} = \frac{l'}{l} \quad \text{............(i)},$$

whence $$e' = \frac{e \sin \alpha}{\cos \gamma \sin \alpha - \sin \gamma (e + \cos \alpha)},$$

and the greatest value of the denominator is

$$[\sin^2 \alpha + (e + \cos \alpha)^2]^{\frac{1}{2}} = (1 + 2e \cos \alpha + e^2)^{\frac{1}{2}}.$$

Further, from (i), putting $\frac{l'}{l} = \lambda$, we have

$$e'^2 = [\lambda (e + \cos \alpha) - \cos \alpha]^2 + (\lambda - 1)^2 \sin^2 \alpha$$
$$= \lambda^2 (1 + 2e \cos \alpha + e^2) - 2\lambda (1 + e \cos \alpha) + 1,$$

whence, substituting for e', we get the required value for l'.

10. Let T, T' be the tensions of the two parts of the string. Then resolving horizontally for the weight,

$$T \cos(\theta - \alpha) = T' \cos(\theta - \beta).$$

Also for the rings,

$$\frac{w}{\cos \theta} = \frac{T}{\sin \alpha}, \quad \frac{w'}{\cos \theta} = \frac{T'}{\sin \beta}.$$

Hence $\quad w \sin \alpha \cos(\theta - \alpha) = w' \sin \beta \cos(\theta - \beta),$

which is equivalent to the given result.

11. Let V_1 be the velocity of C after impact, V_2 and V_3 those of A along and perpendicular to AB. Then B will move in the direction AB with velocity V_2. Hence the equations of momentum are

$$(V - V_1) \cos \theta = 2V_2, \quad (V - V_1) \sin \theta = V_3.$$

Thus if the direction of motion of A makes an angle ϕ with AB,

$$\tan \phi = \frac{V_3}{V_2} = 2 \tan \theta;$$

$$\therefore \tan(\phi - \theta) = \frac{\tan \theta}{1 + 2 \tan^2 \theta} = \frac{1}{\cot \theta + 2 \tan \theta}.$$

12. Let t, t' be the times taken to reach B. Then we have

$$V\cos\alpha\,.\,t = V\cos\alpha'\,.\,t',$$

$$V\sin\alpha\,.\,t - \tfrac{1}{2}gt^2 = V\sin\alpha'\,.\,t' - \tfrac{1}{2}gt'^2,$$

whence $$\frac{t}{\cos\alpha'} = \frac{t'}{\cos\alpha} = k, \text{ say.}$$

$$\therefore\ V\sin(\alpha-\alpha') = \tfrac{1}{2}gk(\cos^2\alpha' - \cos^2\alpha),$$

whence $$k = \frac{2V}{g}\,.\,\frac{1}{\sin(\alpha+\alpha')};$$

$$\therefore\ t - t' = k(\cos\alpha' - \cos\alpha) = \frac{2V}{g}\,.\,\frac{\cos\alpha'-\cos\alpha}{\sin(\alpha+\alpha')}$$

$$= \frac{2V}{g}\,.\,\frac{\sin\frac{1}{2}(\alpha-\alpha')}{\cos\frac{1}{2}(\alpha+\alpha')}.$$

XVII.

1. Take O outside AD on the side of A. Then

(i) $OA\,.\,OB = OM^2 - MB^2 = OM^2 - MC\,.\,MD = OM^2 - (MN^2 - CN^2),$

$OC\,.\,OD = ON^2 - CN^2.$

$$\therefore\ OA\,.\,OB + OC\,.\,OD = OM^2 + ON^2 - MN^2$$

$$= OM^2 + ON^2 - (ON - OM)^2 = 2OM\,.\,ON.$$

(ii) Let $OA = x$, $AC = a$, $CB = b$, $AD = a'$, $BD = b'$.

Then the relation to be proved is

$$-bx + (x + a + b)a' = (x + a)b' + (x + a')a,$$

or $$x(a' - b) + a'b = x(a + b') + ab',$$

which is satisfied since

$$a + b = a' - b' = AB, \text{ and } a/b = a'/b'.$$

2. Let the inscribed sphere touch the generators VA, VA' in L, L' and the section in S, so that S is a focus. Then

VP = sum of tangents from V and P to the sphere

$= VL + SP.$

$$\therefore\ VP + VQ = VL + SP + VL' + SQ$$
$$= 2VL + SP + S'P \quad (\text{since } SQ = S'P)$$
$$= 2VL + AA' = VA + VA'.$$

3. Putting $x + z = p$, $xz = q$, we have to eliminate p, q, y from

$$p = a + 2y \ldots\ldots(\text{i}), \qquad p^3 - 3pq = c^3 + 2y^3 \ldots(\text{iii}),$$
$$p^2 - 2q = b^2 + 2y^2 \ldots(\text{ii}), \qquad \frac{p}{q} = \frac{2}{y} \ldots\ldots\ldots(\text{iv}).$$

Eliminating p and q from (i), (ii) and (iv), we easily find

$$3ay = -(a^2 - b^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{v}).$$

Eliminating the same quantities from (i), (iii) and (iv), we have

$$(a + 2y)^3 - \tfrac{3}{2}y(a + 2y)^2 = c^3 + 2y^3,$$

whence $$a^3 + \tfrac{9}{2}ay + 6ay^2 = c^3.$$

Substituting for y from (v), and reducing, the eliminant is

$$a^4 + a^2b^2 + 4b^4 = 6ac^3.$$

4. We may tabulate as follows :—

Like Things (5)	Unlike Things (4)	Combinations	Permutations
5	1	${}^4C_1$	$4 \times \frac{6!}{5!} = 24$
4	2	${}^4C_2$	$6 \times \frac{6!}{4!} = 180$
3	3	${}^4C_3$	$4 \times \frac{6!}{3!} = 480$
2	4	1	$\frac{6!}{2!} = 360$
			1044

5. Denote the lengths of the diagonals AC, BD by x, y. Let the areas of $\triangle$s ABC, ADC be Δ_1 and Δ_2. Then

$$4R\Delta_1 = abx, \quad 4R\Delta_2 = cdx.$$

$$\therefore\ 4R\Delta = (ab + cd)\,x, \text{ and similarly } 4R\Delta = (bc + ad)\,y.$$

$$\begin{aligned}\therefore\ 16R^2\Delta^2 &= (ab + cd)(bc + ad)\,xy \\ &= (ab + cd)(bc + ad)(ac + bd).\end{aligned}$$

6. If $\cos^2\theta = x$, then

$$\frac{\cos 7\theta}{\cos\theta} \equiv 64x^3 - 112x^2 + 56x - 7 = \phi(x), \text{ say.}$$

Also if $\cos 7\theta = 0$ and $\cos\theta \neq 0$,

$$7\theta = (2r+1)\frac{\pi}{2}, \text{ i.e. } \theta = \frac{(2r+1)\pi}{14}, \quad (r \neq 3).$$

Hence the roots of the equation $\phi(x) = 0$ are

$$\cos^2\frac{(2r+1)\pi}{14}, \quad (r = 0,\ 1,\ 2),$$

and the sum of their reciprocals is $\frac{56}{7} = 8$.

7. Let the equation to the circle be $x^2 + y^2 = a^2$. Then the polar of (x_1, y_1) is $xx_1 + yy_1 = a^2$.

Hence

$$2\Delta' = \frac{\begin{vmatrix} x_1, & y_1, & -a^2 \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \end{vmatrix}^2}{\begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix} . \begin{vmatrix} x_2, & y_2 \\ x_3, & y_3 \end{vmatrix} . \begin{vmatrix} x_3, & y_3 \\ x_1, & y_1 \end{vmatrix}}.$$

Now the numerator is

$$a^4 . \begin{vmatrix} x_1, & y_1, & 1 \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \end{vmatrix}^2 = a^4 . (2\Delta)^2.$$

Hence
$$2\Delta' = \frac{a^4 . (2\Delta)^2}{(2\Delta_1)(2\Delta_2)(2\Delta_3)}.$$

8. If the normal at m passes through (h, k), then m satisfies the equation

$$am^3 + (2a - h)m - k = 0.$$

If the roots are m_1, m_2, m_3, then $\tan\phi_1 = -m_1$, etc.

$$\therefore \tan(\phi_1 + \phi_2 + \phi_3) = \frac{-\Sigma m_1 + m_1 m_2 m_3}{1 - \Sigma m_1 m_2}$$

$$= \frac{\dfrac{k}{a}}{1 - \dfrac{2a-h}{a}} = \frac{k}{h-a}.$$

The sign depends upon the interpretation of the inverse tangent.

9. Let the line $\dfrac{\alpha}{l} + \dfrac{\beta}{m} + \dfrac{\gamma}{n} = 0$ meet the sides of the triangle of reference in D, E, F. Then the co-ordinates of D are

$$\alpha = 0, \quad \frac{\beta}{m} = \frac{\gamma}{-n} = \frac{2\Delta}{bm - cn}.$$

Hence the middle point of AD is

$$\frac{\Delta}{a}, \quad \frac{\Delta m}{bm - cn}, \quad -\frac{\Delta n}{bm - cn},$$

and similarly for the middle point of BE.

Hence the line joining these points is

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ bm - cn, & am, & -an \\ -bl, & cn - al, & bn \end{vmatrix} = 0,$$

which reduces to the given form. The symmetry of the result shews that the middle point of CF also lies on this line.

10. Let O be the centre, and let OA make an angle α with the horizontal. Let R be the pressure at A, R' that at the middle point of BC. Then resolving vertically for the system

$$2R\sin\alpha + R' = W.$$

Taking moments about B for AB,

$$\tfrac{1}{4}W \cdot \frac{a}{2}\sin\alpha = R \cdot a,$$

and from these $$R' = W(1 - \tfrac{1}{4}\sin^2\alpha).$$

Now $$\tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) = \frac{1}{n},$$

whence $$\tan\frac{\alpha}{2} = \frac{n-1}{n+1},$$

and $$\therefore \sin\alpha = \frac{n^2-1}{n^2+1}.$$

Substituting, the result follows.

11. If we replace the weight of the rod AB by two forces at A and B, each equal to half the weight, the rod AB is in equilibrium under two forces at A and two at B, and the resultants of these pairs of forces must be equal and opposite, and both in the line AB. Hence we construct the force diagram as follows:— Draw ab, bc vertically downwards to represent the weights of AB, BC respectively. Bisect ab, bc in the points 1 and 2. Draw $1O$ and $2O$ parallel to AB, CB. Then Ob represents the reaction at B.

But $$\frac{1b}{b2} = \frac{AB}{BC} = \frac{1O}{2O};$$

$\therefore$ Ob bisects the angle $1O2$.

Also $$\frac{R}{W} = \frac{Ob}{ac} = \frac{1}{2} \cdot \frac{Ob}{12} = \frac{1}{2} \cdot \frac{BD}{AC}.$$

12. The equation to the path is

$$y = x\tan\alpha - \tfrac{1}{2}g \cdot \frac{x^2}{V^2\cos^2\alpha}.$$

If the distance of the wall from the man be b, this path passes through the points $(b, a-h)$ and $(2b, -h)$. Hence

$$a - h = b\tan\alpha - \tfrac{1}{2}g \cdot \frac{b^2}{V^2\cos^2\alpha},$$

$$-h = 2b\tan\alpha - \tfrac{1}{2}g \cdot \frac{4b^2}{V^2\cos^2\alpha},$$

whence $\quad 4a - 3h = 2b \tan \alpha, \quad 2a - h = g \cdot \dfrac{b^2}{V^2 \cos^2 \alpha}.$

Eliminating b, these give the required value for $\sin \alpha$.

XVIII.

1. Let O be the intersection of the diagonals of $ABCD$, and let the parallels meet the opposite sides in P, Q, R, S. Draw DU parallel to BC, meeting AC in U. Let the opposite sides intersect in E and F.

Then $\quad \dfrac{AR}{RE} = \dfrac{AO}{OC}, \quad \dfrac{EP}{PD} = \dfrac{BO}{OD} = \dfrac{CO}{OU}, \quad \dfrac{DQ}{QA} = \dfrac{UO}{OA},$

and the ratio compounded of these is negative unity; $\therefore$ P, Q, R (considered as points on the sides of EAD) are collinear, and similarly for any other three.

2. Let PS, RS meet the major axis in N, G. Then

$$SN : PN = OQ : OP = b : a.$$

$\therefore$ S is on the ellipse.

Also $$OG : QS = OR : QR = a + b : a,$$
$$QS : ON = QP : OP = a - b : a,$$
$$\therefore \; OG : ON = a^2 - b^2 : a^2,$$

i.e. $OG = e^2 . ON$ and $\therefore$ SG is the normal at S.

3. The sum of the homogeneous products of n dimensions of a, b, c, d is the coefficient of x^n in

$$\frac{1}{(1 - ax)(1 - bx)(1 - cx)(1 - dx)},$$

$$\text{i.e. in } \Sigma \frac{a^3}{(a-b)(a-c)(a-d)} \cdot \frac{1}{1 - ax},$$

and this coefficient is

$$\sum_{a, b, c, d} \frac{a^{n+3}}{(a-b)(a-c)(a-d)}.$$

The result in question is clearly obtained by putting $d=1$ in this expression, and the last of the four fractions is then equal to

$$\frac{1}{(1-a)(1-b)(1-c)}.$$

4. Assuming $u_n = A\lambda^n$, we have $\lambda^2 - p\lambda + q^2 = 0$. Calling the roots of this equation α and β, we have $u_n = A\alpha^n + B\beta^n$. Hence, if $\alpha > \beta$, $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n-1}} = \alpha$, so that the condition of convergence is $\alpha < 1$.

Now $$\lambda^2 - p\lambda + q^2 \equiv (\lambda - \alpha)(\lambda - \beta);$$

$$\therefore\ 1 - p + q^2 \equiv (1-\alpha)(1-\beta).$$

But, if $1 > \alpha > \beta$, the right-hand side is positive, $\therefore\ p < q^2 + 1$. This assumes α to be real, which will be the case if $p > 2q$. Hence if the series is convergent, p lies between the limits specified.

5. The first equation may be written

$$t_1(t_2 - t_3) = a(1 + t_2 t_3), \text{ where } t_1 = \tan x, \text{ etc.}$$

Putting $$t_2 t_3 = \xi, \quad t_3 t_1 = \eta, \quad t_1 t_2 = \zeta,$$

this becomes $$a\xi + \eta - \zeta = -a,$$

and similarly $$-\xi + b\eta + \zeta = -b,$$

$$\xi - \eta + c\zeta = -c,$$

from which, either $$\xi = \eta = \zeta = -1 \ \dots\dots\dots\dots\dots\dots\text{(i)},$$

or $$\begin{vmatrix} a, & 1, & -1 \\ -1, & b, & 1 \\ 1, & -1, & c \end{vmatrix} = 0, \text{ i.e. } abc + a + b + c = 0 \dots\dots\text{(ii)}.$$

The values (i) lead to $t_2 t_3 = t_3 t_1 = t_1 t_2 = -1$, whence $t_1 = t_2 = t_3$. (The values are imaginary.) These, if admitted, lead to

$$a = b = c = 0.$$

Otherwise the condition (ii) must be satisfied.

6. (i) We have $GP = \frac{1}{3} b \sin C$, etc.

$$\therefore \triangle PQR = \Sigma \triangle GQR = \tfrac{1}{2} \Sigma\, GQ \,.\, GR \sin A$$
$$= \tfrac{1}{18} \,.\, \Sigma\, a \sin C \,.\, a \sin B \,.\, \sin A$$
$$= \tfrac{1}{18} (\Sigma a^2) \sin A \sin B \sin C.$$

(ii) Also $$\frac{PQ}{\sin C} = GC = \tfrac{2}{3} CF.$$

Hence (from the formula $R = abc/4\Delta$), we have

$$\text{radius of circle } PQR = \frac{\frac{8}{27} \,.\, AD \,.\, BE \,.\, CF \,.\, \Pi \sin A}{\frac{2}{9} \,.\, (\Sigma a^2) \,.\, \Pi \sin A}$$
$$= \tfrac{4}{3} \,.\, \frac{AD \,.\, BE \,.\, CF}{\Sigma a^2}.$$

(iii) The sum of the areas of these circles (since their diameters are AG, etc.) is

$$\tfrac{1}{4}\pi \,.\, \Sigma AG^2 = \tfrac{1}{9}\pi \,.\, \Sigma AD^2 = \tfrac{1}{12}\pi \,.\, (\Sigma a^2),$$

since $$AD^2 = \tfrac{1}{2} b^2 + \tfrac{1}{2} c^2 - \tfrac{1}{4} a^2, \text{ etc.}$$

7. The centres are (1, 1) and (4, 4) and the radii 1 and 2. Hence the centres of similitude are (2, 2) and (– 2, – 2). From a figure, the common tangents through (2, 2) are evidently $x = 2$ and $y = 2$.

Let one of the others be

$$y + 2 = m(x + 2).$$

Then, transferring the origin to (– 2, – 2), the line $y = mx$ touches

$$(x - 3)^2 + (y - 3)^2 = 1.$$

Hence the equation

$$(1 + m^2) x^2 - 6(1 + m) x + 17 = 0$$

has equal roots, whence

$$9(1 + m)^2 = 17(1 + m^2), \text{ i.e. } m = \frac{9 \pm \sqrt{17}}{8}.$$

8. The line $$\frac{x-p}{\cos\theta}=\frac{y-q}{\sin\theta}=r$$

meets the conic where

$$r(a\cos^2\theta+b\sin^2\theta)+2r(ap\cos\theta+bq\sin\theta)+ap^2+bq^2-1=0.$$

If the line is a tangent, this quadratic in r has equal roots,

$$\therefore\ (ap\cos\theta+bq\sin\theta)^2=(a\cos^2\theta+b\sin^2\theta)(ap^2+bq^2-1).$$

Also, in this case, each of the values of r is the length of the tangent, so that the square of the tangent is the product of the roots, i.e.

$$(ap^2+bq^2-1)/(a\cos^2\theta+b\sin^2\theta).$$

9. The centre is (4, 2), and the equation referred to the centre is

$$3x^2+8xy-3y^2=50.$$

The axes are now

$$\frac{x^2-y^2}{6}=\frac{xy}{4},\ \text{i.e. } 2x+y=0,\ x-2y=0,$$

and putting $$X=\frac{2x+y}{\sqrt{5}},\quad Y=\frac{x-2y}{\sqrt{5}},$$

the equation becomes $$X^2-Y^2=10.$$

The directrices are now

$$X=\pm\frac{\sqrt{10}}{\sqrt{2}}=\pm\sqrt{5},\ \text{i.e. } 2x+y=\pm 5,$$

or, referring to the original axes,

$$2(x-4)+y-2=\pm 5,$$

$$\text{i.e. } 2x+y=15 \text{ and } 2x+y=5.$$

10. Let A be the highest point of the rod, B the point of contact, G the middle point. Let the horizontal through A and the vertical through G meet in N. Then BN must be the

direction of the friction at B; suppose that it makes an acute angle ϕ with the radius. Then, if $AB = x$,

$$\frac{x}{AN} = \frac{\cos(\theta - \phi)}{\cos\phi}, \quad \frac{AN}{a} = \cos\theta,$$

$$\therefore \frac{x}{a} = \cos\theta(\cos\theta + \sin\theta\tan\phi) \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Also, if F be the friction, $F = \dfrac{W}{\cos(\theta - \phi)}$, where ϕ is given by (i).

In limiting equilibrium $\phi = \pm\lambda$, the angle of friction.

$$\therefore \text{ from (i), } \frac{x}{a} = \cos\theta(\cos\theta \pm \mu\sin\theta).$$

11. The tension of the string in the first case is Mg, and this must remain unaltered when the mass M is replaced by the pulley. In this case, the tension of the lower string is

$$T' = \frac{2mm'}{m + m'} \cdot g,$$

and we must therefore have

$$Mg = pg + 2T',$$

$$\therefore M - p = \frac{4mm'}{m + m'}.$$

12. If $2R$ is the range, and h the greatest height, the equation to the path may be written

$$(x - R)^2 = -\frac{2u^2\cos^2\alpha}{g}(y - h).$$

From symmetry, the co-ordinates of the two lower corners are $\left(R \pm a, \dfrac{\sqrt{3}a}{2}\right)$, and of the two upper corners $\left(R \pm \dfrac{a}{2}, \sqrt{3}a\right)$.

Substituting these we get

$$a^2 = -\frac{2u^2\cos^2\alpha}{g}\left(\frac{\sqrt{3}a}{2} - h\right),$$

$$\frac{a^2}{4} = -\frac{2u^2\cos^2\alpha}{g}(\sqrt{3}a - h).$$

Dividing, we find

$$h = \frac{7}{2\sqrt{3}}\,a, \quad \therefore\ u^2 \sin^2 a = \frac{7}{\sqrt{3}}\,ga,$$

and, by substitution for h,

$$u^2 \cos^2 a = \frac{\sqrt{3}}{4}\,ga.$$

$$\therefore\ u^2 = \frac{31}{4\sqrt{3}} \cdot ga.$$

$$\therefore\ u^2 : u^2 \cos^2 a = 31 : 3.$$

XIX.

1. Let two of the circumcircles meet in O. Then O has the same pedal line for the two triangles, i.e. the feet of the perpendiculars from O on the four sides of the quadrilateral are collinear. Also the line of collinearity bisects the line joining O to each of the four orthocentres. Hence these orthocentres must be collinear.

2. From the circle

$$PQ\,.\,PS = PG\,.\,Pg = CD^2 = SP\,.\,S'P,$$

$$\therefore\ PQ = S'P.$$

Let E be the middle point of SQ, and $S'Z$ the perpendicular from S' on the tangent at P, so that Z lies on the auxiliary circle, and CZ is parallel to SP.

Then $PE = \frac{1}{2}(PS + PQ) = \frac{1}{2}(PS + S'P) = CA = CZ$,

therefore CE is parallel to PZ and is therefore the direction of the diameter conjugate to CP.

3. Solving for $(b-c)x$, etc., by cross-multiplication, we get

$$\frac{(b-c)\,x}{bz-cy} = \frac{(c-a)\,y}{cx-az} = \frac{(a-b)\,z}{ay-bx} = \frac{1}{k}, \text{ say } \quad \text{.........(i).}$$

Eliminating x, y, z, we have

$$\begin{vmatrix} (b-c)k, & c, & -b \\ -c, & (c-a)k, & a \\ b, & -a, & (a-b)k \end{vmatrix} = 0,$$

whence $k=0$, 1 or -1.

If $k=0$, we have $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, and the given equations are identical, and are not satisfied unless two of the quantities a, b, c are equal.

If $k=1$, we clearly have $x=y=z$.

If $k=-1$, the equations (i) give

$$b(z+x)=c(x+y)=a(y+z),$$

$$\text{i.e.}\ \frac{z+x}{ca}=\frac{x+y}{ab}=\frac{y+z}{bc},$$

whence $$\frac{x}{-bc+ca+ab}=\ldots=\ldots, \text{ as stated.}$$

4. (i) From the graphs

$$y=x^3, \quad y=\tfrac{3}{2}x-\tfrac{3}{10},$$

the approximate solutions are $x=-1{\cdot}3$, ${\cdot}2$, $1{\cdot}1$.

(ii) The root clearly lies between 1 and 2. Drawing the graphs $y=\log_e x$ and $y=\frac{1}{x}$ between these limits, the value of the root is about $1{\cdot}76$.

5. Let the bisector of the angle A and the perpendicular from A on BC meet the circumcircle in D and E respectively, and let AL be the diameter.

Then $$B\hat{A}L=90^\circ-C=\frac{A}{2}+\frac{B-C}{2}, \quad B\hat{A}D=\frac{A}{2},$$

$$\therefore DAL=\frac{B-C}{2},$$

$$\therefore p=2R\cos\frac{B-C}{2}.$$

Also $BAE = 90^\circ - B$, $\therefore E\hat{A}L = B - C$, $\therefore u = 2R\cos(B - C)$.

Hence $\Sigma p^2(v - w) = 8R^3 . \Sigma \cos^2 \frac{B-C}{2} (\cos \overline{C - A} - \cos \overline{A - B})$

$$= 4R^3 . \Sigma (1 + \cos \overline{B - C})(\cos \overline{C - A} - \cos \overline{A - B})$$
$$\equiv 0.$$

6. Let $\tan\theta = t$, $\tan\alpha_1 = t_1$, etc. Then the equation is

$$\Sigma \frac{1 + tt_1}{t - t_1} = 0,$$

$$\text{i.e. } \Sigma (1 + tt_1)(t - t_2)(t - t_3) = 0,$$

or

$$t^3 s_1 + t^2(3 - 2s_2) + t(3s_3 - 2s_1) + s_2 = 0,$$

where $s_1 = \Sigma t_1$, $s_2 = \Sigma t_1 t_2$, $s_3 = t_1 t_2 t_3$.

$$\therefore \tan(\theta_1 + \theta_2 + \theta_3) = \frac{\frac{2s_2 - 3}{s_1} + \frac{s_2}{s_1}}{1 - \frac{3s_3 - 2s_1}{s_1}} = \frac{s_2 - 1}{s_1 - s_3}$$

$$= -\cot(\alpha_1 + \alpha_2 + \alpha_3),$$

$$\therefore \theta_1 + \theta_2 + \theta_3 = \alpha_1 + \alpha_2 + \alpha_3 + \text{ an odd multiple of } \frac{\pi}{2}.$$

7. The common focus being the origin, the equation to any one of the parabolas is of the form $y^2 = 4a(x + a)$, and the co-ordinates of any point on it may be taken in the form $a(t^2 - 1)$, $2at$. The tangent at this point is

$$x - ty + a(t^2 + 1) = 0 \text{ .... (i)},$$

and the normal

$$xt + y = a(t^3 + t) \text{ .... (ii)}.$$

Hence, from (i), the parameters of the points, the tangents at which pass through (h, k), are the roots of the equation

$$at^2 - kt + h + a = 0 \text{ .... (iii)}.$$

Calling these t_1, t_2, and supposing that the normals at these points meet in (X, Y), then, from (ii), t_1, t_2 must also be roots of

$$at^3 - (X - a)t - Y = 0 \text{ .... (iv)}.$$

Hence, from (iii) and (iv), we must have

$$at^3 - (X-a)\,t - Y \equiv (at^2 - kt + h + a)\left(t + \frac{k}{a}\right).$$

Identifying coefficients, we obtain

$$X = \frac{k^2}{a} - h, \quad Y = -\frac{k}{a}(h + a),$$

and eliminating a, the locus of (X, Y) is as stated.

8. Let the extremity of one of the semi-diameters be

$$(a \cos \phi, b \sin \phi).$$

Then

$$a^2 \cos^2 \phi + b^2 \sin^2 \phi + 2ga \cos \phi + 2fb \sin \phi + c = 0,$$

or $$a^2 \sin^2 \phi + b^2 \cos^2 \phi - \frac{2ga}{b}(b \cos \phi) + \frac{2fb}{a}(-a \sin \phi)$$
$$- (a^2 + b^2 + c) = 0,$$

proving that $(-a \sin \phi, b \cos \phi)$ lies on the second given circle.

9. The equation may be written

$$a\left(x + \frac{p}{2a}\right)^2 + b\left(y + \frac{q}{2b}\right)^2 = \frac{p^2}{4a} + \frac{q^2}{4b} - r.$$

Transferring the origin to $\left(-\frac{p}{2a}, -\frac{q}{2b}\right)$, this becomes

$$ax^2 + by^2 = \lambda, \text{ where } \lambda = \frac{p^2}{4a} + \frac{q^2}{4b} - r.$$

The semi-axes of this ellipse are $\sqrt{\frac{\lambda}{a}}$ and $\sqrt{\frac{\lambda}{b}}$, and its area is therefore $\frac{\pi\lambda}{\sqrt{ab}}$.

10. Take ac, cb, ba' on the same vertical to represent the weights of the three rods. Bisect these lines in 2, 1, 3 respectively.

Draw $2O$, $1O$ parallel to AC, BC. Then cO must be the direction of the reaction at C, and $3O$ must be parallel to AB.

Let cO make an angle ψ with the vertical, and let p be the perpendicular from O on aba'. Then

$$2c = p(\cot\phi - \cot\psi), \quad c3 = p(\cot\psi + \cot\theta),$$

$$\therefore \frac{\cot\phi - \cot\psi}{\cot\psi + \cot\theta} = \frac{2c}{c3} = \frac{\frac{1}{2}W_2}{W_1 + \frac{1}{2}W_3},$$

leading to the required value for $\cot\psi$.

11. Suppose that, in a limiting position, the ring on AB is about to slip down, and is distant x from A. Let R be the pressure on the rod, T the tension, and θ the inclination of AB to the vertical. Then, for the ring,

$$R = W\sin\theta, \quad T + \mu R = W\cos\theta, \quad \therefore T = W(\cos\theta - \mu\sin\theta).$$

So, for the other ring, $T = W(\sin\theta + \mu\cos\theta)$, $\therefore \tan\theta = \dfrac{1-\mu}{1+\mu}$.

Now the centre of gravity of the whole system is vertically below A. Hence

$$W'.a\sin\theta + W.x\sin\theta = W'.a\cos\theta + W(2l - x)\cos\theta.$$

$$\therefore \frac{1-\mu}{1+\mu} = \frac{W'.a + W(2l-x)}{W'.a + W.x}.$$

$$\therefore x = l + \mu\left(l + \frac{W'}{W}.a\right).$$

Changing the sign of μ, we get the other extreme value, and this is supposed positive, implying the given limitation on the value of μ.

12. If the shell bursts at S, the enveloping paraboloid has focus S, and latus-rectum $4h'$, where $U^2 = 2gh'$.

If it meets the ground in Q, and AS meets the ground in N, then

$$QN^2 = 4AS.AN = 4h'(h + h')$$

$$= \frac{2U^2}{g}\left(h + \frac{U^2}{2g}\right) = \frac{U^2}{g^2}(2gh + U^2),$$

and all the fragments must lie within a circle, centre N and radius QN.

XX.

1. Let O be the orthocentre, and let CO meet the circumcircle in E, and EP, EQ meet AB in K, L and the pedal lines of P and Q respectively in U, V. Then OK, OL are parallel to the pedal lines and U, V are the middle points of PK, QL; therefore UV is parallel to KL. Hence, evidently, from similar triangles, the pedal lines must intersect on EO produced.

2. The remaining tangents TQ, $T'Q'$ to the ellipse must be parallel, since PQ, PQ' are parallel to the conjugate diameters, and thus QQ' must be a diameter.

Produce SP to H, making $PH = PS'$. Then the triangles $S'PT'$, HPT' are congruent, and therefore so also are HTT' and $S'TT'$,

$$\therefore\ H\hat{T}T' = S'TT' = STQ,\ \therefore\ H\hat{T}S = T'TQ.$$

Similarly $H\hat{T}'S = TT'Q'$. Hence S, T, H, T' are cyclic.

Now let the circles in question intersect again in P'.

Then $$S\hat{P}'P = 180^\circ - STP = 180^\circ - SHT',$$

and $$S'P'P = 180^\circ - S'T'P = 180^\circ - HT'P.$$

$$\therefore\ S\hat{P}'S' = 180^\circ - HPT',$$

and is therefore constant. Hence the locus of P' is a circle through the foci.

3. The given series is the absolute term in the product of the expansions of

$$(1-x)^n \text{ and } \left(1-\frac{1}{x}\right)^{-(n+1)}, \quad (x>1),$$

i.e. the absolute term in $(-1)^n \cdot \left(1-\frac{1}{x}\right)^{-1} \cdot x^n$,

i.e. the coefficient of $\frac{1}{x^n}$ in $(-1)^n\left(1-\frac{1}{x}\right)^{-1} = (-1)^n$.

4. If $x = 1 + \frac{1}{1+}\,\frac{1}{1+}\cdots$, then $x = 1 + \frac{1}{x}$,

$$\text{i.e. } x - \frac{1}{x} = 1.$$

Hence $x^2 + \frac{1}{x^2} = 3$, and again, by squaring $x^4 + \frac{1}{x^4} = 7$.

Hence $$x^4 = 7 - \frac{1}{x^4} = 7 - \frac{1}{7-}\,\frac{1}{7-}\cdots\cdots.$$

5. If Δ is the area,

$$16\Delta^2 = 2\Sigma\,(b^2 + y^2)\,(c^2 + z^2) - \Sigma\,(a^2 + x^2)^2.$$

But, by the data,

$$2\Sigma b^2c^2 - \Sigma a^4 = 0 \text{ and } 2\Sigma y^2z^2 - \Sigma x^4 = 0,$$

$$\therefore\ 16\Delta^2 = 2\Sigma\,(b^2z^2 + c^2y^2) - 2\Sigma a^2x^2$$

$$= 2\Sigma\,(b^2 + c^2 - a^2)\,x^2.$$

Also, since $\Sigma a = 0, \quad \therefore\ b^2 + c^2 - a^2 = -2bc,$

$$\therefore\ 4\Delta^2 = -\Sigma bcx^2.$$

Also $-\Sigma bcx^2 = -bcx^2 - cay^2 - ab\,(x+y)^2$

$$= -b\,(c+a)\,x^2 - a\,(b+c)\,y^2 - 2abxy$$

$$= b^2x^2 + a^2y^2 - 2abxy = (bx - ay)^2,$$

and therefore Δ is rational.

6. If x is the circular measure of an acute angle, $\sin x$ differs from x by less than $\frac{1}{6}x^3$.

Now $$\pi^3 < 10\pi < 31{\cdot}416,$$

$$\therefore\ \tfrac{1}{6}\left(\frac{\pi}{36}\right)^3 < \frac{31{\cdot}416}{6^7} < {\cdot}00012, \text{ by division.}$$

Also $\frac{\pi}{36} = {\cdot}0872\ldots.$

Hence the approximation is correct to 3 places of decimals.

7. The polar of (x_1, y_1) is $2ax - yy_1 + 2ax_1 = 0$, and the parallel tangent is $2ax - yy_1 + \frac{1}{2}y_1^2 = 0$.

Hence
$$2\Delta_1 = \frac{\begin{vmatrix} 2a, & -y_1, & 2ax_1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}^2}{\Pi \begin{vmatrix} 2a, & -y_1 \\ 2a, & -y_2 \end{vmatrix}} = \frac{16a^4(2\Delta)^2}{8a^3 \,.\, \Pi\,(y_1 - y_2)},$$

$$2\Delta_2 = \frac{\begin{vmatrix} 2a, & -y_1, & \frac{1}{2}y_1^2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}^2}{8a^3 \,.\, \Pi\,(y_1 - y_2)} = \frac{a^2 \,.\, \begin{vmatrix} 1, & y_1, & y_1^2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}^2}{8a^3 \,.\, \Pi\,(y_1 - y_2)}.$$

Now the determinant in the numerator is $\Pi\,(y_1 - y_2)$,

$$\therefore\ 2\Delta_2 = \frac{1}{8a} \,.\, \Pi\,(y_1 - y_2).$$

$$\therefore\ \Delta_1 \Delta_2 = \tfrac{1}{4}\Delta^2.$$

8. The chord through (x', y') making equal angles with the axes is

$$\frac{x - x'}{\frac{1}{\sqrt{2}}} = \frac{y - y'}{\frac{1}{\sqrt{2}}} = r,$$

and this meets the ellipse where

$$\frac{1}{a^2}\left(\frac{r}{\sqrt{2}} + x'\right)^2 + \frac{1}{b^2}\left(\frac{r}{\sqrt{2}} + y'\right)^2 = 1.$$

If (x', y') is on the curve, this gives $r = 0$ or $r = 2\sqrt{2} \,.\, \dfrac{b^2x' + a^2y'}{a^2 + b^2}$.

But if the line is the normal at $x'y'$, then $a^2y' = b^2x'$,

$$\text{i.e. } \frac{x'}{a^2} = \frac{y'}{b^2} = \frac{1}{\sqrt{a^2 + b^2}}.$$

Substituting above, this gives the required value of r.

9. The equation may be written

$$(3x+5y)^2-206x+246y+393=0 \quad \ldots\ldots\ldots\ldots\text{(i)},$$

and therefore the equation to the directrix is of the form

$$5x-3y+k=0.$$

Hence if (X, Y) is the focus, the parabola will be

$$(x-X)^2+(y-Y)^2=\frac{(5x-3y+k)^2}{34},$$

i.e. $$(3x+5y)^2-2(34X+5k)x+2(3k-34Y)y$$
$$+34(X^2+Y^2)-k^2=0 \quad \ldots\ldots\text{(ii)}.$$

Comparing (i) and (ii), we get

$$34X+5k=103, \quad 3k-34Y=123, \quad 34(X^2+Y^2)-k^2=393.$$

Substituting for X, Y in the third equation, we find $k=7$, whence from the first two, $X=2$, $Y=-3$.

10. Each of the strings makes an angle 45° with the horizontal. Hence, if X be the thrust in BC, we have, taking moments about A for AB,

$$X.l\sin 60^\circ=T.l\sin 75^\circ+w.\frac{l}{2}\cos 60^\circ+Y.l\cos 60^\circ,$$

where Y is the vertical component of the reaction at B.

Also for W, $2T\cos 45^\circ=W$, and for BC, $2Y=w$.

$$\therefore\ X.l\frac{\sqrt{3}}{2}=\frac{W}{\sqrt{2}}.l.\frac{\sqrt{3}+1}{2\sqrt{2}}+w.\frac{l}{4}+\frac{w}{2}.\frac{l}{2},$$

whence $$X=W.\frac{\sqrt{3}+1}{2\sqrt{3}}+\frac{w}{\sqrt{3}}.$$

11. Let f be the acceleration of m_1 horizontally, f_1 and f_2 those of m_2 parallel and perpendicular to the face of m_1, f_3 that of m_3 vertically downwards, R the pressure between m_1 and m_2, R' that between m_2 and m_3.

Then the equations are

$$m_3f_3=m_3g-R' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$
$$m_2f_1=(m_2g+R')\sin\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

$$m_2 f_2 = (m_2 g + R') \cos \alpha - R \quad \text{.............(iii)},$$

$$m_1 f = R \sin \alpha \quad \text{..............................(iv)},$$

$$f_2 = f \sin \alpha \quad \text{..................................(v)},$$

$$f_3 = f_1 \sin \alpha + f_2 \cos \alpha \quad \text{........ ....(vi)}.$$

From (i) and (vi),

$$m_3 (f_1 \sin \alpha + f_2 \cos \alpha) = m_3 g - R',$$

whence from (ii) and (iii), substituting for f_1 and f_2, we find

$$R' \left(\frac{1}{m_2} + \frac{1}{m_3}\right) = \frac{R \cos \alpha}{m_2} \quad \text{..................(vii)}.$$

Also from (iii), (iv) and (v),

$$\left(g + \frac{R'}{m_2}\right) \cos \alpha - \frac{R}{m_2} = \frac{R \sin^2 \alpha}{m_1} \quad \text{............(viii)}.$$

From (vii) and (viii), we find

$$R = \frac{m_1 m_2 (m_2 + m_3)}{(m_1 + m_2)(m_2 + m_3) \sin^2 \alpha + m_1 m_2 \cos^2 \alpha} \cdot g \cos \alpha.$$

Now the pressure on the table is $R \cos \alpha + m_1 g$, and therefore the quantity required is $(m_2 + m_3) g - R \cos \alpha$, which, substituting for R, reduces to the given expression.

12. Let α be the elevation, t and t' the times of going and returning. Then $V \cos \alpha \,.\, t = a$.

After impact the horizontal velocity is $eV \cos \alpha$, while the vertical velocity is unchanged. Hence, since the vertical distance described in time $t + t'$ is zero, we have

$$V \sin \alpha \, (t + t') - \tfrac{1}{2} g \, (t + t')^2 = 0,$$

whence $t + t' = \dfrac{2V \sin \alpha}{g}$. Hence, if the particle strikes the ground again at a distance x from the wall,

$$x = eV \cos \alpha \,.\, t' = eV \cos \alpha \left(\frac{2V \sin \alpha}{g} - \frac{a}{V \cos \alpha}\right)$$

$$= e \,.\, \frac{V^2 \sin 2\alpha}{g} - ae.$$

Hence, if $x > a$, we have

$$\frac{V^2 \sin 2\alpha}{g} > \frac{1+e}{e} \cdot a,$$

which is not possible unless $V^2 > \dfrac{1+e}{e} \cdot ga.$

XXI.

1. If P is a point on DEF such that $\triangle PBE = \triangle PCF$, then each must be equal to $\triangle PAD$.

This may be proved thus: Let p, q, r be the lengths of the perpendiculars from A, B, C on DEF. Then since AEC is a transversal of $\triangle FBD$, we have

$$\frac{FA}{AB} \cdot \frac{BC}{CD} \cdot \frac{DE}{EF} = -1,$$

$$\text{i.e. } \frac{p}{p+q} \cdot \frac{q-r}{r} = \frac{EF}{ED}.$$

But, by hypothesis,

$$q \cdot PE = r \cdot PF, \quad \therefore \frac{q-r}{r} = \frac{EF}{PE}.$$

$$\therefore \frac{p}{p+q} = \frac{PE}{ED}, \quad \therefore \frac{p}{q} = \frac{PE}{PD},$$

$$\text{i.e. } p \cdot PD = q \cdot PE, \quad \therefore \triangle PAD = \triangle PBE.$$

To find the point in question, let BE, CF meet in O. Along CF make $OI = CF$, and along EB make $OJ = EB$, and let K be the middle point of IJ. Then OK is the locus of points for which $\triangle POI = \triangle POJ$, i.e. for which $\triangle PCF = \triangle PBE$. Hence P is the point in which OK meets DEF.

2. Let T_1 be the pole of QR, etc. Then the triangles SRT_1, ST_1Q are similar.

$$\therefore SR : ST_1 = ST_1 : SQ, \text{ i.e. } ST_1^2 = SQ \cdot SR,$$

with two similar results. Hence, multiplying,

$$ST_1 \cdot ST_2 \cdot ST_3 = SP \cdot SQ \cdot SR.$$

3. From the last two equations

$$\frac{a}{y^2z^2(y^2-z^2)} = \ldots = \ldots .$$

But $y+z=-x$, $\quad \therefore \; \dfrac{a}{yz(y-z)} = \ldots = \ldots$

$$= \frac{b+c}{x(y-z)(x-y-z)} = \frac{b+c}{2x^2(y-z)} = \ldots .$$

Hence, if x, y, z are not all equal,

$$\frac{b+c}{a} = \frac{2x^2}{yz}, \text{ etc.}$$

Multiplying we have $\Pi(b+c) = 8abc$, provided none of the quantities $b+c$, etc. is zero.

If $b+c=0$, the system is satisfied by $x=0$, $y=1$, $z=-1$.

4. Let u_n be the number produced from one seed in the nth year. Then, from the data,

$$u_n = pu_{n-1} + pu_{n-2}.$$

Assuming $u_n = Ax^n$, we have $x^2 = px + p$, whence

$$x = \frac{p \pm q}{2}.$$

We therefore have

$$u_n = A\left(\frac{p+q}{2}\right)^n + B\left(\frac{p-q}{2}\right)^n.$$

Now $\quad u_1 = 1, \; u_2 = p,$

$$\therefore \; A(p+q) + B(p-q) = 2,$$
$$A(p+q)^2 + B(p-q)^2 = 4p,$$

whence $A = \dfrac{1}{q}$, $B = -\dfrac{1}{q}$, giving the required value for u_n.

5. We have $t_1{}^2 = (\tfrac{1}{2}R + r_1)^2 - r_1{}^2 = \tfrac{1}{4}R^2 + Rr_1$,

$$\therefore \; \Sigma \frac{t_1{}^2}{r_1} = \tfrac{1}{4}R^2 . \Sigma \frac{1}{r_1} + 3R$$
$$= \tfrac{1}{4}R^2 . \frac{1}{r} + 3R = \frac{R^2 + 12Rr}{4r},$$

whence the result, remembering that $r = 4R . \Pi \sin \dfrac{A}{2}$.

6. Putting $\tan\frac{\theta}{2} = t$, we have

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2},$$

$$\sin 2\theta = \frac{4t(1-t^2)}{(1+t^2)^2}, \quad \cos 2\theta = \frac{1-6t^2+t^4}{(1+t^2)^2},$$

whence, substituting and reducing, we find

$$t^4(a\cos 2\alpha - b\cos\beta + c) - 2t^3(2a\sin 2\alpha - b\sin\beta)$$
$$+ 2t^2(c - 3a\cos 2\alpha) + 2t(2a\sin 2\alpha + b\sin\beta)$$
$$+ a\cos 2\alpha + b\cos\beta + c = 0.$$

If the roots of this equation be t_1, t_2, etc , then

$$\tan\frac{s}{2} = \frac{\Sigma t_1 - \Sigma t_1 t_2 t_3}{1 - \Sigma t_1 t_2 + t_1 t_2 t_3 t_4}$$

$$= \frac{8a\sin 2\alpha}{(a\cos 2\alpha - b\cos\beta + c) - 2(c - 3a\cos 2\alpha) + (a\cos 2\alpha + b\cos\beta + c)}$$

$= \tan 2\alpha$.

$$\therefore \frac{s}{2} = n\pi + 2\alpha.$$

7. Since the orthogonal circles pass through the limiting points, the equation to any one may be taken as

$$x^2 + y^2 + 2Gx + 2Fy = 0,$$

with the condition $\qquad 2Gg + 2Ff - c = 0,$

or, putting $G = \lambda c$, $F = \lambda' c$,

$$2\lambda g + 2\lambda' f = 1,$$

and the equation to the circle is

$$x^2 + y^2 + 2c(\lambda x + \lambda' y) = 0,$$

or $\qquad (x^2 + y^2)(\lambda g + \lambda' f) + c(\lambda x + \lambda' y) = 0,$

whence, putting $\frac{\lambda'}{\lambda} = \mu$, the result follows.

8. Suppose the conics intersect at a point whose eccentric angle (for the first conic) is θ. The tangents are

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,$$

$$x(\alpha a\cos\theta+b\gamma\sin\theta)+y(a\gamma\cos\theta+b\beta\sin\theta)=1.$$

If these are perpendicular, then

$$b(\alpha a\cos\theta+b\gamma\sin\theta)\cos\theta+a(a\gamma\cos\theta+b\beta\sin\theta)\sin\theta=0,$$

$$\text{i.e. } ab\beta\tan^2\theta+\gamma(a^2+b^2)\tan\theta+ab\alpha=0 \quad \text{.........(i).}$$

Also, since the point θ is on the second conic,

$$\therefore\ \alpha a^2\cos^2\theta+\beta b^2\sin^2\theta+2\gamma ab\sin\theta\cos\theta=1=\cos^2\theta+\sin^2\theta,$$

whence $$(\beta b^2-1)\tan^2\theta+2\gamma ab\tan\theta+(\alpha a^2-1)=0 \quad \text{......(ii).}$$

The equations (i) and (ii) to determine $\tan\theta$ must be identical. Hence, comparing coefficients, the result follows.

9. The equation may be written in the form

$$(1-h)(x^2+y^2)+h\left(x+y+\frac{g}{h}\right)^2=0,$$

shewing that the origin is a focus.

Again, the centre is given by $x+hy+g=0$, $hx+y+g=0$, leading to $x=y=-\dfrac{g}{1+h}$, and since this is the middle point of the line joining the origin to the second focus, that focus must be

$$x=y=-\frac{2g}{1+h}.$$

10. Let R be the reaction at B, then

$$\text{for } BC,\quad 2R\cos\phi=W,$$

$$\text{,, } AB,\quad \frac{R}{\sin\theta}=\frac{W}{\sin(\phi-\theta)},$$

$$\therefore\ 2\cos\phi=\frac{\sin(\phi-\theta)}{\sin\theta},\ \text{giving } \tan\phi=3\tan\theta.$$

Also the reactions at A and B must meet in a point O vertically below G, the middle point of AB. Hence

$$\frac{AG}{GO} = \frac{\sin\theta}{\sin(\alpha-\theta)}, \qquad \frac{BG}{GO} = \frac{\sin\phi}{\sin(\phi-\alpha)}.$$

Equating these ratios, we find

$$\cot\theta + \cot\phi = 2\cot\alpha, \quad \text{whence} \quad \cot\alpha = \tfrac{2}{3}\cot\theta.$$

11. Let v be the velocity of either of the upper particles, when the rods make an angle θ with the vertical. Then the velocity of either of the lower particles is $2v\cos\theta$ vertically.

Hence the energy equation is

$$mv^2 + m(2v\cos\theta)^2 = 2mga\sin\theta + 2mg \,.\, 2a\sin\theta,$$

where m is the mass of each particle.

This gives $\qquad v^2(1+4\cos^2\theta) = 6ga\sin\theta,$

so that, when $\theta = 60°$, $2v^2 = 3\sqrt{3}\,ga$, and the required velocity is

$$2v\cos 60° = v = \left(\frac{3\sqrt{3}}{2}ga\right)^{\frac{1}{2}}.$$

12. With the usual notation, the equations for the impact are

$$nv\cos\theta + v'\cos\phi = nu\cos\beta \quad \text{.................(i)},$$

$$v\cos\theta - v'\cos\phi = -eu\cos\beta \quad \text{.................(ii)},$$

$$v\sin\theta = u\sin\beta \quad \text{.................(iii)},$$

$$v'\sin\phi = 0 \quad \text{.................(iv)}.$$

From (i) and (ii) $(n+1)v\cos\theta = (n-e)u\cos\beta$;

$\therefore$ from (iii) $\qquad \tan\theta = \dfrac{n+1}{n-e}\,.\,\tan\beta.$

By the question, $\theta = \alpha + \beta$, and we therefore have

$$(n-e)(\tan\alpha + \tan\beta) = (n+1)\tan\beta(1-\tan\alpha\tan\beta),$$

i.e. $\quad (n+1)\tan\alpha\tan^2\beta - (1+e)\tan\beta + (n-e)\tan\alpha = 0.$

This will give two values of β if the roots are real, i.e. if

$$(1+e)^2 > 4(n+1)(n-e)\tan^2\alpha,$$

i.e. $\quad (1+e)^2\operatorname{cosec}^2\alpha > (1+e)^2 + 4(n+1)(n-e) > (2n+1-e)^2,$

i.e. $\quad n > \dfrac{1+e}{2}\operatorname{cosec}\alpha - \dfrac{1-e}{2}.$

If the quadratic has equal roots, this is an equality, and if $e=1$, $\alpha=\frac{\pi}{3}$, we have $n=\frac{2}{\sqrt{3}}$.

In this case

$$\tan\beta=\tfrac{1}{2}\cdot\frac{1+e}{(n+1)\tan\alpha}=\frac{1}{2+\sqrt{3}};$$

$$\therefore\ \beta=\frac{\pi}{21}\ \text{and}\ \theta=\alpha+\beta=\frac{5\pi}{12}.$$

Hence, returning to the equations (iii) and (ii), we easily get the remaining result, noting that from (iv), $\phi=0$.

XXII.

1. Let TQ meet PP' in U, MM' in V, and the circle in Q'.

Then
$$\frac{VR}{VQ}=\frac{PU}{UQ}\ \text{and}\ \frac{VR'}{VQ}=\frac{UP'}{UQ};$$

$$\therefore\ \frac{VR\,.\,VR'}{VQ^2}=\frac{PU\,.\,UP'}{UQ^2}=\frac{QU\,.\,UQ'}{UQ^2}=\frac{UQ'}{UQ}.$$

Also, since $(TQ'UQ)$ is harmonic,

$$\therefore\ \frac{TQ}{TQ'}=\frac{QU}{UQ'}=\frac{TQ+QU}{TQ'+UQ'}=\frac{2VQ}{2VT};$$

$$\therefore\ \frac{VR\,.\,VR'}{VQ^2}=\frac{VT}{VQ},\ \text{i.e.}\ VR\,.\,VR'=VQ\,.\,VT;$$

therefore the points T, R', Q, R are concyclic.

2. Draw the other tangents from P and Q to the ellipse, meeting in T'. Then, since P is on the director circle, PT' is perpendicular to PT and therefore parallel to TQ. Hence $TPT'Q$ is a parallelogram circumscribing the ellipse, and therefore its diagonals are conjugate diameters. But one of these is PQ and the other is TT'.

3. Suppose the given expression equal to

$$\frac{A}{(1-ax)^2}+\frac{B}{1-ax}+\frac{C}{1-bx}.$$

Then, clearing of fractions, and putting $x=\frac{1}{a}$, we find $A=\frac{a}{a-b}$, and putting $x=\frac{1}{b}$, $C=\frac{b^2}{(a-b)^2}$.

Also $\quad A+B+C=1$, whence $B=-\frac{ab}{(a-b)^2}$.

The coefficient of x^n in the expansion is

$$A(n+1)a^n+Ba^n+Cb^n.$$

Substituting and reducing, this gives the expression required.

4. Since m^2+1 is to be divisible by 5, m must be of one of the forms $5p\pm 2$, and m being odd, p must be odd, say $2n+1$. Thus m must be of one of the forms $10n+7$ or $10n+3$. Hence, putting $n=3k+k'$ ($k'=0, 1, 2$), all possible forms are included in $30k+10k'+7$ and $30k+10k'+3$. In the first form we must exclude the case $k'=2$, and in the second $k'=0$, each of which makes m a multiple of 3. The remaining forms are

$$30k+7, \quad 30k+17, \quad 30k+13, \quad 30k+23,$$

$$\text{i.e. } 30k\pm 7, \quad 30k\pm 13.$$

In all these cases m^2+1 is even, and is therefore divisible by 10.

Also, since $m^2-1=(m-1)(m+1)$, and these factors are consecutive even integers, and one of them is divisible by 3, since m is not a multiple of 3, it follows that in all cases m^2-1 is a multiple of 24.

To prove the first part independently of the second, we note that, by Fermat's Theorem, $m^4-1=M(5)$.

Also $m^4-1=(m-1)(m+1)(m^2+1)$, and these factors are all even integers, two of them being consecutive. Also, as before, either $m+1$ or $m-1$ is $M(3)$. Hence m^4-1 is divisible by $5\times 16\times 3=240$.

5. Let PL, PM, PN be the perpendiculars from P on the sides of the equilateral triangle ABC, their lengths being α, β, γ. Then $\dfrac{MN}{\sin M\hat{P}N} = PA$, each being the diameter of the circle $AMPN$.

Hence $$\rho_1^2 = \frac{\beta^2+\gamma^2+2\beta\gamma\cos\dfrac{\pi}{3}}{\sin^2\dfrac{\pi}{3}},$$

i.e. $\beta^2+\gamma^2+\beta\gamma = \frac{3}{4}\rho_1^2$, and two similar equations ...(i).

Also $$(\alpha+\beta+\gamma)\,a = 2\,.\,\triangle ABC = \frac{\sqrt{3}}{2}a^2;$$

$$\therefore\ \alpha+\beta+\gamma = \frac{\sqrt{3}}{2}a \text{..............(ii).}$$

Hence, subtracting two of the set (i), we get

$$\beta-\gamma = -\frac{\sqrt{3}}{2}\,.\,\frac{\rho_2^2-\rho_3^2}{a} \text{..................(iii).}$$

Also $$(\Sigma\alpha)^2 \equiv \Sigma(\beta^2+\gamma^2+\beta\gamma) - \tfrac{1}{2}\Sigma(\beta-\gamma)^2.$$

Substituting from (i), (ii), (iii) and reducing, we obtain the form given.

6. Retaining squares of small quantities, the given relation is

$$(1-\tfrac{1}{2}\theta^2)\cos\alpha - \theta\,.\,\sin\alpha = (1-\tfrac{1}{2}\phi^2)\cos\alpha - \phi\,.\,\cos\beta\sin\alpha \text{ ...(i).}$$

Now suppose $\theta = A\phi + B\phi^2$. Then (i) becomes

$$\cos\alpha - A\,.\,\sin\alpha\,.\,\phi - (\tfrac{1}{2}A^2\cos\alpha + B\sin\alpha)\,\phi^2$$
$$= \cos\alpha - \cos\beta\sin\alpha\,.\,\phi - \tfrac{1}{2}\cos\alpha\,.\,\phi^2.$$

This must be an identity. Hence

$$A\sin\alpha = \cos\beta\sin\alpha;\quad \therefore\ A = \cos\beta.$$

$$\tfrac{1}{2}A^2\cos\alpha + B\sin\alpha = \tfrac{1}{2}\cos\alpha;\quad \therefore\ B = \tfrac{1}{2}\cot\alpha\sin^2\beta.$$

7. If $x\cos\alpha + y\sin\alpha - p = 0$ and $y = mx$ are inclined at an angle $\frac{\pi}{3}$, we have

$$\frac{\cos\alpha + m\sin\alpha}{m\cos\alpha - \sin\alpha} = \pm\sqrt{3},$$

whence, putting $\frac{y}{x}$ for m, the equation to the other two sides must be

$$(x\cos\alpha + y\sin\alpha)^2 = 3\,(y\cos\alpha - x\sin\alpha)^2,$$

i.e. $x^2(\cos^2\alpha - 3\sin^2\alpha) + 4xy\sin 2\alpha + y^2(\sin^2\alpha - 3\cos^2\alpha) = 0,$

or $\qquad x^2(1 - 2\cos 2\alpha) - 4xy\sin 2\alpha + y^2(1 + 2\cos 2\alpha) = 0.$

8. Let λ_1 and λ_2 be the values of λ for which the circles become point-circles (viz. the limiting points). Then the coordinates of the limiting points are

$$\left(-\frac{g}{1+\lambda_1},\ -\frac{\lambda_1 f}{1+\lambda_1}\right),\quad \left(-\frac{g}{1+\lambda_2},\ -\frac{\lambda_2 f}{1+\lambda_2}\right),$$

and if the lines joining these to the origin are perpendicular, we have

$$g^2 + \lambda_1\lambda_2 f^2 = 0.$$

But λ_1, λ_2 are the roots of

$$g^2 + \lambda^2 f^2 - (c + \lambda c')(1 + \lambda) = 0\,;$$

$$\therefore\ \lambda_1\lambda_2 = \frac{g^2 - c}{f^2 - c'}.$$

Hence the required condition is

$$g^2(f^2 - c') + f^2(g^2 - c) = 0,$$

or

$$\frac{c}{g^2} + \frac{c'}{f^2} = 2.$$

9. The eccentric angles of the feet of the normals satisfy the equation

$$a\xi\sin\theta - b\eta\cos\theta = (a^2 - b^2)\sin\theta\cos\theta,$$

or $(a^2 - b^2)^2\cos^4\theta - 2a\xi\,(a^2 - b^2)\cos^3\theta$

$$+ \left[a^2\xi^2 + b^2\eta^2 - (a^2 - b^2)^2\right]\cos^2\theta + \ldots = 0,$$

whence, putting $a\cos\theta = x$, the abscissae of the four feet are the roots of

$$(a^2-b^2)^2 x^4 - 2a^2\xi(a^2-b^2)x^3 + a^2\left[a^2\xi^2 + b^2\eta^2 - (a^2-b^2)^2\right]x^2 + \ldots = 0.$$

Hence $$\Sigma x_1 = \frac{2a^2\xi}{a^2-b^2},$$

$$\Sigma x_1 x_2 = \frac{a^2(a^2\xi^2+b^2\eta^2)}{(a^2-b^2)^2} - a^2;$$

$$\therefore\ \Sigma x_1^2 = \frac{2a^2(a^2\xi^2-b^2\eta^2)}{(a^2-b^2)^2} + 2a^2.$$

Hence $$\Sigma(\xi - x_1)^2 = 4\xi^2 - \frac{4a^2\xi^2}{a^2-b^2} + \frac{2a^2(a^2\xi^2-b^2\eta^2)}{(a^2-b^2)^2} + 2a^2$$

$$= -\frac{4b^2\xi^2}{a^2-b^2} + \frac{2a^2(a^2\xi^2-b^2\eta^2)}{(a^2-b^2)^2} + 2a^2.$$

Similarly $$\Sigma(\eta - y_1)^2 = -\frac{4a^2\eta^2}{b^2-a^2} + \frac{2b^2(b^2\eta^2-a^2\xi^2)}{(b^2-a^2)^2} + 2b^2.$$

Adding these, we obtain the result given.

10. Let AB, ED meet the line of the reaction at C in M, N. Then M and N must be vertically below the centres of the rods. Hence, if R be the reaction,

$$\frac{R}{\sin\alpha} = \frac{w_1}{\sin(\theta-\alpha)},\quad \frac{R}{\sin\beta} = \frac{w_2}{\sin(\beta+\theta)},$$

$$\therefore\ w_1\frac{\sin(\beta+\theta)}{\sin\beta} = w_2\frac{\sin(\theta-\alpha)}{\sin\alpha},$$

i.e. $$w_1(\cos\theta + \cot\beta\sin\theta) = w_2(\sin\theta\cot\alpha - \cos\theta),$$

or $$(w_1+w_2)\cot\theta = w_2\cot\alpha - w_1\cot\beta.$$

11. Suppose that the particles coalesce after time t. The sum of the vertical distances described by them in this time is $u\sin\alpha\,.\,t$, and this must be equal to the distance of the point of projection from the directrix,

$$\therefore\ u\sin\alpha\,.\,t = \frac{u^2}{2g},\quad \text{i.e. } t = \frac{u}{2g\sin\alpha}.$$

If, after time t, the direction of motion makes an angle θ with the downward vertical, we have

$$-\cot\theta = \frac{u\sin\alpha - gt}{u\cos\alpha} = \tan\alpha - \operatorname{cosec} 2\alpha = -\cot 2\alpha,$$

$$\therefore \frac{\theta}{2} = \alpha.$$

But since the particles are equal, the new direction of motion makes an angle $\frac{\theta}{2}$ with the vertical, and is therefore at right angles to the original direction of projection.

Again, the height of the new directrix above the point of union is $\dfrac{\left(v\cos\frac{\theta}{2}\right)^2}{2g}$, where $v = gt = \dfrac{u}{2\sin\alpha}$, so that this height is $\frac{1}{8}\dfrac{u^2}{g}\cot^2\alpha$.

Also the height of the point of union above the point of projection is

$$u\sin\alpha \,.\, t - \tfrac{1}{2}gt^2 = \frac{u^2}{2g} - \frac{1}{8}\frac{u^2}{g}\operatorname{cosec}^2\alpha.$$

Hence the required height is

$$\frac{u^2}{2g} - \frac{1}{8}\frac{u^2}{g} = \frac{3}{4}\left(\frac{u^2}{2g}\right).$$

12. Let v_1 be the velocity of m along the line of centres, v_2 that of m' horizontally.

The impulse of the blow on m is $m(u\cos\theta - v_1)$ along the line of centres. Hence, resolving horizontally,

$$m(u\cos\theta - v_1)\sin\theta = m'v_2 \quad\text{.................(i).}$$

Also, by Newton's Law,

$$v_1 - v_2\sin\theta = -eu\cos\theta \quad\text{.................(ii).}$$

Solving (i) and (ii), we obtain the required value for v_2.

XXIII.

1. Let O be the given point, A the nearest corner, and let the line required cut the sides through A in P and Q. Then the difference of the parts is

$$(\text{area of parallelogram}) - 2\triangle APQ.$$

Since this is to be a maximum, $\triangle APQ$ must be a minimum, and therefore PQ must be bisected at O. Hence the construction: Join AO and produce to B, making $OB = AO$. Through B draw lines parallel to the sides, meeting the sides through A in P and Q. Then PQ is the line required.

2. Let TP, TQ be the tangents. Then T, P, N, Q are concyclic.

Therefore $T\hat{N}P = T\hat{Q}P = 90° - S\hat{T}Q$, since TQ subtends a right angle at S. Therefore $S\hat{T}Q = N\hat{T}P$. But $S\hat{T}Q = S'\hat{T}P$. Therefore TN must pass through S'.

3. First suppose m, n positive integers. Take m quantities each equal to a, and n each equal to b. Then, since the A.M. of these is $>$ their G.M., we have

$$\frac{ma+nb}{m+n} > (a^m b^n)^{\frac{1}{m+n}},$$

which is equivalent to the given result.

If m and n are not integers, choose k so that km, kn are integers. Then, as above,

$$\frac{kma+knb}{km+kn} > (a^{km}b^{kn})^{\frac{1}{km+kn}},$$

from which k disappears.

4. We have immediately

$$x = \frac{a}{a'+y}, \quad y = \frac{b}{b'+x},$$

$$\text{i.e. } a'x + xy = a,$$

$$b'y + xy = b.$$

Subtracting, the result follows.

5. If $\tan\theta = t$, the equation in question is

$$(at - c)^2 = b^2(1 + t^2),$$

or $$(a^2 - b^2)t^2 - 2act + c^2 - b^2 = 0.$$

The roots of this are $\tan\alpha$, $\tan\beta$. Hence

$$\tan\alpha + \tan\beta = \frac{2ac}{a^2 - b^2}, \quad \tan\alpha\tan\beta = \frac{c^2 - b^2}{a^2 - b^2}.$$

$$\therefore \quad \tan(\alpha + \beta) = \frac{\dfrac{2ac}{a^2 - b^2}}{1 - \dfrac{c^2 - b^2}{a^2 - b^2}} = \frac{2ac}{a^2 - c^2}.$$

Further $$(\tan\alpha - \tan\beta)^2 = \frac{4a^2c^2 - 4(c^2 - b^2)(a^2 - b^2)}{(a^2 - b^2)^2}$$

$$= \frac{4b^2(a^2 + c^2 - b^2)}{(a^2 - b^2)^2},$$

whence the second result follows.

6. Put $\beta = \alpha + x$, where x is small. Then, neglecting x^2 and higher powers, the expression is

$$\frac{\alpha\sin(\alpha + x) - (\alpha + x)\sin\alpha}{\alpha\cos(\alpha + x) - (\alpha + x)\cos\alpha} = \frac{\alpha(\sin\alpha + x\cos\alpha) - (\alpha + x)\sin\alpha}{\alpha(\cos\alpha - x\sin\alpha) - (\alpha + x)\cos\alpha}$$

$$= \frac{\sin\alpha - \alpha\cos\alpha}{\alpha\sin\alpha + \cos\alpha} = \frac{\tan\alpha - \alpha}{1 + \alpha\tan\alpha}$$

$$= \tan(\alpha - \tan^{-1}\alpha).$$

7. Let U_r be the result of putting $y = mx$ in u_r, and dividing by x^r. Then we shall have

$$u_0 + U_1 x + U_2 x^2 = 0, \qquad v_0 + V_1 x + V_2 x^2 = 0,$$

and to determine the appropriate values of m we have to eliminate x from these equations. The result is

$$(u_0 V_1 - U_1 v_0)(U_1 V_2 - U_2 V_1) = (u_0 V_2 - U_2 v_0)^2.$$

Now, putting $m = \dfrac{y}{x}$, and clearing of fractions, we get the equation to the lines in the form given.

If the conics are $u_1 + u_2 = 0$, $v_1 + v_2 = 0$, then the equation to the three chords from the origin to the other points of intersection is $u_1v_2 - u_2v_1 = 0$.

In the case of the conics in question, this is

$$f'y\,(ax^2 + by^2) = gx\,(a'x^2 + b'y^2),$$

or
$$ga'x^3 - f'a\,x^2y + gb'xy^2 - f'by^3 = 0.$$

If the conics have contact of the second order, these three lines coincide. Hence the expression on the left is a perfect cube. The conditions for this may be given in a variety of forms, the simplest being

$$\frac{a}{b} = \frac{9a'}{b'}, \quad \frac{f'^2}{g^2} = \frac{3a'b'}{a^2}.$$

8. The rectangular hyperbola through the feet of the normals is

$$(a^2 - b^2)\,xy + b^2kx - a^2hy = 0,$$

and therefore any conic through these four points is of the form

$$b^2x^2 + a^2y^2 - a^2b^2 + \lambda\left[(a^2 - b^2)\,xy + b^2kx - a^2hy\right] = 0.$$

If this is a parabola,

$$a^2b^2 = \tfrac{1}{4}\lambda^2\,(a^2 - b^2)^2, \quad \text{i.e. } \lambda = \pm\frac{2ab}{a^2 - b^2}.$$

Hence there are two such parabolas, their equations being

$$(a^2 - b^2)\,(bx \pm ay)^2 \pm 2ab\,(b^2kx - a^2hy) - a^2b^2\,(a^2 - b^2) = 0.$$

Also the latus-rectum of $(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$ is

$$2\,(f\alpha - g\beta)/(\alpha^2 + \beta^2)^{\frac{3}{2}},$$

giving the required values in these cases.

9. Let the osculating circle be

$$x^2 + y^2 + \lambda x + \mu y + \nu = 0.$$

To find where this meets the hyperbola, put $x = \dfrac{k}{t}$, $y = kt$.

We then have

$$k^2t^4 + \mu kt^3 + \nu t^2 + \lambda kt + k^2 = 0.$$

Now if P is m, and Q is m', this equation must have three roots equal to m, and the remaining root equal to m', i.e. it must be identical with

$$(t-m)^3(t-m')=0.$$

Hence, noticing that the product of the roots is unity, we must have $m^3m'=1$.

But the equation to PQ is

$$mm'x+y=k(m+m'),$$

and substituting for m', this takes the form

$$mx+m^3y=k(1+m^4).$$

Comparing with $xy'+x'y=2k^2$, which is the polar of (x', y'), we have

$$\frac{y'}{m}=\frac{x'}{m^3}=\frac{2k}{1+m^4},$$

and, eliminating m, we obtain the required locus for (x', y').

10. Let A, B be the centres of the spheres, G their combined C. of G., O the point of support, and suppose that the strings make angles α and β with the vertical (i.e. with OG). Then, if T and T' are the tensions,

$$\frac{T}{\sin\beta}=\frac{T'}{\sin\alpha}=\frac{P+Q}{\sin\omega}.$$

Also
$$\frac{OG}{AG}=\frac{\sin A}{\sin\alpha},\quad \frac{OG}{BG}=\frac{\sin B}{\sin\beta}.$$

$$\therefore\quad \frac{BG}{AG}=\frac{\sin\beta}{\sin\alpha}\cdot\frac{b}{a},\qquad \text{i.e.}\quad \frac{\sin\beta}{\sin\alpha}=\frac{Pa}{Qb}.$$

Also, since $\omega=\alpha+\beta$, we have

$$\sin^2\omega=\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta\cos\omega.$$

Hence
$$\frac{T}{Pa}=\frac{T'}{Qb}=\frac{P+Q}{(Q^2b^2+P^2a^2+2PQab\cos\omega)^{\frac{1}{2}}}.$$

11. Let u_r be the velocity of the system just before the rth bead leaves the table, v_r that just after. Then, since the total

momentum is unaltered by the jerk which sets the rth bead in motion,

$$\therefore \quad (r-1)\, u_r = r v_r.$$

Also $u_r^2 = v_{r-1}^2 - 2ga$, whence

$$(r-1)^2 u_r^2 - (r-2)^2 u_{r-1}^2 = -2ga\,(r-1)^2.$$

Taking this equation for all values of r from n to 3, and adding, we obtain

$$(n-1)^2 u_n^2 - u_2^2 = -2ga \,.\, \sum_{r=3}^{r=n} (r-1)^2$$

$$= -2ga \,.\, \left[\frac{n\,(n-1)\,(2n-1)}{6} - 1\right].$$

Now $u_2 = V$, and by the condition of the question $u_n = 0$. Making these substitutions, the result follows.

12. Suppose the particle strikes the plane after time t. Then, by the condition of the question, we must have

$$u \cos(\alpha - \beta) - g \sin\beta \,.\, t = -\left[u \sin(\alpha-\beta) - g\cos\beta \,.\, t\right],$$

α being the angle of projection.

Hence

$$u\left[\cos(\alpha-\beta) + \sin(\alpha-\beta)\right] = gt\,(\cos\beta + \sin\beta).$$

But $\quad t = \dfrac{2u \sin(\alpha-\beta)}{g\cos\beta}, \qquad \therefore \quad \cot(\alpha-\beta) = 1 + 2\tan\beta,$

i.e. $\quad \dfrac{\cos(\alpha-\beta)}{1+2\tan\beta} = \dfrac{\sin(\alpha-\beta)}{1} = \dfrac{1}{(2+4\tan\beta+4\tan^2\beta)^{\frac{1}{2}}} = \dfrac{1}{k^{\frac{1}{2}}}$, say.

Hence

$$\cos\alpha = \cos(\overline{\alpha-\beta}+\beta) = \frac{(1+2\tan\beta)\cos\beta - \sin\beta}{k^{\frac{1}{2}}} = \frac{\cos\beta+\sin\beta}{k^{\frac{1}{2}}},$$

and $$t = \frac{2u}{g} \,.\, \frac{1}{k^{\frac{1}{2}}\cos\beta}.$$

Hence the horizontal distance described in time t is

$$u\cos\alpha \,.\, t = \frac{2u^2}{g} \,.\, \frac{1+\tan\beta}{k}$$

and the vertical distance is $\tan\beta$ times this expression, as given.

XXIV.

1. Let A be the fixed vertex, and draw through A the diameter ADD' of the fixed circle on which B lies. On AD describe a triangle ADE similar to ABC, and through D' draw $D'E'$ parallel to DE, to meet AE produced in E'.

Then $AD : AE = AB : AC$, i.e. $AD : AB = AE : AC$, and $D\hat{A}B = E\hat{A}C$, therefore the triangles DAB, EAC are similar. So also are the triangles ABD', ACE'. Hence C lies on the circle on EE' as diameter, and the intersections of this circle with the second given circle are the possible positions of C.

2. Let the tangent at P meet the asymptote in L, and draw SK parallel to the asymptote. Then since LP and the asymptote are the tangents from L,

$$\therefore \quad L\hat{S}P = LSK = SLQ,$$

$$\therefore \quad SQ = QL \text{ and similarly } S'R = RL,$$

$$\therefore \quad S'R - SQ = QR,$$

$$\therefore \quad PR + PQ + QR = S'P - SP = AA'.$$

3. The number of permutations of the N figures among themselves is

$$\frac{N!}{m!\,n!\,p!\ldots}.$$

In $\dfrac{(N-1)!}{(m-1)!\,n!\,p!\ldots}$ of these, the figure a will occupy the rth place.

In $\dfrac{(N-1)!}{m!\,(n-1)!\,p!\ldots}$ of these, the figure b will occupy the rth place, and so on.

Hence the sum represented by the figures in the rth place in all possible decimals is

$$\left[\frac{(N-1)!}{(m-1)!\,n!\,p!\ldots}\,.\,a + \frac{(N-1)!}{m!\,(n-1)!\,p!\ldots}\,.\,b + \ldots\right].\,\frac{1}{10^r}$$

$$= \frac{(N-1)!}{m!\,n!\,p!\ldots}\,(ma + nb + \ldots)\,.\,\frac{1}{10^r}.$$

Hence, since $\sum\limits_{r=1}^{r=\infty}.\,\dfrac{1}{10^r} = \dfrac{1}{9}$, the result follows.

4. By Fermat's Theorem $a^{b-1}-1=M(b)$, and $b^{a-1}-1=M(a)$, whence the result follows immediately.

5. With the usual notation

$$\cos\theta=(OP^2+PI^2-OI^2)/2OP\,.\,PI.$$

Now denoting the radius of the in-circle by ρ, we have

$$OP^2=R^2(1+8r),\quad PI^2=2\rho^2+4R^2r,\quad OI^2=R^2-2R\rho.$$

Also

$$\rho=4R\,.\,\Pi\sin\frac{A}{2}\,;\ \therefore\ \rho^2=2R^2\,.\,\Pi(1-\cos A)=2R^2(1+p+q+r),$$

whence $$PI^2=4R^2(1+p+q+2r).$$

Further,

$$\rho=R(\Sigma\cos A-1)=-R(p+1).$$

Making these substitutions, we obtain the result given.

6. If in the formula

$$\sin n\theta=2^{n-1}\sin\theta\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left(\theta+\frac{\overline{n-1}\pi}{n}\right)$$

we change θ to $\theta+\dfrac{\pi}{2n}$, and then write $2n$ for n, we obtain

$$\cos 2n\theta=2^{2n-1}\sin\left(\theta+\frac{\pi}{4n}\right)\sin\left(\theta+\frac{3\pi}{4n}\right)\ldots\sin\left(\theta+\frac{\overline{4n-1}\pi}{4n}\right).$$

If now we put $\theta=0$, this becomes

$$\sin\frac{\pi}{4n}\sin\frac{3\pi}{4n}\ldots\sin\frac{(4n-1)\pi}{4n}=2^{-2n+1}.$$

There are $2n$ factors on the left, and they are equal in pairs. Hence taking the square root we have the result stated, the positive sign being taken, since the angles involved in the final formula are all acute.

7. (i) $\cot^{-1}(2n^2)\equiv\cot^{-1}(2n-1)-\cot^{-1}(2n+1)$,

$$\therefore\ S_n=\frac{\pi}{4}-\cot^{-1}(2n+1)=\cot^{-1}\left(\frac{n+1}{n}\right).$$

(ii) Denoting the given series by S, and the series

$$1 - 2\cos\alpha \,.\, \cos\alpha + \frac{4\cos^2\alpha \,.\, \cos 2\alpha}{2\,!} - \ldots$$

by C, we have

$$C - iS = 1 - 2\cos\alpha \,.\, e^{i\alpha} + \frac{4\cos^2\alpha \,.\, e^{2i\alpha}}{2\,!} - \ldots$$

$$= e^{-2\cos\alpha \,.\, e^{i\alpha}} = e^{-2\cos\alpha\,(\cos\alpha + i\sin\alpha)}$$

$$= e^{-2\cos^2\alpha}\left[\cos(\sin 2\alpha) - i\sin(\sin 2\alpha)\right];$$

$$\therefore\; S = e^{-2\cos^2\alpha} \,.\, \sin(\sin 2\alpha).$$

8. The area of the triangle formed by the polars of the given points

$$\Delta' = \frac{1}{2} \cdot \frac{\begin{vmatrix} \dfrac{x_1}{a^2}, & \dfrac{y_1}{b^2}, & 1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}^2}{\begin{vmatrix} \dfrac{x_1}{a^2}, & \dfrac{y_1}{b^2} \\ \dfrac{x_2}{a^2}, & \dfrac{y_2}{b^2} \end{vmatrix} \begin{vmatrix} \cdots \\ \cdots \end{vmatrix} \begin{vmatrix} \cdots \\ \cdots \end{vmatrix}}$$

$$= \frac{1}{2} \cdot \frac{a^2b^2(2\Delta)^2}{\Pi \begin{vmatrix} x_1, & y_1 \\ x_2, & y_2 \end{vmatrix}}$$

where Δ is the area of the triangle formed by the three points themselves. But, if the triangle is self-conjugate, $\Delta' = \Delta$, whence the result.

9. The equation to the director circle is

$$(ab - h^2)(x^2 + y^2) + c(a + b) = 0,$$

and the pole of $lx + my + 1 = 0$ is $\dfrac{bcl - chm}{ab - h^2}, \; \dfrac{-chl + cam}{ab - h^2}$. If this lies on the director circle, we obtain, by substitution and reduction,

$$(b^2+h^2)\,l^2-2h\,(a+b)\,lm+(a^2+h^2)\,m^2+\lambda=0,$$

where $\lambda=(a+b)\,(ab-h^2)/c$.

This is the tangential equation to the envelope, and the corresponding Cartesian equation is that given.

10. Let the line of the reaction at B meet AC in O. Then O is vertically above the centre of BC. Also A is vertically over the C. of G. of the two rods, which is the middle point of the line joining their centres. Hence if a be the length of either rod, θ and ϕ the inclinations of AB and BO to the vertical,

$$\tan\theta=\tfrac{1}{4}a/\tfrac{3}{4}a=\tfrac{1}{3}.$$

Also
$$\frac{\sin(45^\circ-\theta)}{\sin 45^\circ}=\frac{\sin\phi}{\cos(\phi-\theta)},$$

i.e. $(\cos\theta-\sin\theta)(\cot\phi\cos\theta+\sin\theta)=1$, whence $\cot\phi=\dfrac{4}{3}$.

Now, for equilibrium, $\dfrac{T}{\sin\phi}=\dfrac{R}{\sin(45^\circ-\theta)}=\dfrac{W}{\sin(45^\circ-\theta+\phi)}$,

whence substituting the known values of the trigonometrical ratios, we find

$$R=\frac{W}{2},\quad T=\frac{3\sqrt{5}}{10}W.$$

11. The surface-areas of the pot and the two parts of the lid are

$$2\pi rh+\pi r^2,\quad 2\pi r^2,\quad \pi r^2.$$

Hence their weights are in the ratio $2h+r:2nr:r$.

Suppose that in any position the lid makes an angle α with the horizontal. The pot will stand upright in this position if the vertical through the C. of G. of the whole falls within the base. This will be the case if

$$2nr\,(r\cos\alpha+\tfrac{1}{2}r\sin\alpha)+r\,.\,r\cos\alpha<(2h+r)\,r,$$

$$\text{i.e. } (2n+1)\cos\alpha+n\sin\alpha<\frac{2h}{r}+1.$$

Now the maximum value of the expression on the left is

$$\sqrt{(2n+1)^2+n^2}=\sqrt{5n^2+4n+1}.$$

Hence the condition required is

$$\sqrt{5n^2+4n+1} < \frac{2h}{r} + 1,$$

or
$$\frac{h}{r} > \frac{\sqrt{5n^2+4n+1}-1}{2}.$$

12. Let P be the point of projection, E and F the points of impact. Let EF make angles α and β with CA and AB, and FP an angle γ with AB. Draw PL and EM perpendicular to AB. Then

$$\tan\alpha = e\tan 60^\circ, \quad \tan\gamma = e\tan\beta,$$

and $\beta = 120^\circ - \alpha$, whence

$$\tan\gamma = \frac{\sqrt{3}e(1+e)}{3e-1}.$$

Also $$\frac{PL}{FL} = \tan\gamma,\ \frac{EM}{FM} = \tan\beta;\quad \therefore FM = e\,.\,FL.$$

$$\therefore PE = LM = (1+e)\,FL = (1+e)\,PF\cos\gamma.$$

Also $$\frac{BP}{PF} = \frac{\sin\gamma}{\sin 60^\circ} = \frac{2}{\sqrt{3}}\sin\gamma;$$

$$\therefore \frac{BP}{PE} = \frac{2\tan\gamma}{\sqrt{3}(1+e)} = \frac{2e}{3e-1},$$

and $PE = PC$, whence the result.

XXV.

1. The circles AQR, BRP, CPQ will intersect in a fixed point O. For

$$B\hat{O}C - B\hat{A}C = OBA + OCA = OPR + OPQ.$$

Hence $B\hat{O}C = \hat{A} + \hat{P}$ and is $\therefore$ given. Similarly $C\hat{O}A = \hat{B} + \hat{Q}$. Now let S be the circumcentre of PQR. Then $O\hat{P}R = O\hat{B}R$, and $O\hat{R}P = O\hat{B}P$, $\therefore \triangle OPR$ is of given species. Hence $OP:PR$ is constant. Also, by hypothesis, $SP:PR$ is constant. $\therefore OP:PS$ is constant. Further $O\hat{P}S = SPR - OPR$, and is $\therefore$ given. Hence

$\triangle OPS$ is of given species, and one vertex O is fixed, while a second vertex P moves along a given straight line, viz. BC. Hence the third vertex S also describes a straight line.

2. Let the plane of the paper be that containing V, the vertex of the cone, and the axis of the section. Let the focal sphere touch the generators in this plane in L, L' and let SA meet LL' produced in X, A being the vertex of the section. Then SX is half the latus-rectum, and is $\therefore$ given. Also SL' is a diameter of the focal sphere, and is equal to the perpendicular from V on the section. If O is the centre of the focal sphere, OA is parallel to $L'X$, and the triangles XSL', AOV are similar, $\therefore SX : SL' = AO : OV$. Now draw a parallel to the axis of the cone at a distance SX from it, cutting VA in H, and draw HK perpendicular to the axis. Then $VK = SL'$. Hence the construction :—With centre V and radius VK describe a sphere, and draw the tangent plane to this sphere parallel to any generator. This tangent plane will give a parabolic section with the given latus-rectum.

3. The sum of the products required is the coefficient of x^n in

$$(1 + a_1 x + \dots + a_1^{m-1} x^{m-1})(1 + a_2 x + \dots + a_2^{m-1} x^{m-1})\dots,$$

i.e. in

$$\frac{1 - a_1^m x^m}{1 - a_1 x} \cdot \frac{1 - a_2^m x^m}{1 - a_2 x} \dots .$$

But

$$\frac{1}{(1 - a_1 x)(1 - a_2 x)\dots} = 1 + \Sigma H_n x^n .$$

Hence the above expression is

$$(1 - \Sigma a_1^m x^m + \Sigma a_1^m a_2^m x^{2m} - \dots)(1 + \Sigma H_n x^n),$$

and the coefficient of x^n in this is the expression given.

4. If the denominators of the convergents are q_1, q_2, etc., then

$$w_{n+1} - w_n = \frac{(-1)^n}{q_{n+1} q_n} .$$

Hence
$$w_{n+1} - w_{n-1} = \frac{(-1)^n}{q_{n+1} q_n} + \frac{(-1)^{n-1}}{q_n q_{n-1}} = \frac{(-1)^n (q_{n-1} - q_{n+1})}{q_{n-1} q_n q_{n+1}} .$$

Thus the fraction in question is

$$-\frac{q_n\, q_{n-1}}{(q_{n-1}-q_{n+1})\,(q_{n-2}-q_n)}.$$

But $\quad q_{n+1}-q_{n-1}=p_{n+1}\,q_n$ and $\therefore$ this fraction is $-\dfrac{1}{p_{n+1}\,p_n}$.

5. Let the diagonals intersect in O, and let $AC=78$, $BD=50$.

Then $\quad \dfrac{78}{\sin B}=130, \quad \therefore \sin B=\dfrac{3}{5}, \quad \therefore A\hat{B}C=A\hat{O}B,$

$$\therefore A\hat{C}B=A\hat{B}O=A\hat{C}D, \quad \therefore AB=AD.$$

Further

$$\frac{50}{\sin C}=130, \therefore \sin C=\frac{5}{13}, \therefore \cos C=\frac{12}{13}, \text{ whence } \sin\frac{C}{2}=\frac{1}{\sqrt{26}}.$$

But $\quad \dfrac{AB}{\sin\dfrac{C}{2}}=130, \quad \therefore AB=5\sqrt{26}.$

Again $\sin BAC=\sin\left(B+\dfrac{C}{2}\right)=\dfrac{3}{5}\cdot\dfrac{5}{\sqrt{26}}+\dfrac{4}{5}\cdot\dfrac{1}{\sqrt{26}}=\dfrac{19}{5\sqrt{26}}.$

But $\quad \dfrac{BC}{\sin BAC}=130, \quad \therefore BC=19\sqrt{26}.$

Similarly $\quad \sin DAC=\dfrac{11}{5\sqrt{26}}$ and $CD=11\sqrt{26}.$

6. Supposing the angles in circular measure, let S denote the given series, and C the corresponding series in cosines. Then

$$C+iS=\frac{b}{c}\,.\,e^{iA}+\tfrac{1}{2}\,.\,\frac{b^2}{c^2}e^{2iA}+\ldots$$

$$=-\log\left(1-\frac{b}{c}e^{iA}\right).$$

Now

$$c-be^{iA}=c-b(\cos A+i\sin A)=a\cos B-i\,.\,a\sin B=ae^{-iB};$$

$$\therefore C+iS=-\left(\log\frac{a}{c}-iB\right),$$

$$\therefore S=B.$$

7. Any tangent to one parabola is $x - my + am^2 = 0$, and to the other $y - m'x + bm'^2 = 0$.

If these are perpendicular $m' = -m$, so that the equations are

$$am^2 - my + x = 0, \quad bm^2 + mx + y = 0.$$

Eliminating m, we obtain the locus given for the intersection.

8. Suppose the circle through α, β, γ cuts the ellipse again in δ', and that the normals at α, β, γ meet in δ. Then $\alpha + \beta + \gamma + \delta'$ is an even multiple of π, and $\alpha + \beta + \gamma + \delta$ an odd multiple. Hence $\delta' - \delta$ is an odd multiple.

The centre of the circle is

$$X = \frac{a^2 - b^2}{4a}(\cos\alpha + \cos\beta + \cos\gamma + \cos\delta')$$

$$= \frac{a^2 - b^2}{4a}(\cos\alpha + \cos\beta + \cos\gamma - \cos\delta),$$

and $$Y = \frac{b^2 - a^2}{4b}(\sin\alpha + \sin\beta + \sin\gamma - \sin\delta).$$

Also, for the four normals, $\Sigma \cos\alpha = \dfrac{2a^2}{a^2 - b^2}\cos\delta$ (see XXII. 9),

$$\therefore \cos\alpha + \cos\beta + \cos\gamma - \cos\delta = \frac{2b^2}{a^2 - b^2}\cos\delta.$$

Hence $$X = \frac{b^2}{2a}\cos\delta, \quad \text{i.e. } a^3X = \frac{a^2b^2}{2}\cos\delta.$$

Similarly $b^3Y = \dfrac{a^2b^2}{2}\sin\delta$, so that the locus is that stated.

9. Any one of the conics is of the form $\beta\gamma = k\alpha^2$, and the tangent at $(\alpha', \beta', \gamma')$ is

$$2k\alpha'\alpha - \gamma'\beta - \beta'\gamma = 0, \quad \text{or} \quad 2\beta'\gamma'\alpha - \gamma'\alpha'\beta - \alpha'\beta'\gamma = 0.$$

If this is parallel to $l\alpha + m\beta + n\gamma = 0$, we have

$$\begin{vmatrix} 2\beta'\gamma', & -\gamma'\alpha', & -\alpha'\beta' \\ l, & m, & n \\ a, & b, & c \end{vmatrix} = 0,$$

whence the locus of $(\alpha', \beta', \gamma')$ is the conic

$$2(mc - nb)\beta\gamma + (lc - na)\gamma\alpha + (ma - lb)\alpha\beta = 0.$$

10. Let R and R' be the normal pressures at the lower and upper ends of the side in question, W the weight, a the side of the square, and θ the inclination required. Then resolving horizontally and vertically, we obtain the equations

$$R + \mu' R' = W, \quad R' = \mu R,$$

whence
$$\frac{R}{1} = \frac{R'}{\mu} = \frac{W}{1 + \mu\mu'} \quad \text{..................(i).}$$

Also taking moments about the lowest corner,

$$R' \,.\, a \sin\theta + \mu' R' \,.\, a \cos\theta = W \,.\, \frac{a}{\sqrt{2}} \,.\, \cos(45^\circ + \theta),$$

$$\text{i.e. } R' \sin\theta + \mu' R' \cos\theta = \tfrac{1}{2} W (\cos\theta - \sin\theta).$$

Substituting from (i), we obtain the required value for $\tan\theta$.

11. Let f be the acceleration of m relative to M down the plane, F that of M horizontally. Then the horizontal acceleration of m is $f\cos\alpha - F$, and its vertical acceleration is $f \sin\alpha$. Hence the equations are

$$mg - R\cos\alpha = mf\sin\alpha,$$

$$R\sin\alpha = m(f\cos\alpha - F),$$

$$R\sin\alpha = MF.$$

From these we find

$$f = \frac{(M+m)\operatorname{cosec}\alpha}{M\operatorname{cosec}^2\alpha + m} \,.\, g, \quad F = \frac{m\cot\alpha}{M\operatorname{cosec}^2\alpha + m} \,.\, g.$$

The time is given by $h = \frac{1}{2} f \sin\alpha \,.\, t^2$, whence

$$t^2 = \frac{2h}{g} \,.\, \frac{M\operatorname{cosec}^2\alpha + m}{M+m},$$

and in this time the wedge moves a distance

$$\tfrac{1}{2} F t^2 = \frac{mh}{M+m} \,.\, \cot\alpha.$$

12. At each impact the velocity perpendicular to the plane is altered in the ratio $e : 1$. Hence the times in the successive trajectories are

$$\frac{2u\sin\beta}{g\cos\alpha}, \quad \frac{2eu\sin\beta}{g\cos\alpha}, \text{ etc.,}$$

so that the time for the first n trajectories is

$$\frac{2u \sin \beta}{g \cos \alpha} \cdot \frac{1 - e^n}{1 - e} = t, \text{ suppose.}$$

If in this time the particle has returned to the point of projection, we have

$$u \cos \beta \, . \, t - \tfrac{1}{2} g \sin \alpha \, . \, t^2 = 0,$$

$$\therefore \; t = \frac{2u \cos \beta}{g \sin \alpha}.$$

Equating the values of t, we obtain

$$e^n = 1 - (1 - e) \cot \alpha \cot \beta,$$

whence, taking logarithms, the result follows.

XXVI.

1. Let P, Q be any two points conjugate for a circle, centre O. Then, if QY be drawn perpendicular to OP, QY will be the polar of P, therefore $OY \, . \, OP = r^2$, where r is the radius. But Y lies on the circle on PQ as diameter, therefore the tangent to this circle from $O = r$, i.e. the two circles cut orthogonally. Now the circles described on the diagonals of a complete quadrilateral are coaxal, and two of them cut a given circle orthogonally, therefore so also does the third. Hence, conversely, the two remaining vertices are conjugate for the circle. Further, since the given circle cuts a coaxal system (viz. the circles on the diagonals) orthogonally, it belongs to a coaxal system whose radical axis is the line of centres of the other system, viz. the line through the middle points of the diagonals.

2. Draw PN, QM perpendicular to the major axis. Then since P is on the ellipse

$$CP^2 = b^2 + e^2 \, . \, CN^2.$$

Now if r is the radius of the circle, we have from similar triangles

$$CQ^2 : CP^2 = QM^2 : CN^2 = CQ^2 - r^2 : r^2,$$

$$\therefore \; CQ^2 - e^2 \, . \, QM^2 : CP^2 - e^2 \, . \, CN^2 = CQ^2 - r^2 : r^2,$$

$$\text{i.e. } CQ^2 - e^2 \, . \, QM^2 : b^2 = CQ^2 - r^2 : r^2.$$

Also $QM^2 = CQ^2 - CM^2$. Hence

$$(1-e^2)\,CQ^2 + e^2 \,.\, CM^2 : b^2 = CQ^2 - r^2 : r^2,$$

$$\text{i.e. } \frac{b^2}{a^2} \,.\, CQ^2 + b^2 + e^2 \,.\, CM^2 : b^2 = CQ^2 : r^2,$$

$$\therefore \quad \left(\frac{b^2}{r^2} - \frac{b^2}{a^2}\right) CQ^2 = b^2 + e^2 \,.\, CM^2.$$

Now if the tangents from A and A' to the circle meet the minor axis in B', and if $CB' = b'$, then

$$a^2 - r^2 : r^2 = a^2 : b'^2.$$

Hence $$\frac{b^2}{b'^2} \,.\, CQ^2 = b^2 + e^2 \,.\, CM^2,$$

or $$CQ^2 = b'^2 + e'^2 \,.\, CM^2,$$

where $e' : e = b' : b$.

Hence the locus of Q is a concentric and coaxal ellipse, of minor axis b' and eccentricity e'. It will coincide with the original ellipse if $b' = b$, i.e. if the given circle is that inscribed in the rhombus formed by the lines joining the extremities of the axes.

3. Calling the given determinant Δ_m, it is evident on expanding that

$$\Delta_m = (1 + x^2)\,\Delta_{m-1} - x^2\,\Delta_{m-2},$$

$$\text{i.e. } \Delta_m - \Delta_{m-1} = x^2\,(\Delta_{m-1} - \Delta_{m-2}).$$

Similarly $$\Delta_{m-1} - \Delta_{m-2} = x^2\,(\Delta_{m-2} - \Delta_{m-3}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta_3 - \Delta_2 = x^2\,(\Delta_2 - \Delta_1).$$

Hence, multiplying,

$$\Delta_m - \Delta_{m-1} = x^{2m-4}\,(\Delta_2 - \Delta_1) = x^{2m},$$

$$\therefore \; \Delta_m - \Delta_1 = x^4 + x^6 + \ldots + x^{2m},$$

and $\Delta_1 = 1 + x^2$. Hence the result follows.

4. We have

$$\log(n+1)-\log n=\log\left(1+\frac{1}{n}\right)$$

$$=\sum_{r=1}(-1)^{r-1}\cdot\frac{1}{r}\cdot\frac{1}{n^r}.$$

Similarly

$$\log n-\log(n-1)=\sum_{r=1}(-1)^{r-1}\cdot\frac{1}{r}\cdot\frac{1}{(n-1)^r},$$

....

$$\log 2-\log 1=\sum_{r=1}(-1)^{r-1}\cdot\frac{1}{r},$$

the series being all convergent. Adding we obtain

$$\log(n+1)=\sum_{r=1}(-1)^{r-1}\cdot\frac{1}{r}\cdot S_r.$$

5. We have

$$A'A_1=c\cos B\sin B,\quad A'A_2=b\cos C\sin C,$$

$$\therefore\ \Delta_1=\tfrac{1}{2}bc\cos B\sin B\cos C\sin C\,.\sin A$$

$$=\Delta\,.\sin B\cos B\sin C\cos C.$$

But $\qquad a=\dfrac{\Delta}{R\sin B\sin C},\quad\therefore\ a^2.\Delta_1=\dfrac{\Delta^3}{R^2}.\cot B\cot C.$

Also $\Sigma\cot B\cot C=1$. Hence the result follows.

6. We have

$$\cos\alpha+\cos 5\alpha=2\cos 3\alpha\cos 2\alpha=2\cos\frac{6\pi}{13}\cos\frac{4\pi}{13},$$

$$\cos 2\alpha+\cos 3\alpha=2\cos\frac{5\pi}{13}\cos\frac{\pi}{13},$$

$$\cos 4\alpha+\cos 6\alpha=2\cos\frac{10\pi}{13}\cos\frac{2\pi}{13}=-2\cos\frac{3\pi}{13}\cos\frac{2\pi}{13}.$$

Hence the first expression is equal to

$$-8\,.\prod_{r=1}^{r=6}\cos\frac{r\pi}{13}.$$

Now if in the identity

$$\frac{x^{13}-1}{x-1} = \prod_{r=1}^{r=6}\left(x^2 - 2x\cos\frac{2r\pi}{13} + 1\right)$$

we put $x = -1$, we get $1 = 2^{12} . \prod_{r=1}^{r=6} \cos^2 \frac{r\pi}{13}$.

On taking the square root, the positive sign must be taken, since all the angles involved are acute. Hence

$$\prod_{r=1}^{r=6} \cos\frac{r\pi}{13} = \frac{1}{2^6},$$

and the first result follows.

The second may be obtained similarly, by using $\frac{x^9-1}{x-1}$.

7. Since

$$1 - 2ax + a^2x^2\sec^2\theta = (1 - ax\sec\theta . e^{i\theta})(1 - ax\sec\theta . e^{-i\theta}),$$

we have

$$\frac{\sin\theta}{1-2ax+a^2x^2\sec^2\theta} = \frac{1}{2i}\left(\frac{e^{i\theta}}{1-ax\sec\theta . e^{i\theta}} - \frac{e^{-i\theta}}{1-ax\sec\theta . e^{-i\theta}}\right)$$

$$= \frac{1}{2i}\left(e^{i\theta} . \sum_0^\infty a^nx^n\sec^n\theta . e^{ni\theta} - e^{-i\theta} . \sum_0^\infty a^nx^n\sec^n\theta . e^{-ni\theta}\right)$$

$$= \frac{1}{2i}\sum_0^\infty a^nx^n\sec^n\theta\left(e^{\overline{n+1}\,i\theta} - e^{-\overline{n+1}\,i\theta}\right)$$

$$= \sum_0^\infty a^nx^n\sec^n\theta . \sin(n+1)\theta.$$

8. The centre of the circle through α, β, γ is

$$X = \frac{a^2-b^2}{4a}[\Sigma\cos\alpha + \cos(\alpha+\beta+\gamma)],$$

$$Y = -\frac{a^2-b^2}{4b}[\Sigma\sin\alpha - \sin(\alpha+\beta+\gamma)].$$

But since the centroid is at the centre of the ellipse,

$$\Sigma\cos\alpha = \Sigma\sin\alpha = 0,$$

$$\therefore\ aX = \tfrac{1}{4}(a^2-b^2)\cos(\alpha+\beta+\gamma),$$

$$bY = \tfrac{1}{4}(a^2-b^2)\sin(\alpha+\beta+\gamma),$$

and the locus of (X, Y) is as stated.

9. Take the diameter through P and the tangent at P as axes. Then, if $CP = a$, and the conjugate semi-diameter $= b$, the equation to the ellipse is

$$\frac{(x+a)^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2x}{a} = 0.$$

Let O be (x', y'). Then the tangents from O meet the axis of y, where

$$\frac{y^2}{b^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{2x'}{a}\right) = \left(\frac{yy'}{b^2} + \frac{x'}{a}\right)^2,$$

or
$$\frac{y^2}{b^2}\left(\frac{x'}{a} + 2\right) - \frac{2yy'}{b^2} - \frac{x'}{a} = 0.$$

If the difference of the values of y given by this equation is constant and equal to $2k$, we find

$$\frac{y'^2}{b^2} + \frac{x'^2}{a^2} + \frac{2x'}{a} = \frac{k^2}{b^2}\left(\frac{x'}{a} + 2\right)^2,$$

shewing that the locus of (x', y') is a conic having four-point contact with the given conic, the common tangent being $\frac{x}{a} + 2 = 0$, i.e. the tangent at P'.

10. Let AC, BD intersect in O. Then, if W is the whole weight, the weights of the rods are equivalent to a weight W at O, and for equilibrium AO must be vertical. Hence the equation of virtual work is

$$W \,.\, \delta(AO) - T \,.\, \delta(AC) = 0.$$

But $AC = 2AO$, $\quad \therefore \; T = \frac{1}{2} W.$

11. Let OA, OB represent u, U respectively. Then AB represents ρ. Also, if C be a point in AB such that $w \,.\, AC = W \,.\, CB$, then OC represents V.

We have then $\quad \dfrac{AC}{W} = \dfrac{CB}{w} = \dfrac{\rho}{W+w}$ ....(i).

Also the angle between AB and OC (i.e. $O\hat{C}A$) is θ.

But $$OB^2 = OC^2 + CB^2 + 2OC \,.\, CB \cos\theta,$$
and $$OA^2 = OC^2 + AC^2 - 2OC \,.\, CA \cos\theta.$$
Substituting from (i), we obtain the results given.

12. Suppose the particle leaves the sphere at a point P at an angular distance θ from the highest point. Then, if v be the velocity of projection,

$$\frac{v^2 - 2ga(1 + \cos\theta)}{a} = g\cos\theta,$$

$$\text{i.e. } v^2 = ga(2 + 3\cos\theta) \qquad \text{(i).}$$

The equation to the path referred to horizontal and vertical axes through P is

$$y = x\tan\theta - \tfrac{1}{2}g \,.\, \frac{x^2}{V^2\cos^2\theta} \qquad \text{(ii),}$$

where V is the velocity at P, i.e. $V^2 = ga\cos\theta$.

The co-ordinates of the lowest point are $a\sin\theta$, $-a(1 + \cos\theta)$. Substituting in (ii) and reducing, we find

$$(2\cos\theta - 1)(\cos\theta + 1)^2 = 0,$$

whence $\cos\theta = \frac{1}{2}$, and (i) gives the required value of v.

XXVII.

1. Invert the system from P, and let q, q', o be the inverses of Q, Q', O. Then the circles become two perpendicular straight lines through q and q', the circle OQQ' becomes a circle touching these lines at q and q' and passing through o, and the circles OPQ, OPQ' become the lines oq and oq'. But if C is the centre of the circle oqq', then $q\hat{o}q' = \frac{1}{2}q\hat{C}q'$, and is therefore half a right angle. Hence the theorem.

2. Let $AS = p$, so that the latus-rectum is $4p$. Let P and P' be the points of contact on the same side of the axis, PG, $P'G'$ the normals, PN, $P'N'$ the ordinates. We then have

$$NG = N'G' = 2p, \qquad NN' = GG' = d,$$

$$PG = a, \quad P'G' = b.$$

But $$PG^2 = PN^2 + NG^2 = 4AS \,.\, AN + NG^2,$$

$$\text{i.e. } a^2 = 4p \,.\, AN + 4p^2 = 4p\,(AN + p).$$

Similarly $$b^2 = 4p\,(AN' + p),$$

$$\therefore\ b^2 - a^2 = 4p \,.\, NN' = 4pd.$$

3. Since $x^2 = (z+y)(z-y)$, and x is prime, therefore $z+y = x^2$ and $z-y=1$, therefore $y = \frac{1}{2}(x^2-1) = \frac{1}{2}(x-1)(x+1)$. Now $x-1$, $x+1$ are consecutive even numbers, therefore their product is divisible by 8, therefore y is a multiple of 4.

Also, x being a prime other than 3, either $x-1$ or $x+1$ is a multiple of 3, therefore y is a multiple of 12.

Further $2(y+x+1) = x^2 - 1 + 2x + 2 = (x+1)^2$.

4. By the rules of partial fractions, we find

$$\frac{1+x}{(1+x^2)(1-x)^2} = -\frac{1}{2}\cdot\frac{1-x}{1+x^2} + \frac{1}{2}\cdot\frac{1}{1-x} + \frac{1}{(1-x)^2}$$

$$= -\tfrac{1}{2}(1-x)(1+x^2)^{-1} + \tfrac{1}{2}(1-x)^{-1} + (1-x)^{-2}.$$

The coefficient of x^{2p} in $(1+x^2)^{-1}$ is $(-1)^p$.

Hence the coefficient of x^{2p} in the first expression is $-\frac{1}{2}(-1)^p$, and the coefficient of x^{2p-1} is $\frac{1}{2}(-1)^{p-1} = -\frac{1}{2}(-1)^p$. Hence the coefficient of x^n is $-\frac{1}{2}(-1)^p$, where p is $\frac{1}{2}n$ if n is even and $\frac{1}{2}(n+1)$ if n is odd. The coefficients of x^n in the second and third expressions are respectively $\frac{1}{2}$ and $(n+1)$. Hence the result.

5. We have

$$\frac{1-\cos 2\alpha}{1-\cos\alpha} = \frac{2\sin^2\alpha}{2\sin^2\dfrac{\alpha}{2}} = 4\cos^2\frac{\alpha}{2}.$$

Hence the given expression is

$$\left(\frac{\cos\dfrac{7\theta}{2}}{\cos\dfrac{\theta}{2}}\right)^2.$$

Also $\dfrac{\cos\dfrac{7\theta}{2}}{\cos\dfrac{\theta}{2}} = \dfrac{\sin 4\theta - \sin 3\theta}{\sin\theta} = 4\cos\theta\cos 2\theta - (3 - 4\sin^2\theta)$

$$= 2(\cos 3\theta + \cos\theta) - (2\cos 2\theta + 1).$$

6. We have

$$\begin{aligned}\sin\theta\,.\,\sqrt[3]{\sec\theta} &= (\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{120}\theta^5 - \ldots)(1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{24}\theta^4 - \ldots)^{-\frac{1}{3}} \\ &= (\theta - \tfrac{1}{6}\theta^3 + \tfrac{1}{120}\theta^5 - \ldots)(1 + \tfrac{1}{6}\theta^2 + \tfrac{1}{24}\theta^4 + \ldots) \\ &= \theta + \tfrac{1}{45}\theta^5 + \ldots.\end{aligned}$$

Clearly the succeeding coefficients will all be numerically less than $\frac{1}{45}$. Hence the error is less than

$$\frac{1}{45}(\theta^5 + \theta^7 + \theta^9 + \ldots \textit{ad inf.}), \quad \text{i.e.} < \frac{1}{45}\cdot\frac{\theta^5}{1-\theta^2},$$

$$\text{i.e.} \quad < \frac{1}{45}\cdot\frac{\left(\dfrac{\pi}{18}\right)^4}{1-\left(\dfrac{\pi}{18}\right)^2} \times \frac{\pi}{18} \times \frac{180}{\pi} \times 3600 \text{ seconds},$$

and, on calculation, this quantity is less than unity.

7. Let $\left(\dfrac{c}{m}, cm\right)$ be a point of intersection, so that

$$\frac{c^2}{a^2m^2} + \frac{c^2m^2}{b^2} = 1 \quad \text{........................(i).}$$

The tangents at this point are

$$m^2x + y - 2cm = 0 \text{ and } \frac{cx}{ma^2} + \frac{cmy}{b^2} = 1,$$

and they coincide if $\dfrac{m^3a^2}{c} = \dfrac{b^2}{cm}$, i.e. if $m^2 = \dfrac{b}{a}$, whence, from (i), $2c^2 = ab$.

If ω is the angle between the axes, the tangents are at right angles if

$$cm\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \left(\frac{m^3c}{b^2} + \frac{c}{ma^2}\right)\cos\omega = 0,$$

or $$\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - \left(\frac{m^2}{b^2} + \frac{1}{a^2m^2}\right)\cos\omega = 0,$$

whence, from (i), $$\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}\cos\omega = 0.$$

8. The normal at α is

$$\frac{e \sin \alpha}{1 + e \cos \alpha} \cdot \frac{l}{r} = e \sin \theta + \sin (\theta - \alpha),$$

and for the intersections with the conic we have

$$\frac{e \sin \alpha}{1 + e \cos \alpha} (1 + e \cos \theta) = e \sin \theta + \sin (\theta - \alpha),$$

leading to

$$e (\sin \theta - \sin \alpha) + (1 + e \cos \alpha + e^2) \sin (\theta - \alpha) = 0.$$

Dividing by $\sin \dfrac{\theta - \alpha}{2}$, the equation for the other intersection is

$$e \cos \frac{\theta + \alpha}{2} + (1 + e \cos \alpha + e^2) \cos \frac{\theta - \alpha}{2} = 0,$$

$$\text{i.e. } (1 + e \overline{1 + \cos \alpha} + e^2) \cos \frac{\theta}{2} \cos \frac{\alpha}{2}$$

$$+ (1 - e \overline{1 - \cos \alpha} + e^2) \sin \frac{\theta}{2} \sin \frac{\alpha}{2} = 0,$$

whence the result follows.

9. If the normals at the extremities of

$$\frac{lx}{a} + \frac{my}{b} = 1 \text{ and } \frac{l'x}{a} + \frac{m'y}{b} = 1$$

are concurrent in (X, Y), then, for some value of λ,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda \left(\frac{lx}{a} + \frac{my}{b} - 1\right) \left(\frac{l'x}{a} + \frac{m'y}{b} - 1\right) = 0$$

must coincide with

$$(a^2 - b^2) xy + b^2 xY - a^2 yX = 0.$$

The identification at once gives $\lambda = 1$, $ll' + 1 = 0$, $mm' + 1 = 0$, so that the second of the two lines may be written $\dfrac{x}{al} + \dfrac{y}{bm} = -1$.

If this passes through α, then $\dfrac{\cos \alpha}{l} + \dfrac{\sin \alpha}{m} = -1$.

Eliminating l, the equation to the first line may be written

$$m^2 \cdot \frac{y}{b} - m\left(\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha + 1\right) - \sin\alpha = 0,$$

so that its envelope is

$$\left(\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha + 1\right)^2 + 4\frac{y}{b}\sin\alpha = 0.$$

10. Let x be the length of the string. Then the equation of virtual work is

$$2W \cdot \delta\left(\frac{x}{2}\right) + 2W \cdot \delta\left(\frac{3x}{2}\right) - T\,\delta x = 0,$$

whence $T = 4W$.

Resolve the reaction at C into two components X, Y horizontally and vertically. Then, taking moments about A for AB, BC combined, we have

$$2W \cdot \frac{x}{2} - X \cdot 2x = 0, \quad \therefore \; X = \tfrac{1}{2}W.$$

Also taking moments about B for BC, we have

$$X \cdot x + Y \cdot x = (T - W)\frac{x}{2} = 3W \cdot \frac{x}{2},$$

$$\therefore \; Y = W.$$

Hence the reaction at $C = \sqrt{X^2 + Y^2} = \dfrac{\sqrt{5}}{2} \cdot W$.

11. Let AB, AC be any two of the tubes on opposite sides of the vertical through A. Then the horizontal line through A is the common directrix of all the subsequent parabolic paths. Let BM be the perpendicular from B on this line. Make $A\hat{B}S = A\hat{B}M$, and $BS = BM$. Then S is the focus of the path of the particle emerging at B. But $AS = l\cos BAS$. Hence, if (r, θ) be the polar co-ordinates of S referred to A as pole, and AM as initial line, we have

$$r = l\cos\tfrac{1}{2}\theta, \text{ since } B\hat{A}S = \tfrac{1}{2}M\hat{A}S.$$

Similarly for AC the locus will be

$$r = l\cos\tfrac{1}{2}(\pi - \theta), \text{ i.e. } r = l\sin\tfrac{1}{2}\theta.$$

12. Let X and Y be the components of the pressure of the earth on a particle of mass m at the point, perpendicular to and along the radius. Let a be the radius of the earth, ω the angular velocity about its axis. Then

$$X \cos \alpha + (mg - Y) \sin \alpha = m\omega^2 \,.\, a \sin \alpha,$$

$$X \sin \alpha - (mg - Y) \cos \alpha = 0.$$

From these

$$X = m\omega^2 a \sin \alpha \cos \alpha, \quad Y = m\,(g - \omega^2 a \sin^2 \alpha),$$

$$\therefore \; \tan \theta = \frac{X}{Y} = \frac{\sin \alpha \cos \alpha}{\dfrac{g}{\omega^2 a} - \sin^2 \alpha}.$$

Now the quantity $\dfrac{\omega^2 a}{g}$ is small $\left(\text{about } \dfrac{1}{289}\right)$ and therefore neglecting squares of this quantity, we have

$$\sqrt{X^2 + Y^2} = mg\left(1 - \frac{\omega^2 a}{g} \sin^2 \alpha\right),$$

and the ratio of the values of this at the equator and pole is $1 - \dfrac{\omega^2 a}{g}$, so that $c = \dfrac{g}{\omega^2 a}$, and we get the value of $\tan \theta$ as given.

XXVIII.

1. Let LMN be the pedal line of P. Then $P\hat{L}N = P\hat{B}N$. Hence the inclinations of the pedal lines of P and Q to BC are $90° - P\hat{B}A$ and $90° - Q\hat{B}A$, so that the angle between them is PBQ, i.e. half the angle subtended by PQ at the centre.

It follows from this that the pedal lines at the extremities of a diameter are at right angles. Let O be the orthocentre, and let OP, OQ meet the pedal lines in Y and Y'. Then Y and Y' are on the nine-point circle, and since they bisect OP and OQ, $\therefore \; YY' = \frac{1}{2}PQ$, i.e. YY' is a diameter of the nine-point circle. Hence since the pedal lines pass through Y and Y' and are at right angles, they must intersect on the nine-point circle.

2. Let the given tangent be BQP, touching the parabola at B, and meeting the tangent at the vertex in Q and the axis in P, and let C be the fixed point on the axis. Then $BQ = QP$, and hence, if D be the middle point of BC, QD is parallel to the axis, and therefore perpendicular to QA, where A is the vertex. It therefore follows that the envelope of QA is a parabola of which D is the focus, and BP the tangent at the vertex.

3. The coefficient of x^n is $n^2(n+1)^2$, which may be written in the form

$$n(n+1)[2+(n-1)(n+2)] = 4 \cdot \frac{n(n+1)}{2!} + 24 \cdot \frac{(n-1)n(n+1)(n+2)}{4!}.$$

Now $\qquad \dfrac{n(n+1)}{2!}$ is the coefficient of x^{n-1} in $(1-x)^{-3}$,

and $\dfrac{(n-1)n(n+1)(n+2)}{4!}$ is the coefficient of x^{n-2} in $(1-x)^{-5}$.

Hence $n^2(n+1)^2$ is the coefficient of x^n in

$$4x(1-x)^{-3} + 24x^2(1-x)^{-5},$$

which is therefore the value of the given series.

4. Adding the rows, we may remove the factor $2a + 2b + c$. Now subtracting the first column from the second, third and fourth, and the fourth from the fifth, and putting $b - c = \alpha$, etc. (so that $\alpha + \beta + \gamma = 0$), the determinant becomes

$$\begin{vmatrix} -\alpha, & 0, & \gamma, & 0 \\ \alpha, & -\beta, & -\beta, & -\gamma \\ \gamma, & \gamma, & 0, & -\alpha \\ 0, & -\gamma, & \beta, & \alpha \end{vmatrix}$$

$$= -\alpha \begin{vmatrix} -\beta, & -\beta, & -\gamma \\ \gamma, & 0, & -\alpha \\ -\gamma, & \beta, & \alpha \end{vmatrix} + \gamma \begin{vmatrix} \alpha, & -\beta, & -\gamma \\ \gamma, & \gamma, & -\alpha \\ 0, & -\gamma, & \alpha \end{vmatrix}$$

$$= -\alpha \begin{vmatrix} -\beta, & -\beta, & -\gamma \\ \gamma, & 0, & -\alpha \\ 0, & \beta, & 0 \end{vmatrix} + \gamma \begin{vmatrix} \alpha, & -\beta, & -\gamma \\ \gamma, & 0, & 0 \\ 0, & -\gamma, & \alpha \end{vmatrix}$$

$$= -\alpha(-\alpha\beta^2 - \beta\gamma^2) + \gamma(\alpha\beta\gamma + \gamma^3)$$
$$= (\alpha\beta + \gamma^2)^2,$$

which, on substituting for α, β, γ, is the second factor given.

5. Let the angles of $ABCD$ be 2α, 2β, 2γ, 2δ. Then we have

$$\Delta = R^2 . \Sigma \cot \alpha, \quad \Delta' = \tfrac{1}{2}R^2 . \Sigma \sin 2\alpha.$$

Now

$$\begin{aligned}\Sigma \sin 2\alpha &= 2 \sin (\alpha + \beta) \cos (\alpha - \beta) + 2 \sin (\gamma + \delta) \cos (\gamma - \delta)\\ &= 2 \sin (\alpha + \beta) [\cos (\alpha - \beta) + \cos (\gamma - \delta)], \text{ since } \Sigma\alpha = \pi,\\ &= 2 \sin (\alpha + \beta)[\cos (\alpha - \beta) - \cos (\alpha + \beta) + \cos (\gamma - \delta) - \cos (\gamma + \delta)]\\ &= 4 \sin (\alpha + \beta) \sin \gamma \sin \delta + 4 \sin (\gamma + \delta) \sin \alpha \sin \beta.\end{aligned}$$

Hence

$$\frac{\Sigma \sin 2\alpha}{\Pi \sin \alpha} = \frac{4 \sin (\alpha + \beta)}{\sin \alpha \sin \beta} + \frac{4 \sin (\gamma + \delta)}{\sin \gamma \sin \delta}$$

$$= 4\Sigma \cot \alpha,$$

$$\therefore \frac{\Delta}{\Delta'} = \frac{2\Sigma \cot \alpha}{\Sigma \sin 2\alpha} = \tfrac{1}{2}\Pi \operatorname{cosec} \alpha.$$

But $\operatorname{cosec} \alpha = \dfrac{OA}{R}$, etc. Hence $\dfrac{\Delta}{\Delta'} = \tfrac{1}{2} . \dfrac{OA . OB . OC . OD}{R^4}$.

6. We have

$$\xi + i\eta = \tan \{\sin^{-1} (x + iy)\} = \frac{x + iy}{\sqrt{1 - (x + iy)^2}}$$

$$= \frac{x + iy}{\sqrt{1 - x^2 + y^2 - 2ixy}}.$$

Similarly $$\xi - i\eta = \frac{x - iy}{\sqrt{1 - x^2 + y^2 + 2ixy}}.$$

Multiplying these together, we have

$$(\xi^2 + \eta^2)^2 = \frac{(x^2 + y^2)^2}{(1 - x^2 + y^2)^2 + 4x^2y^2} = \frac{(x^2 + y^2)^2}{(1 + x^2 + y^2)^2 - 4x^2}.$$

7. If the normals at m, m' pass through μ, then m, m' are the roots of $m^2 + m\mu + 2 = 0$.

The pair of lines through the origin parallel to the normals is

$$(mx + y)(m'x + y) = 0, \text{ i.e. } 2x^2 - \mu xy + y^2 = 0,$$

and the equation to the bisectors of the angles between these is

$$\frac{x^2 - y^2}{1} = \frac{xy}{-\frac{1}{2}\mu},$$

i.e. $\mu(x^2 - y^2) + 2xy = 0$, or $(\mu x + y)^2 - (1 + \mu^2) y^2 = 0$.

These form a harmonic pencil with $\mu x + y = 0$ and $y = 0$, which are respectively the line through the origin parallel to the normal at P, and the diameter through P.

8. The intersection of the tangents to the circle at

$$(r \cos \alpha,\ r \sin \alpha) \text{ and } (r \cos \beta,\ r \sin \beta)$$

is

$$\frac{r \cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}},\quad \frac{r \sin \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}}.$$

If this and the corresponding point for α, γ lie on the given hyperbola, we have

$$\frac{\cos^2 \frac{\alpha + \theta}{2}}{a^2} - \frac{\sin^2 \frac{\alpha + \theta}{2}}{b^2} = \frac{\cos^2 \frac{\alpha - \theta}{2}}{r^2}, \quad (\theta = \beta, \gamma),$$

whence $b^2 r^2 (1 + \cos \overline{\alpha + \theta}) - a^2 r^2 (1 - \cos \overline{\alpha + \theta}) = a^2 b^2 (1 + \cos \overline{\alpha - \theta})$

or $(a^2 r^2 + b^2 r^2 - a^2 b^2) \cos \alpha \cos \theta - (a^2 r^2 + b^2 r^2 + a^2 b^2) \sin \alpha \sin \theta$

$$= a^2 r^2 - b^2 r^2 + a^2 b^2.$$

Now, if β, γ satisfy $A \cos \theta + B \sin \theta = C$, then

$$\frac{A}{\cos \frac{\beta + \gamma}{2}} = \frac{B}{\sin \frac{\beta + \gamma}{2}} = \frac{C}{\cos \frac{\beta - \gamma}{2}},$$

and $\therefore$ the intersection of tangents at β, γ is $\left(r \cdot \frac{A}{C},\ r \cdot \frac{B}{C}\right)$,

i.e. in this case

$$r \cdot \frac{a^2 r^2 + b^2 r^2 - a^2 b^2}{a^2 r^2 - b^2 r^2 + a^2 b^2} \cdot \cos \alpha, \quad -r \cdot \frac{a^2 r^2 + b^2 r^2 + a^2 b^2}{a^2 r^2 - b^2 r^2 + a^2 b^2} \cdot \sin \alpha,$$

and the locus of this point is the ellipse in question.

9. Let the given conic be $ax^2 + 2hxy + by^2 = 1$, and the fixed direction that of $y = mx$. Then any one of the conics in question is of the form

$$ax^2 + 2hxy + by^2 - 1 + \lambda (y - mx - c)^2 = 0,$$

and the centre is given by

$$ax + hy - \lambda m (y - mx - c) = 0,$$

$$hx + by + \lambda (y - mx - c) = 0.$$

Eliminating λ, the locus of the centres is

$$ax + hy + m (hx + by) = 0$$

or $y = m'x$, where $m' = -\dfrac{hm + a}{h + bm}$,

$$\text{i.e. } a + h (m + m') + bmm' = 0,$$

which is the condition that the diameters $y = mx$, $y = m'x$ should be conjugate.

10. Let $OABC$ be the rhombus, B being the lowest joint. By symmetry X, the reaction at B, is horizontal. Let R be the pressure between the sphere and either of the upper rods, W the weight of each rod.

Then, resolving vertically for the system,

$$2R \sin \theta = 4W \quad \text{(i)}.$$

Taking moments about O for OA, AB combined,

$$W . a \sin \theta + X . 2a \cos \theta = R . r \cot \theta \quad \text{(ii)},$$

and taking moments about A for AB,

$$W . \frac{a}{2} \sin \theta = X . a \cos \theta \quad \text{(iii)}.$$

From (ii) and (iii),

$$2 Wa \sin \theta = R . r \cot \theta$$

$$= 2 Wr \cot \theta \operatorname{cosec} \theta, \text{ from (i)},$$

$$\therefore \frac{a}{r} = \cot \theta \operatorname{cosec}^2 \theta = \cot \theta + \cot^3 \theta.$$

11. At the impact the horizontal velocity of each ball is multiplied by e, the vertical velocity remaining unchanged.

Let t be the time to the impact, t' the time of return, so that

$$t = \frac{a}{v \cos \alpha}, \quad t' = \frac{a}{ev \cos \alpha}.$$

Also the vertical distance described by either ball in time $t + t'$ is zero.

$$\therefore \; v \sin \alpha \, (t + t') - \tfrac{1}{2} g \, (t + t')^2 = 0,$$

$$\therefore \; \frac{2v \sin \alpha}{g} = t + t' = \frac{a}{v \cos \alpha}\left(1 + \frac{1}{e}\right),$$

$$\text{i.e. } ev^2 \sin 2\alpha = ga \, (1 + e).$$

12. Suppose that the circular motion ceases at P, and let S be the focus of the subsequent parabolic path, the tangent at P making an angle θ both with the vertical and with SP.

Since the tension vanishes at P, we have

$$m \, . \, \frac{2g \, (h - a \sin \theta)}{a} = mg \sin \theta, \text{ i.e. } \sin \theta = \frac{2h}{3a},$$

and the height of P above O is $a \sin \theta = \dfrac{2h}{3}$.

Again, since the velocity of P in the parabola is that due to a fall from the directrix,

$$\therefore \; SP = \text{height of directrix above } P = h - a \sin \theta,$$

and $PN = SP \sin 2\theta$, if PN is perpendicular to the vertical through S.

Hence the distance of O from the axis

$$= a \cos \theta - (h - a \sin \theta) \sin 2\theta$$

$$= a \cos \theta - \tfrac{1}{2} a \sin \theta \sin 2\theta = a \cos^3 \theta$$

$$= a \left(1 - \frac{4h^2}{9a^2}\right)^{\frac{3}{2}}.$$

XXIX.

1. Considering the hexagons $XX'YY'ZZ'$ and $YZ'ZX'XY'$, we see from Pascal's Theorem that each of the triangles in question is in perspective with ABC. If the triangles are $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, then again, considering the hexagon $Z'ZX'YY'X$, we see that A, α, α' are collinear, and similarly for B, β, β' and C, γ, γ'. Hence the triangles have a common centre of perspective.

2. Let PN, $P'N'$ be the ordinates of P and P'. Then

$$1-\frac{SX^2}{ST^2}=\frac{TX^2}{ST^2}=\frac{SN^2}{SP^2} \text{ (since } T \text{ is on the directrix)}$$

$$=\frac{SN'^2}{SP'^2}=\frac{e^2(SN^2-SN'^2)}{e^2(SP^2-SP'^2)}.$$

Now $\quad e(SN+SN')=e\,.\,NN'=e(NX-N'X)=SP-SP'$

and $\quad e(SN-SN')=e(NX+N'X-2SX)$

$$=SP+SP'-2l,$$

where $2l$ is the latus-rectum;

$$\therefore\ 1-\frac{SX^2}{ST^2}=\frac{SP+SP'-2l}{e^2(SP+SP')},$$

$$\therefore\ e^2-\frac{l^2}{ST^2}=1-\frac{2l}{SP+SP'}$$

$$=1-\frac{l^2}{SP\,.\,SP'},$$

since l is the harmonic mean between SP and SP'.

$$\therefore\ l^2\left(\frac{1}{SP\,.\,SP'}-\frac{1}{ST^2}\right)=1-e^2,$$

and $(1-e^2)\,b^2=l^2$.

3. When two large beads are together, the other large one may be in $3n+1$ places (taking account of the cases which are the same on turning over). When two large beads are separated by one small one, to exclude the former case we must add two more small beads, one on each side, before filling the remaining

places. Thus in this case we get $(3n-1)$ necklaces. The next case will give $(3n-2)$, the next $(3n-4)$, the next $(3n-5)$, the next $(3n-7)$, and so on. Hence the total number of necklaces is

$$(3n+1)+(3n-2)+(3n-5)+\dots\dots\dots\dots+[3n-(3n-1)]$$
$$+(3n-1)+(3n-4)+(3n-7)+\dots\dots\dots\dots+[3n-(3n-1+1)],$$

i.e. $3n^2+3n+1$, and this includes the original one. Hence the number of re-arrangements is $3n(n+1)$.

4. By the ordinary rule of partial fractions,

$$\frac{n!}{(y+n)(y+n+1)\dots(y+2n)}=\sum_{r=0}^{r=n}(-1)^r\frac{{}^nC_r}{y+n+r},$$

$$\therefore\ \frac{n!}{y^n}\left(1+\frac{n}{y}\right)^{-1}\left(1+\frac{n+1}{y}\right)^{-1}\dots\dots\left(1+\frac{2n}{y}\right)^{-1}$$
$$=\sum_{r=0}^{r=n}(-1)^r\,.\,{}^nC_r\left(1+\frac{n+r}{y}\right)^{-1}.$$

Equating coefficients of $\frac{1}{y^{n+1}}$ on each side, we find that the given series is equal to

$$n![n+(n+1)+(n+2)+\dots+2n]=n!\,.\,\frac{3n(n+1)}{2}.$$

5. Putting $\tan\theta=t$, the given equation is

$$\Pi(\cos x-t\sin x)+\Pi(\sin x'-t\cos x')=0.$$

The coefficient of t is

$$-(\Sigma\sin x\cos y\cos z+\Sigma\cos x'\sin y'\sin z')$$
$$=-\sin(x+y+z)-\sin x\sin y\sin z-\cos x'\cos y'\cos z'$$
$$+\cos(x'+y'+z')$$
$$=-(\sin x\sin y\sin z+\cos x'\cos y'\cos z'),$$
$$\text{since } x+y+z=\frac{\pi}{2}-(x'+y'+z').$$

Similarly the coefficient of t^2 is $\cos x\cos y\cos z+\sin x'\sin y'\sin z'$. Hence the equation is

$$(\cos x\cos y\cos z+\sin x'\sin y'\sin z')(1+t^2)$$
$$-(\sin x\sin y\sin z+\cos x'\cos y'\cos z')(t+t^3)=0.$$

Dividing by $1+t^2$, the result follows.

6. (i) $\tan^{-1}\left(\frac{2}{n^2}\right) = \tan^{-1}\left(\frac{1}{n-1}\right) - \tan^{-1}\left(\frac{1}{n+1}\right)$,

$$\therefore \sum_1^\infty \tan^{-1}\left(\frac{2}{n^2}\right) = \tan^{-1}\infty + \tan^{-1}1 = \frac{\pi}{2} + \frac{\pi}{4};$$

(ii) $\left(\frac{1}{n} - \frac{1}{n+1}\right)^2 = \frac{1}{n^2} + \frac{1}{(n+1)^2} - 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$.

But $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. Hence the sum in question is

$$\frac{\pi^2}{6} + \left(\frac{\pi^2}{6} - 1\right) - 2 = \frac{\pi^2}{3} - 3.$$

7. Let the parameters of the angular points be m_1, m_2, m_3, these being the roots of the equation

$$m^3 - pm^2 + qm - r = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

The orthocentre is the intersection of the lines

$$(m_1 + m_2)\,x + 2y = a\,(m_1 + m_2)\,m_3^2 + 4am_3,$$

$$(m_1 + m_3)\,x + 2y = a\,(m_1 + m_3)\,m_2^2 + 4am_2,$$

whence, on solving, its co-ordinates are

$$-a\,(q + 4), \qquad \tfrac{1}{2}\,[a\,(pq - r) + 4ap].$$

If this is the focus, $q = -5$ and $r = -p$.

Hence the equation (i) becomes

$$m^3 - pm^2 - 5m + p = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

Now let the equation to the circumcircle be

$$x^2 + y^2 + \lambda x + \mu y + \nu = 0.$$

To find where this meets the parabola put $x = am^2$, $y = 2am$. We then have

$$a^2m^4 + (4a + \lambda)\,am^2 + 2\mu am + \nu = 0.$$

Since all the roots of (ii) are roots of this equation, it must be identical with

$$(m + p)\,(m^3 - pm^2 - 5m + p) = 0,$$

since it contains no term in m^3. Comparing the two forms, we find

$$-4p = \frac{2\mu}{a}, \text{ and } \quad p^2 = \frac{\nu}{a^2},$$

$$\text{i.e.} \quad \mu = -2ap, \quad \nu = a^2p^2.$$

Hence, putting $x = 0$ in the equation to the circle, we find that the resulting equation in y has equal roots, and the result follows.

8. Let P be the point, and draw perpendiculars SY, $S'Y'$ on either of the tangents. Suppose $S\hat{P}S' = 2\beta$.

Then $\quad SP + S'P = 2a_1, \qquad SP - S'P = 2a_2,$

$$\therefore \; SP = a_1 + a_2, \qquad S'P = a_1 - a_2.$$

Also the normal at P to the confocal ellipse bisects both the angle SPS' and the angle between the tangents. Hence

$$\frac{SY}{SP} = \sin(\alpha - \beta), \qquad \frac{S'Y'}{S'P} = \sin(\alpha + \beta).$$

Multiplying these, we have

$$\frac{b^2}{a_1^2 - a_2^2} = \sin^2\alpha - \sin^2\beta \text{.....................(i)}.$$

Now $\quad \cos 2\beta = \dfrac{SP^2 + S'P^2 - SS'^2}{2SP \,.\, SP'} = \dfrac{2(a_1^2 + a_2^2) - 4(a^2 - b^2)}{2(a_1^2 - a_2^2)},$

whence $\quad \sin^2\beta = \frac{1}{2}(1 - \cos 2\beta) = \dfrac{a^2 - b^2 - a_2^2}{a_1^2 - a_2^2}.$

Hence from (i) we obtain $\sin^2\alpha = \dfrac{a^2 - a_2^2}{a_1^2 - a_2^2}$, which is equivalent to the result given.

9. The equation to the tangents from $(\alpha', \beta', \gamma')$ is

$$(\alpha^2 + 4\beta\gamma\cos A)(\alpha'^2 + 4\beta'\gamma'\cos A) = [\alpha\alpha' + 2(\beta\gamma' + \beta'\gamma)\cos A]^2.$$

Now the lines

$$u\alpha^2 + \ldots + \ldots + 2u'\beta\gamma + \ldots + \ldots = 0$$

are at right angles if $\Sigma u - 2\Sigma u' \cos A = 0$.

Using this condition here, we get

$$\beta'\gamma' \cos A - \gamma'^2 \cos^2 A - \beta'^2 \cos^2 A - (\alpha'^2 + 2\beta'\gamma' \cos A) \cos^2 A$$
$$+ \cos A \cos B\alpha'\beta' + \cos A \cos C\gamma'\alpha' = 0,$$

which reduces to the required form.

10. Let O be the circumcentre of ABC. Then G, the C. of G. of the tetrahedron, is in DO, such that $OG = \frac{1}{4} DO$. Let the vertical through G meet the plane ABC in S: then evidently S is on OA. Let AO produced meet BC in E, so that E is the middle point of BC. Let T, T' be the tensions of the strings at B and A respectively.

Now $DO = \sqrt{\frac{2}{3}}a$, where a is the length of an edge.

$$\therefore\ OS = \frac{1}{4}\sqrt{\frac{2}{3}}\,a\,\frac{1}{\sqrt{3}} = \frac{a}{6\sqrt{2}},$$

$$\therefore\ AS = \frac{a}{\sqrt{3}} - \frac{a}{6\sqrt{2}} = \frac{2\sqrt{6}-1}{6\sqrt{2}}\,a,$$

and
$$SE = \frac{\sqrt{3}}{2}a - AS = \frac{\sqrt{6}+1}{6\sqrt{2}}\,a.$$

But
$$\frac{2T}{T'} = \frac{AS}{SE} = \frac{2\sqrt{6}-1}{\sqrt{6}+1}.$$

Hence the result.

11. Let f be the negative acceleration due to the resistance, u the velocity with which the shot enters the partition at height h. Then

$$h = \tfrac{1}{2}g \cdot \frac{d^2}{u^2} = \tfrac{1}{2}g \cdot \frac{x^2}{u^2 - 2bf}, \quad \text{where} \quad x = a - c - b - d,$$

$$\text{i.e. } \frac{2bf}{u^2} = 1 - \frac{x^2}{d^2}.$$

Similarly, in the second case,

$$\frac{2(b-t)f}{u^2} = 1 - \frac{(x+c)^2}{d^2},$$

the thickness being diminished by t.

Hence $$\frac{b}{b-t} = \frac{d^2 - x^2}{d^2 - (x+c)^2},$$

$$\therefore \quad \frac{t}{b} = \frac{(x+c)^2 - x^2}{d^2 - x^2} = \frac{2cx}{d^2 - x^2}$$

$$= \frac{2c\,(a-d)}{d^2 - (a-d)^2}, \text{ neglecting squares of small quantities.}$$

12. In the position of relative equilibrium, let the normal PG make an angle θ with the vertical, and let PN be the ordinate. Then, if R is the pressure on the curve,

$$R \sin\theta = m\omega^2 \,.\, PN, \qquad R\cos\theta = mg,$$

$$\therefore \quad \tan\theta = \frac{\omega^2 \,.\, PN}{g} = \frac{\omega^2 \,.\, NG \tan\theta}{g},$$

$$\therefore \quad \omega^2 = \frac{g}{NG}.$$

But $NG = l\cos\phi$, where ϕ is the eccentric angle of P; i.e. $NG < l$, unless P is at the lowest point.

$$\therefore \quad \omega^2 > \frac{g}{l}.$$

XXX.

1. Let A and B be the points of intersection, ACD the line. Then, since A and B are fixed, the angles ACB and ADB are given, i.e. the triangle BCD is of constant shape. Let BN be the perpendicular on CD, and O the orthocentre. For a triangle of given shape, the ratio $BO : BN$ is evidently constant. But the locus of N is a circle, viz. the circle on AB as diameter. Hence, since B is fixed, the locus of O is also a circle.

2. Let S be the orthocentre, S' the other focus. Then

$$S'\hat{A}B = SAC = 90^\circ - C.$$

Similarly $$S'\hat{B}A = 90^\circ - C.$$

$$\therefore \quad S'A = S'B = S'C, \quad \text{similarly.}$$

Hence S' is the circumcentre.

3. Putting each of the equal quantities equal to k, and solving the equations for λ^2, λ and 1, we have

$$\frac{\lambda^2}{\begin{vmatrix} k, & b_1, & c_1 \\ k, & b_2, & c_2 \\ k, & b_3, & c_3 \end{vmatrix}} = \frac{\lambda}{\begin{vmatrix} a_1, & k, & c_1 \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1, & b_1, & k \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \end{vmatrix}}$$

whence, eliminating λ and dividing by k^2, the result follows.

4. We have

$$\sqrt{5} = 2 + \frac{1}{4+}\,\frac{1}{4+}\,\frac{1}{4+}\dots$$

and the convergents are

$$\tfrac{2}{1},\quad \tfrac{9}{4},\quad \tfrac{38}{17},\quad \tfrac{161}{72},\quad \tfrac{682}{305},\quad \dots\dots$$

The odd convergents are all $< \sqrt{5}$, and form an increasing series, and the intermediate convergents between $\frac{38}{17}$ and $\frac{682}{305}$ are

$$\frac{38+161}{17+72},\quad \frac{38+2\times 161}{17+2\times 72},\quad \frac{38+3\times 161}{17+3\times 72},$$

and of these, the one whose denominator most nearly equals 200 without exceeding it is the second, viz. $\frac{360}{161}$. There is also no intermediate in the series of even convergents with a denominator between 161 and 200. Hence the fraction required is $\frac{360}{161}$.

5. Let AII_1 meet BC in D. Then if $A'B'C'$ be the triangle formed by the remaining common tangents, $B'C'$ passes through D, and BC, $B'C'$ make equal angles with II_1. Hence $A'B'$ and $A'C'$ make equal angles with BC, so that the triangle formed by these three lines is isosceles, and has the same inscribed circle as ABC.

$\therefore$ IA' is perpendicular to BC. Also $\hat{A}' = \pi - 2A$,

$$\therefore\ a' = B'C' = r\left(\cot\frac{B'}{2} + \cot\frac{C'}{2}\right)$$

$$= r(\tan B + \tan C) = r\,\frac{\sin A}{\cos B\cos C}.$$

Hence the area is

$$\tfrac{1}{2}b'c'\sin A' = \tfrac{1}{2}r^2 \cdot \frac{\sin B}{\cos C\cos A}\cdot\frac{\sin C}{\cos A\cos B}\cdot \sin 2A$$

$$= r^2\tan A\tan B\tan C,$$

and the radius of the circumcircle is

$$\frac{a'}{2\sin A'}=\tfrac{1}{2}r\cdot\frac{\sin A}{\cos B\cos C}\cdot\frac{1}{\sin 2A}=\tfrac{1}{4}r\sec A\sec B\sec C.$$

6. Suppose $z=x+iy$. Then

$$\cos(x+iy)=a,$$

$$\therefore\ \cos(x-iy)=a,\quad\text{since } a \text{ is real,}$$

$$\therefore\ \cos(x-iy)-\cos(x+iy)=0$$

and
$$\cos(x+iy)+\cos(x-iy)=2a.$$

These give

$$\sin x\sinh y=0\quad\text{and}\quad\cos x\cosh y=a.$$

From the first, $\sin x=0$, since $\sinh y$ cannot be zero unless $y=0$, which clearly cannot be the case. Hence $\cos x=\pm 1$. Also, from the second equation, since $\cosh y$ is essentially positive and a is positive by hypothesis, $\therefore$ $\cos x$ must be positive. Hence $\cos x=1$, and $x=2n\pi$. The second equation now gives $\cosh y=a$, and the result follows.

7. Let the ellipses be

$$\frac{l}{r}=1+e\cos\theta,\quad\frac{l'}{r}=1+e'\sin\theta,$$

and let α be the vectorial angle corresponding to $r=c$.

The tangents at α are

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha),\quad\frac{l'}{r}=e'\sin\theta+\cos(\theta-\alpha).$$

If these coincide,

$$\frac{e+\cos\alpha}{\cos\alpha}=\frac{\sin\alpha}{e'+\sin\alpha}=\frac{l}{l'},$$

$$\therefore\ e=\left(\frac{l}{l'}-1\right)\cos\alpha,\quad e'=\left(\frac{l'}{l}-1\right)\sin\alpha.$$

But
$$e\cos\alpha=\frac{l}{c}-1,\quad e'\sin\alpha=\frac{l'}{c}-1,$$

$$\therefore\ e^2=\left(\frac{l}{l'}-1\right)\left(\frac{l}{c}-1\right),\quad e'^2=\left(\frac{l'}{l}-1\right)\left(\frac{l'}{c}-1\right).$$

8. The tangent at (x', y') is $\frac{x}{x'} + \frac{y}{y'} = 2$, and $\therefore$ the circle of curvature is of the form

$$xy - x'y' + \lambda\left(\frac{x}{x'} + \frac{y}{y'} - 2\right)(px + qy - 1) = 0,$$

where $px' + qy' = 1$. The conditions for a circle are

$$\frac{p}{x'} = \frac{q}{y'}, \quad 1 + \lambda\left(\frac{q}{x'} + \frac{p}{y'}\right) = 0.$$

Hence $$\frac{p}{x'} = \frac{q}{y'} = \frac{1}{x'^2 + y'^2}, \quad \lambda = -x'y'.$$

Substituting and reducing, we obtain the equation given.

Further, the normal is $xx' - yy' = x'^2 - y'^2$, which meets the hyperbola again at $\left(-\frac{y'^2}{x'}, \ -\frac{x'^2}{y'}\right)$, and the centre of curvature being

$$\tfrac{1}{2}\left(3x' + \frac{y'^2}{x'}\right), \ \tfrac{1}{2}\left(3y' + \frac{x'^2}{y'}\right),$$

the second result follows immediately.

9. The conditions that $\Sigma\lambda\alpha = 0$ should be a tangent to both conics are

$$\Sigma\frac{\lambda^2}{l} = 0, \quad \Sigma\frac{\lambda^2}{l'} = 0,$$

whence, solving,

$$\frac{\lambda^2}{\frac{1}{mn'} - \frac{1}{m'n}} = \ldots = \ldots,$$

$$\text{i.e.} \quad \frac{\lambda^2}{ll'(mn' - m'n)} = \ldots = \ldots \quad \ldots\ldots\ldots \text{(i)}.$$

The points of contact are $\left(\frac{\lambda}{l}, \ldots\right)$ and $\left(\frac{\lambda}{l'}, \ldots\right)$ and from the identities

$$\Sigma l^2(m^2n'^2 - m'^2n^2) = 0, \quad \Sigma l'^2(m^2n'^2 - m'^2n^2) = 0,$$

together with (i), it is evident that these points lie on

$$\Sigma ll'(mn' + m'n)\alpha^2 = 0.$$

10. The reaction at A must be in the direction AB. Hence if the direction of the string meets AB, CD in H, K, then considering the equilibrium of the rod AC, the reaction at C must be in the direction CH. Hence, drawing BG parallel to CH, meeting DC produced in G, BCG is the triangle of forces for the joint C. Thus if R is the stress in BC, S the force along CH, we have $\dfrac{R}{S} = \dfrac{BC}{BG}$.

Also BCH is the force-triangle for the rod AC, $\therefore \dfrac{T}{S} = \dfrac{BC}{CH}$.

Now let BA, CD meet in O. Then

$$\frac{R}{T} = \frac{CH}{BG} = \frac{OH}{OB} = \frac{HK}{BC} = \tfrac{1}{2} \cdot \frac{AD + BC}{BC},$$

since H, K are the middle points of AB, CD.

11. Let T_1, T_2 be the impulsive tensions, u and v the velocities of m after impact, parallel and perpendicular to its original direction. Then the initial velocity of m' is

$$u \cos 45^\circ - v \sin 45^\circ = \frac{u - v}{\sqrt{2}},$$

and that of m'' is $\dfrac{u + v}{\sqrt{2}}$. Hence we have

$$(T_1 - T_2) \cos 45^\circ = mv, \qquad T_1 = m' \cdot \frac{u - v}{\sqrt{2}}, \qquad T_2 = m'' \cdot \frac{u + v}{\sqrt{2}},$$

leading to
$$\frac{u}{v} = \frac{m' + m'' + 2m}{m' - m''},$$

$$\text{i.e. } \frac{u - v}{u + v} = \frac{m'' + m}{m' + m}.$$

12. Let α be the required angle, v the velocity of projection, a the side of the square. The velocity parallel to AB is $ev \cos \alpha$ after the first impact, $e^2 v \cos \alpha$ after the third. Hence the time of flight is

$$\frac{a}{v \cos \alpha} + \frac{a}{ev \cos \alpha} + \frac{a}{e^2 v \cos \alpha} \quad \text{..................(i).}$$

The velocity perpendicular to AB is $v \sin \alpha$ until the second impact, $ev \sin \alpha$ afterwards; therefore the time of flight is

$$\frac{a}{v \sin \alpha} + \frac{a}{ev \sin \alpha} \quad \text{.....................(ii).}$$

Equating the expressions (i) and (ii), the result follows.

XXXI.

1. Let O be the orthocentre, and let AO meet the circumcircle in D. Draw OE parallel to the given line, meeting BC in E. Join DE and produce to meet the circumcircle in P. Then P is the point required. For, drawing perpendiculars PL, PN on BC, BA, we have

$$P\hat{L}N = P\hat{B}N = P\hat{D}A = E\hat{O}D,$$

and therefore OE is parallel to LN.

2. Reciprocating with respect to E, we obtain a parabola, focus E, and four tangents, while the four conics become the circles circumscribing the triangles formed by these four; and since these circles pass through the focus E, the original conics must be parabolas.

Further, if the circumcircles of the triangles formed by four lines all pass through a point E, then the four circumcentres lie on a circle through E.

This may be proved thus: Let PQR be the triangle formed by three of the lines, and let the other line meet its sides in P', Q', R'. Denote the centres of the circles PQR, $PQ'R'$, $QR'P'$, $RP'Q'$ by α, β, γ, δ respectively. Then $\alpha P = \alpha E$, $\beta P = \beta E$.

$$\therefore \ \alpha\hat{E}\beta = \alpha\hat{P}\beta = \alpha\hat{P}Q - \beta\hat{P}Q$$
$$= (90^\circ - P\hat{R}Q) - (90^\circ - P\hat{Q'}R') = Q'\hat{P'}R.$$

Also since the common chord of two circles is perpendicular to the line joining their centres,

$$\therefore \ \alpha\hat{\gamma}\beta = Q\hat{E}R' = Q\hat{P'}R'.$$
$$\therefore \ \alpha\hat{E}\beta = \alpha\hat{\gamma}\beta,$$

i.e. α, β, γ, E are concyclic, and similarly δ lies on the same circle.

The theorem in question is the reciprocal of this.

3. We have to shew that

$$n(1+a^2+\ldots+a^{2n})-(n+1)(a+a^3+\ldots+a^{2n-1})$$

is positive. Calling this expression $f(n)$, we have

$$\begin{aligned} f(n)-af(n-1) &= n(1+2a^2+2a^4+\ldots+2a^{2n-2}+a^{2n}) \\ &\qquad -2n(a+a^3+\ldots+a^{2n-1}) \\ &= n(1-a)(1-a+a^2-a^3+\ldots-a^{2n-1}) \\ &= n(1-a)^2(1+a^2+a^4+\ldots+a^{2n-2}), \end{aligned}$$

an essentially positive quantity. Hence, if $f(n-1)$ is positive, $f(n)$ must be positive. But $f(1)=(1-a)^2$. Hence the result follows by induction.

4. The total number of possibilities is p^3, and the number of cases in which the sum is $2p$ is the coefficient of x^{2p} in

$$(x^1+x^2+\ldots+x^p)^3,$$

i.e. the coefficient of x^{2p-3} in $(1-x^p)^3(1-x)^{-3}$, which is

$$-3\frac{(p-1)(p-2)}{2!}+\frac{(2p-2)(2p-1)}{2!}=\frac{(p-1)(p+4)}{2}.$$

Hence the required chance is

$$\frac{(p-1)(p+4)}{2p^3}.$$

5. Let $\tan x=t, \quad \cot\alpha=a$, etc.

Then since $\alpha+\beta+\gamma=\pi$, and

$$\cos(\alpha+\beta+\gamma)=\cos\alpha\cos\beta\cos\gamma-\Sigma\cos\alpha\sin\beta\sin\gamma,$$

we have

$$\frac{1}{\sin\alpha\sin\beta\sin\gamma}=-abc+\Sigma a.$$

Hence, dividing the given equation by $\cos^3 x\sin\alpha\sin\beta\sin\gamma$, it may be written in the form

$$(a-t)(b-t)(c-t)+\Sigma a-abc=0.$$

But $\Sigma bc = 1$. Hence this equation takes the form

$$-t + (\Sigma a)\, t^2 - t^3 + \Sigma a = 0,$$

or
$$(1 + t^2)(t - \Sigma a) = 0,$$

$$\therefore\ t = \Sigma a,$$

and since $\Sigma bc = 1$,
$$t^2 = \Sigma a^2 + 2.$$

6. If $y = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, then $e^{2x} = \dfrac{1 + y}{1 - y}$. Now whatever real value x may have, e^{2x} is positive, and therefore y must always lie between 1 and -1, and clearly both $y = 1$ and $y = -1$ are asymptotes to the curve.

Hence both the curves $y = \sin x$ and $y = \tanh x$ lie entirely between the lines $y = \pm 1$, and the second curve cuts each undulation of the first in the first and third quadrants in two points. Hence the given equation has an infinite number of real roots. Further, when x has a very large positive value, the intersections are very near the asymptote, and therefore occur in pairs in the neighbourhood of the points in which $y = \sin x$ touches the asymptote, i.e. points for which $x = (2n + \frac{1}{2})\pi$.

7. Suppose the Cartesian equation of the line to be

$$lx + my + n = 0.$$

Then we must have

$$l\,(at^2 + bt + c) + m\,(a't^2 + b't + c') + n\,(At^2 + Bt + C) = 0$$

for all values of t. Hence

$$la + ma' + nA = 0,$$

$$lb + mb' + nB = 0,$$

$$lc + mc' + nC = 0.$$

Eliminating l, m, n from these, we get the required condition.

In the case given the condition is satisfied, and $\dfrac{l}{1} = \dfrac{m}{-5} = \dfrac{n}{7a}$, the Cartesian equation being $x - 5y + 7a = 0$.

We also have

$$(2x - a)\, t^2 - (6x + 2a)\, t + 5\,(x + a) = 0,$$

and if t is real, this implies that

$$(3x+a)^2 - 5(2x-a)(x+a) > 0,$$

$$\text{i.e. } x^2 - ax - 6a^2 < 0,$$

so that x lies between $3a$ and $-2a$, the corresponding values of y being $2a$ and a. Hence real values of t correspond to a length

$$\sqrt{(3a+2a)^2 + (2a-a)^2} = \sqrt{26} \,.\, a.$$

8. If the chord α, β touches $4xy = c^2$, we have

$$\frac{c^2}{ab} \cos \frac{\alpha+\beta}{2} \sin \frac{\alpha+\beta}{2} = \cos^2 \frac{\alpha-\beta}{2},$$

or
$$\frac{c^2}{ab} \sin(\alpha+\beta) = 1 + \cos(\alpha-\beta).$$

Putting $\dfrac{c^2}{ab} = k$, $\tan \dfrac{\beta}{2} = t$, this equation becomes

$$t^2(1 - \cos\alpha + k \sin\alpha) - 2t(k\cos\alpha - \sin\alpha) + 1 + \cos\alpha - k\sin\alpha = 0.$$

If the roots are t_1, t_2, the equation to the chord QR is

$$\frac{x}{a}(1 - t_1 t_2) + \frac{y}{b}(t_1 + t_2) = 1 + t_1 t_2,$$

or
$$\cos\alpha\left(-\frac{x}{a} + \frac{ky}{b}\right) + \sin\alpha\left(\frac{kx}{a} - \frac{y}{b}\right) = 1,$$

and the envelope of this is

$$\left(-\frac{x}{a} + \frac{ky}{b}\right)^2 + \left(\frac{kx}{a} - \frac{y}{b}\right)^2 = 1,$$

$$\text{i.e. } (1+k^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - 4k \,.\, \frac{xy}{ab} = 1.$$

9. Let the triangle of reference be that formed by the lines α, β, γ. Then any other line is of the form $l\alpha + m\beta + n\gamma = 0$. Hence, since in the equation $\alpha = 0$, the expression may represent any arbitrary multiple of the perpendicular, we may, without loss of generality, take the relation between α, β, γ, δ to be

$$\alpha + \beta + \gamma + \delta = 0 \quad \text{.........................(i).}$$

Now the equation to any conic touching α, β, γ is

$$\Sigma l^2\alpha^2 - 2\Sigma mn\,\beta\gamma = 0 \qquad \text{(ii)},$$

and if, in addition, this touches $\delta = 0$, or $\alpha + \beta + \gamma = 0$, we have the condition $\Sigma l = 0$.

Now, from (i) we have $\alpha^2 = -(\alpha\beta + \alpha\gamma + \alpha\delta)$, etc. Hence, eliminating the square terms, (ii) becomes

$$(m+n)^2\beta\gamma + (n+l)^2\gamma\alpha + (l+m)^2\alpha\beta + l^2\alpha\delta + m^2\beta\delta + n^2\gamma\delta = 0,$$

or, since $\Sigma l = 0$,

$$l^2(\beta\gamma + \alpha\delta) + m^2(\gamma\alpha + \beta\delta) + n^2(\alpha\beta + \gamma\delta) = 0,$$

and we may further write $l = \mu - \nu$, $m = \nu - \lambda$, $n = \lambda - \mu$, giving the required form.

10. The particle will be somewhere on the line of greatest slope, i.e. the line joining the centre of the disc to its lowest point. Let R be the pressure between the particle and the disc, p the perpendicular from the centre of the sphere on the disc, θ the angle the disc makes with the horizontal, x the distance of the particle from the centre of the disc.

Then, for the particle $R = w\cos\theta$, $\mu R = w\sin\theta$, therefore $\tan\theta = \mu$, and for the disc, taking moments about the centre of the sphere,

$$R\,.\,x = \mu R\,.\,p + W\,.\,p\sin\theta,$$

$$\text{i.e. } w\cos\theta\,.\,x = \mu w\cos\theta\,.\,p + W\,.\,p\sin\theta,$$

$$\text{i.e. } x = \mu p + \frac{W}{w}\,.\,p\tan\theta$$

$$= \mu p\left(1 + \frac{W}{w}\right),$$

and $p = \sqrt{b^2 - a^2}$.

11. Let I, I' be the impulses between the striking sphere and the spheres at rest. Then since the direction of motion is unaltered, we must have

$$M_1(u - v) = I\cos\beta + I'\cos\gamma \qquad \text{(i)},$$

$$I\sin\beta = I'\sin\gamma \qquad \text{(ii)}.$$

Also the initial velocities of the spheres M_2 and M_3 are $v \cos \beta$, $v \cos \gamma$,

$$\therefore\ I = M_2 v \cos \beta, \quad I' = M_3 v \cos \gamma.$$

Substituting these values in (i) and (ii), the results follow.

12. Let u be the initial velocity of each ball. Then the velocity of B after the first impact with the wall is $-eu$. Hence, if u_1, v_1 are the velocities of A, B after the first impact, m and m' their masses, we have

$$mu_1 + m'v_1 = mu + m'(-eu),$$

$$u_1 - v_1 = -e(1+e)u,$$

whence

$$u_1 = \frac{m - (2e + e^2)m'}{m + m'} \cdot u, \quad v_1 = \frac{(1 + e + e^2)m - em'}{m + m'} \cdot u.$$

There will be a second impact between the balls if $ev_1 > -u_1$, i.e.

$$e(1 + e + e^2)m - e^2m' > -m + (2e + e^2)m',$$

$$\text{i.e. } (1 + e + e^2 + e^3)m > (2e + 2e^2)m',$$

$$\text{i.e. } \frac{m}{m'} > \frac{2e}{1 + e^2}.$$

If the masses are equal, and if their velocities after the second impact be u_2 and v_2, we have

$$u_2 + v_2 = u_1 - ev_1 = \frac{1 - 3e - e^2 - e^3}{2} \cdot u,$$

$$u_2 - v_2 = -e(u_1 + ev_1) = -\frac{e - e^2 - e^3 + e^4}{2} \cdot u,$$

whence

$$u_2 = \frac{1 - 4e - e^4}{4} \cdot u, \qquad v_2 = \frac{1 - 2e - 2e^2 - 2e^3 + e^4}{4} \cdot u,$$

and there will be a third impact between the balls if $ev_2 > -u_2$,

$$\text{i.e. } e(1 - 2e - 2e^2 - 2e^3 + e^4) > -1 + 4e + e^4,$$

$$\text{i.e. } 1 - 3e - 2e^2 - 2e^3 - 3e^4 + e^5 > 0,$$

$$\text{i.e. } (1 + e + e^2 + e^3)(1 - 4e + e^2) > 0,$$

$$\text{i.e. } [e - (2 + \sqrt{3})][e - (2 - \sqrt{3})] > 0,$$

$$\text{i.e. } e < 2 - \sqrt{3}.$$

XXXII.

1. Inscribe in the given circle a triangle DEF similar to the given triangle. Through D, E, F draw straight lines parallel to the opposite sides, forming a triangle ABC. This is the triangle required, since evidently D, E, F are the middle points of the sides. It is, moreover, easy to prove directly that the given circle passes through the feet of perpendiculars of ABC, for if the circle meet BC again in L, it is evident from symmetry that $FL = DE$, i.e. $FL = FB$, and therefore $A\hat{L}B$ is a right angle.

2. Let PQ, PQ' be equally inclined to the normal at P; and let the normal and tangent at P meet QQ' in U', U respectively. Then since PU, PU' are the bisectors of $Q\hat{P}Q'$, therefore $P(UQU'Q')$ is harmonic, therefore the polar of U passes through U', i.e. PU' is the polar of U. But if T is the pole of QQ', then PT is the polar of U. Hence PT, PU' are the same line, i.e. T lies on the normal at P.

3. Let $$f(x) = e^x - 1 - 2x.$$

Then $$f(1) = e - 3, \quad f(2) = e^2 - 5,$$

and knowing that $e = 2{\cdot}718\ldots$, it is clear from a mental approximation that $f(1)$ is negative and $f(2)$ positive, and that $f(2)$ is much greater numerically than $f(1)$.

We conclude that the equation has a root between 1 and 2, and that its value is considerably nearer the former than the latter. Further, for values of x greater than 2, e^x increases much more rapidly than $1 + 2x$, so that $f(x)$ will always be positive. Hence the equation has no other positive root, as can also be shewn graphically.

Now, either from Tables, or by direct calculation, we have

$$e^{1{\cdot}1} = 3{\cdot}004, \quad e^{1{\cdot}2} = 3{\cdot}320, \quad e^{1{\cdot}3} = 3{\cdot}669.$$

Using these, we find that $f(1{\cdot}1)$ and $f(1{\cdot}2)$ are both negative, and $f(1{\cdot}3)$ is positive. Hence the root lies between $1{\cdot}2$ and $1{\cdot}3$.

Suppose we assume $x = 1{\cdot}2 + \xi$, and retain only the first power of ξ. Then the equation becomes

$$e^{1\cdot2}(1+\xi) - 1 = 2{\cdot}4 + 2\xi,$$

$$\text{i.e. } 3{\cdot}32 + 3{\cdot}32\xi - 1 = 2{\cdot}4 + 2\xi,$$

whence $$1{\cdot}32\xi = {\cdot}08,$$

$$\text{i.e. } \xi = {\cdot}05\ldots.$$

Hence a nearer approximation to the root is $1{\cdot}25$.

4. Let d be a quantity such that axd, byd, czd are all integers. Then writing X for xd, etc., we have

$$aX + bY + cZ = X + Y + Z.$$

Now take aX quantities each equal to $\frac{1}{a}$, bX each equal to $\frac{1}{b}$, etc. Then since the A.M. of these quantities is greater than their G.M., we have

$$\frac{X+Y+Z}{aX+bY+cZ} > \left\{\left(\frac{1}{a}\right)^{aX}\left(\frac{1}{b}\right)^{bY}\left(\frac{1}{c}\right)^{cZ}\right\}^{\frac{1}{aX+bY+cZ}}$$

$$\text{i.e. } 1 > \left(\frac{1}{a}\right)^{aX}\left(\frac{1}{b}\right)^{bY}\left(\frac{1}{c}\right)^{cZ}.$$

$$\therefore\ 1 > \left(\frac{1}{a}\right)^{ax}\left(\frac{1}{b}\right)^{by}\left(\frac{1}{c}\right)^{cz},$$

or $$a^{ax}\, b^{by}\, c^{cz} > 1.$$

5. Denoting either of the angles in question by θ, we have

$$\frac{a}{OC} = \frac{\sin C}{\sin\theta}, \qquad \frac{OC}{b} = \frac{\sin(A-\theta)}{\sin A},$$

$$\therefore\ \frac{a}{b} = \frac{\sin C\sin(A-\theta)}{\sin\theta\sin A},$$

$$\therefore\ \frac{\sin A}{\sin B} = \frac{\sin C}{\sin A}(\sin A\cot\theta - \cos A),$$

$$\therefore\ \frac{\sin^2 A}{\sin B\sin C} + \cos A = \sin A\cot\theta,$$

$$\therefore\ \cot\theta = \cot A + \frac{\sin A}{\sin B \sin C}$$

$$= \cot A + \frac{\sin (B+C)}{\sin B \sin C} = \Sigma \cot A.$$

Hence $\operatorname{cosec}^2\theta = 1 + \cot^2\theta = \Sigma\cot^2 A + 3$ (since $\Sigma \cot A \cot B = 1$)

$$= \Sigma \operatorname{cosec}^2 A.$$

6. (i) We have

$$\tanh\frac{a}{2^{n-1}} = 2\coth\frac{a}{2^{n-2}} - \coth\frac{a}{2^{n-1}},$$

since $\tanh\theta = 2\coth 2\theta - \coth\theta$.

Adding this series of equations, beginning with $n = 1$, we find that the sum of the given series is

$$2\coth 2a - \frac{1}{2^{n-1}}\coth\frac{a}{2^{n-1}}.$$

(ii) Multiplying by $2\cos\theta$, and turning each product into a sum, we easily find that

$$2(1-\cos\theta) S = (n+1)\sin n\theta - n\sin(n+1)\theta.$$

In the third case, the given series is the real part of

$$i\log(1 - ie^{i\theta}\cos\theta) = i\log(1 + \sin\theta\cos\theta - i\cos^2\theta),$$

and therefore its value is

$$\tan^{-1}\left(\frac{\cos^2\theta}{1+\sin\theta\cos\theta}\right) = \cot^{-1}\left(\frac{\cos^2\theta + \sin^2\theta + \sin\theta\cos\theta}{\cos^2\theta}\right)$$

$$= \cot^{-1}(1 + \tan\theta + \tan^2\theta).$$

7. The directrix being perpendicular to the axis, let its equation be $x\sqrt{b} - y\sqrt{a} + k = 0$, the focus being (X, Y).

Then the equation to the parabola is

$$(x - X)^2 + (y - Y)^2 = \frac{(x\sqrt{b} - y\sqrt{a} + k)^2}{b + a},$$

reducing to

$$(x\sqrt{a} + y\sqrt{b})^2 - 2x[(a+b)X + k\sqrt{b}] - 2y[(a+b)Y - k\sqrt{a}]$$
$$+ (X^2 + Y^2)(a+b) - k^2 = 0.$$

Identifying this with the given equation, we have

$$(a+b)X + k\sqrt{b} = -g,$$

$$(a+b)Y - k\sqrt{a} = -f,$$

$$(X^2+Y^2)(a+b) - k^2 = \frac{g^2+f^2}{a+b}.$$

Substituting for X, Y in the last equation, we find that $k=0$, giving the required values for X and Y.

8. Writing down the equation to the tangents to the ellipse from (h, k), we find from the usual formula that the angle between them is given by

$$\tan\alpha = \frac{2\sqrt{b^2h^2+a^2k^2-a^2b^2}}{h^2+k^2-a^2-b^2}.$$

Now $$4(b^2h^2+a^2k^2-a^2b^2)+(h^2+k^2-a^2-b^2)^2$$
$$=(h^2+k^2+a^2-b^2)^2-4h^2(a^2-b^2)$$
$$=\rho^2\rho'^2,$$

since $$\rho^2=(h-ae)^2+k^2, \quad \rho'^2=(h+ae)^2+k^2.$$

Hence

$$\sin\alpha = \frac{2\sqrt{b^2h^2+a^2k^2-a^2b^2}}{\rho\rho'} \quad \text{..................(i)}.$$

Also the x co-ordinates of the intersections of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{hx}{a^2}+\frac{ky}{b^2}=1$ are the roots of

$$(b^2h^2+a^2k^2)x^2-2a^2b^2hx+a^4(b^2-k^2)=0.$$

Hence if the points of intersection are (x_1, y_1), (x_2, y_2), we find

$$(x_1-x_2)^2=\frac{4a^4k^2(b^2h^2+a^2k^2-a^2b^2)}{(b^2h^2+a^2k^2)^2}.$$

Writing down the corresponding value of $(y_1-y_2)^2$ from symmetry, and adding, we find

$$l^2=\frac{4(b^4h^2+a^4k^2)(b^2h^2+a^2k^2-a^2b^2)}{(b^2h^2+a^2k^2)^2} \quad \text{............(ii)},$$

l being the length of the chord of contact.

But the radius of the circumcircle is $\dfrac{l}{2 \sin a}$, for which (i) and (ii) give the required value.

9. Let the conics be $\Sigma \sqrt{la} = 0$ and $\Sigma \sqrt{l'a} = 0$. Then the equations to EF, $E'F'$ are

$$- la + m\beta + n\gamma = 0, \quad - l'a + m'\beta + n'\gamma = 0,$$

and since AP joins the intersection of these to the point of reference A, its equation is

$$(lm' - l'm)\,\beta - (nl' - n'l)\,\gamma = 0.$$

Writing down the corresponding equations to BQ, CR, we see that the three lines meet in the point

$$\frac{a'}{mn' - m'n} = \ldots = \ldots.$$

Further, if the fourth common tangent be $La + M\beta + N\gamma = 0$, then $\Sigma \dfrac{l}{L} = 0$, and $\Sigma \dfrac{l'}{L} = 0$, whence

$$\frac{1/L}{mn' - m'n} = \ldots = \ldots,$$

$$\text{i.e. } \frac{L}{1/a'} = \frac{M}{1/\beta'} = \frac{N}{1/\gamma'}.$$

10. Suppose that, in the twisted position, each string makes an angle ϕ with the vertical. Let A, A' be the upper and lower ends of one of the strings, O' the centre of the ring, V and L the middle points of AA' and OO'. Draw AK vertical to meet the ring in K, and let N be the middle point of $A'K$. Then

$$l \sin \phi = A'K = 2a \sin \frac{\theta}{2} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Also $VL = O'N = a \cos \dfrac{\theta}{2}$.

Resolving vertically $3T \cos \phi = W$, and the moment of the required couple is

$$3T \sin \phi \,.\, VL = W \tan \phi \,.\, a \cos \frac{\theta}{2}.$$

Substituting for ϕ from (i), we obtain the result given.

11. Let R be the reaction between a ring and a rod, T the tension of the string.

Then taking moments about the hinge for either of the lower rods, we obtain

$$W \cdot \frac{a}{2} \sin\theta + R \cdot b \operatorname{cosec}\theta = Ta \sin(\alpha + \theta) \;\ldots\ldots\ldots\text{(i)}$$

(since α is evidently the angle which either string makes with the vertical).

Resolving vertically for the cross-rod,

$$2R \sin\theta = W' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and for the whole system

$$2T \cos\alpha = 2W + W' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Eliminating R and T from (i), (ii), and (iii), the result follows.

12. We suppose the end B to be above the vertex. The directrix of the free path is evidently the horizontal through A, since the velocity at B is that due to the vertical distance between A and B. Let S' be the focus of the free path, and let BS' meet the path again in P. Draw PN vertical to meet the directrix in N, and produce it to N', so that NN' = the distance between A and the directrix of the tube. Then the horizontal through N' is a fixed line. Also since the parabolas have a common tangent at B, SBS' is a straight line, S being the focus of the tube.

Evidently SS' = the sum of the perpendiculars from S, B on the directrices $= NN'$. Also $S'P = PN$, therefore $SP = PN'$. Again, the tangent at P is equally inclined to SP and PN'. Hence, if we draw a parabola, focus S and directrix the horizontal through N', it will touch the parabolic path at P. This fixed parabola is therefore the envelope of all the parabolic paths.

XXXIII.

1. Let EF meet BC in O, and let $E'O$ and AB produced meet in F''. Then the pencil $O\,(CEAE')$ is harmonic, therefore the range $(BFAF'')$ is harmonic. But $(BFAF')$ is harmonic, by hypothesis; therefore F'' coincides with F, i.e. EF, $E'F'$ meet on BC.

2. Let A, A' be the vertices of the elliptic section, PNP' the common double ordinate, so that PN is a radius of the circular section. Then

$$PN^2 : AN \,.\, NA' = b^2 : a^2.$$

But if AD, $A'D'$ are the perpendiculars on the axis of the cone, then $AN : AD = A'N : A'D' = 2 : \surd 3$.

Also $\qquad AD \,.\, A'D' = b^2, \quad \therefore\ AN \,.\, NA' = \dfrac{4}{3} b^2,$

$$\therefore\ PN^2 : ab = \frac{4}{3} b^3 : a^3.$$

But evidently $\quad e = \dfrac{1}{\sqrt{3}}, \quad \therefore\ \dfrac{b^2}{a^2} = \dfrac{2}{3}, \quad \therefore\ \dfrac{b^3}{a^3} = \dfrac{2\sqrt{2}}{3\sqrt{3}}.$

Hence $\qquad PN^2 : ab = 8\sqrt{2} : 9\sqrt{3}.$

3. We have, by the Binomial Theorem, as long as the series is convergent,

$$(1 - xy)^{-\frac{1}{x}} = 1 + \sum_1^\infty \frac{(1+x)(1+2x) \ldots\ldots (1+\overline{r-1}x)}{r!} y^r,$$

and the coefficient of x^p in the numerator of the fraction written is ${}^{r-1}S_p$.

Hence the given series is the coefficient of $x^n y^{2n}$ in the product of the series written above, and the series

$$1 - \frac{y}{1!} + \frac{y^2}{2!} - \ldots,$$

i.e. the coefficient of $x^n y^{2n}$ in $e^{-y}(1 - xy)^{-\frac{1}{x}} = e^{-y - \frac{1}{x}\log(1-xy)}$.

Now, with the same condition of convergency,

$$-\frac{1}{x}\log(1 - xy) = y + \frac{xy^2}{2} + \frac{x^2y^3}{3} + \ldots.$$

Hence we require the coefficient of $x^n y^{2n}$ in $e^{\frac{xy^2}{2} + \frac{x^2y^3}{3} + \ldots}$.

But in this expansion $x^n y^{2n}$ occurs only once, viz. in the term

$$\left(\frac{xy^2}{2} + \frac{x^2y^3}{3} + \ldots\right)^n \Big/ n!$$

and its coefficient there is $\dfrac{1}{2^n \,.\, n!}$.

4. The total number of tickets is

$$1+2+3+\ldots+(n^2+1)=\frac{(n^2+1)(n^2+2)}{2}.$$

Since there are m^2 tickets marked m^2, and each of these confers a prize of m shillings, the value of the prizes from these tickets is m^3 shillings. Hence the total value of all the prizes is

$$1^3+2^3+3^3+\ldots+n^3=\frac{n^2(n+1)^2}{4} \text{ shillings.}$$

Hence the expectation is

$$\frac{n^2(n+1)^2}{4} \div \frac{(n^2+1)(n^2+2)}{2} \text{ shillings.}$$

5. Let r be the radius. The angle between $AB\,(=a)$ and CD is $180^\circ-(A+D)$, $\quad\therefore\ x=2r\sin\dfrac{A+D}{2}$.

But $$d=r\left(\cot\frac{A}{2}+\cot\frac{D}{2}\right)=r\,.\,\frac{\sin\dfrac{A+D}{2}}{\sin\dfrac{A}{2}\sin\dfrac{D}{2}},$$

$$\therefore\ x=2d\sin\frac{A}{2}\sin\frac{D}{2}.$$

Similarly $$x=2b\sin\frac{B}{2}\sin\frac{C}{2},$$

$$\therefore\ x^2=4bd\,.\,\Pi\sin\frac{A}{2},$$

and similarly $$y^2=4ac\,.\,\Pi\sin\frac{A}{2},$$

$$\therefore\ x^2:y^2=bd:ac.$$

6. Let the vertices, counting from A, be A_1, A_2, ... A_{n-1}. Then

$$PA_r{}^2=a^2+\left(\frac{a}{2}\right)^2-2a\left(\frac{a}{2}\right)\cos\frac{2r\pi}{n}$$

$$=a^2\left(\frac{5}{4}-\cos\frac{2r\pi}{n}\right).$$

Hence $$\Sigma PA_r^4 = a^4\left(\frac{25}{16} - \frac{5}{2}\cdot\sum_1^n \cos\frac{2r\pi}{n} + \sum_1^n \cos^2\frac{2r\pi}{n}\right).$$

Now $$\sum_1^n \cos^2\frac{2r\pi}{n} = \tfrac{1}{2}\cdot\sum_1^n\left(\cos\frac{4r\pi}{n} + 1\right) = \tfrac{1}{2}n,$$

since $$\sum_1^n \cos\frac{4r\pi}{n} = 0. \quad \text{Also } \sum_1^n \cos\frac{2r\pi}{n} = 0.$$

Hence $$\Sigma PA_r^4 = a^4\left(\frac{25}{16}n + \tfrac{1}{2}n\right) = \frac{33n}{16}\cdot a^4.$$

7. Let α be the eccentric angle of $P\ (x, y)$, and θ the angle the tangent at P makes with the major axis. Then since M and N are on the directrices,

$$\therefore\ PM\,.\,PN = \left(\frac{a^2}{e^2} - x^2\right)\sec^2\theta.$$

Now $$\tan\theta = -\frac{b}{a}\cot\alpha, \qquad \therefore\ \sec^2\theta = \frac{1 - e^2\cos^2\alpha}{\sin^2\alpha}.$$

$$\therefore\ PM\,.\,PN = a^2\left(\frac{1}{e^2} - \cos^2\alpha\right)\frac{1 - e^2\cos^2\alpha}{\sin^2\alpha}$$

$$= \frac{a^2}{e^2}\left(\frac{1 - e^2\cos^2\alpha}{\sin\alpha}\right)^2.$$

Now $\dfrac{1 - e^2\cos^2\alpha}{\sin\alpha} = \dfrac{1 - e^2}{\sin\alpha} + e^2\sin\alpha$, and the minimum value of this is $2e\sqrt{1 - e^2}$, occurring when $\dfrac{1 - e^2}{\sin\alpha} = e^2\sin\alpha$, i.e. when $\dfrac{1 - e^2}{e^2} = \sin^2\alpha$. This only gives a possible value of α when $1 - e^2 \ngtr e^2$, i.e. $e \nless \dfrac{1}{\sqrt{2}}$, and the required minimum is

$$\frac{a^2}{e^2}\cdot 4e^2(1 - e^2) = 4b^2.$$

In other cases, the quantity decreases as α increases, and has its least value when $\alpha = \dfrac{\pi}{2}$, that value being $\dfrac{a^2}{e^2}$.

8. An extremity of the latus-rectum for any conic is the point $\left(c, a-\dfrac{c^2}{a}\right)$ and the normal at this point is

$$a^2x\left(a-\frac{c^2}{a}\right)-b^2cy=(a^2-b^2)\,c\left(a-\frac{c^2}{a}\right),$$

$$\text{i.e.}\quad a^2x-acy=c^3,$$

and the envelope of this line for different values of a is $y^2+4cx=0$. Similarly the envelope of the normals at the other extremities is $y^2-4cx=0$.

9. If we change to the principal axes, the equation takes the form

$$\alpha X^2+\beta Y^2+c'=0,\quad \text{where } c'=\frac{\Delta}{C}.$$

We now find (as in XXXII. 8) that the square of the length of the chord of contact of tangents from (X, Y) is

$$-\frac{4c'(\alpha^2X^2+\beta^2Y^2)(\alpha X^2+\beta Y^2+c')}{\alpha\beta(\alpha X^2+\beta Y^2)^2},$$

while the perpendicular from (X, Y) on the chord is $\dfrac{\alpha X^2+\beta Y^2+c'}{\sqrt{\alpha^2X^2+\beta^2Y^2}}$. Hence the area of the triangle is

$$\frac{(\alpha X^2+\beta Y^2+c')^{\frac{3}{2}}}{\alpha X^2+\beta Y^2}\cdot\sqrt{-\frac{c'}{\alpha\beta}}.$$

Now $\alpha\beta=C$, and hence, transferring to the original axes, the area is

$$\frac{S^{\frac{3}{2}}}{S-c'}\cdot\frac{\sqrt{-\Delta}}{C}=\frac{S^{\frac{3}{2}}.\sqrt{-\Delta}}{CS-\Delta}.$$

10. Let G be the C. of G. of the man and ladder, and let the directions of limiting friction at A and B, the lower and upper ends of the ladder, meet in O. Then

$$\frac{AG}{GO}=\frac{\sin\epsilon}{\cos(\theta+\epsilon)},\quad \frac{BG}{GO}=\frac{\cos\eta}{\sin(\theta+\eta)},$$

$$\therefore\ \frac{AG}{BG}=\frac{\sin\epsilon\sin(\theta+\eta)}{\cos\eta\cos(\theta+\epsilon)},$$

$$\therefore\ \frac{AG}{AB}=\frac{\sin\epsilon\sin(\theta+\eta)}{\cos\theta\cos(\epsilon-\eta)}.$$

Now if the man is at height h', we have

$$(n+1)\, AG = \tfrac{1}{2} AB \,.\, n + h' \operatorname{cosec} \theta.$$

Substituting for AG and putting $AB = h \operatorname{cosec} \theta$, we get the required expression for h'.

If $h < h'$, the man will reach the top of the ladder.

11. Let R be the reaction between the rod and the table, R' that between the ring and the cylinder, P the point of attachment of the rod to the ring. Then resolving vertically for the ring and rod,

$$W + w = R \cos \theta,$$

and horizontally, $\quad R' = R \sin \theta = (W + w) \tan \theta.$

Taking moments about P for the rod,

$$W \,.\, l \cos \theta = R \,.\, c \sec \theta = (W + w)\, c \sec^2 \theta,$$

$$\therefore \cos^3 \theta = \frac{W + w}{W} \cdot \frac{c}{l}.$$

Again, let x be the vertical distance between the two points of contact of the ring. Then, taking moments about P for the ring, $R'x = wr$. The condition that the cylinder should not upset is

$$R'(c \tan \theta - x) < W'r,$$

$$\text{i.e. } (W + w)\, c \tan^2 \theta - wr < W'r.$$

12. Let m be the mass of each particle, I the impulse between C and B, T the impulsive tension of the string, U the initial velocity of A, V_1 that of C after impact, u and v those of B along and perpendicular to the groove after impact.

Then for the motion of C we have

$$I = m(V - V_1) \quad \text{......................(i)},$$

for that of B along and perpendicular to CB,

$$I - T \cos 45^\circ = mv \quad \text{....................(ii)},$$

$$T \cos 45^\circ = mu \quad \text{..................(iii)},$$

and for that of A along the groove

$$T \cos 45^\circ = mU \quad \text{..................(iv)}.$$

Also, since the string is inextensible

$$(v-u)\cos 45^\circ = U\cos 45^\circ \ldots\ldots\ldots\ldots\ldots\ldots \text{(v)},$$

and since B and C are inelastic,

$$v = V_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)}.$$

From (iii) and (iv), $u = U$, and thence from (v), $v = 2u$. Hence by (ii) and (iii), $\quad I = m(v+u) = 3mu,$

and by (i) and (vi), $\quad I = m(V - 2u),$

$$\therefore\ 3u = V - 2u, \quad \text{i.e. } u = \tfrac{1}{5}V.$$

XXXIV.

1. Invert so that the given line and the given circle become two concentric circles. Then the inverse of the variable circle is a circle touching one of these concentric circles and cutting the other at a constant angle. It is therefore evident that it always touches a third concentric circle, the inverse of which is a circle coaxal with the inverses of the other two, i.e. such that the given line is the radical axis of it and the other fixed circle.

2. Project the ellipse orthogonally into a circle. The triangles then project into triangles having the same property for the circle, and therefore all equilateral. But all equilateral triangles inscribed in a circle are equal in area, and since the area of any triangle bears a constant ratio to the area of its projection, it follows that all the triangles in the ellipse have the same area, viz. $\dfrac{a}{b}\cdot\dfrac{3\sqrt{3}b^2}{4} = \dfrac{3\sqrt{3}}{4}ab.$

3. Let $S = u_0 + u_1x + u_2x^2 + \ldots + u_nx^n + \ldots$. Then evidently, from the data,

$$S^2 = u_0^{\,2} + u_2x + u_3x^2 + \ldots + u_{n+1}x^n + \ldots,$$

i.e. $$xS^2 = S - 1 \quad (\text{since } u_0 = 1,\ u_1 = 1),$$

provided the series S is convergent.

Hence, solving, $S = \dfrac{1 \pm \sqrt{1 - 4x}}{2x}$,

and since S only contains positive powers of x,

$$\therefore\ S = \frac{1}{2x}\{1 - (1 - 4x)^{\frac{1}{2}}\}.$$

Hence u_n = coefficient of x^{n+1} in $-\frac{1}{2}(1 - 4x)^{\frac{1}{2}}$

$$= -\tfrac{1}{2}\,.\,\frac{\frac{1}{2}(\frac{1}{2} - 1)\ldots(\frac{1}{2} - n)}{(n+1)\,!}(-4)^{n+1}$$

$$= \frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots 2n}\,.\,\frac{4^n}{n+1}.$$

4. The nth term is

$$\frac{1}{(a_1+1)\ldots(a_{n-1}+1)} - \frac{1}{(a_1+1)\ldots(a_n+1)}.$$

Hence the sum of n terms is

$$1 - \frac{1}{(a_1+1)(a_2+1)\ldots(a_n+1)} \quad \text{.................(i).}$$

Now since the a's are all positive,

$$\therefore\ (a_1+1)(a_2+1)\ldots(a_n+1) > 1 + \sum_1^n a_r.$$

But since the series $a_1 + a_2 + a_3 + \ldots$ is divergent, $1 + \sum_1^n a_r$ can be made as great as we please by sufficiently increasing n. Hence, since

$$\frac{1}{(a_1+1)(a_2+1)\ldots(a_n+1)} < \frac{1}{1 + \sum_1^n a_r},$$

it follows that the sum (i) can be made to differ from unity by as small a quantity as we please, provided n is taken sufficiently great. Hence the series is convergent and equal to unity.

5. We have, with the usual notation,

$$\triangle OIP = \triangle OAI + \triangle IAP - \triangle OAP$$

$$= \tfrac{1}{2}AI\,(AO + AP)\sin\frac{B-C}{2} - \tfrac{1}{2}AO\,.\,AP\sin(B - C).$$

Now $$AI = 4R \sin\frac{B}{2}\sin\frac{C}{2}, \quad AP = 2R\cos A.$$

$$\therefore \triangle OIP = 2R^2 \sin\frac{B-C}{2}\left[\sin\frac{B}{2}\sin\frac{C}{2}(1+2\cos A) - \cos\frac{B-C}{2}\cos A\right].$$

The quantity in the square brackets is

$$\sin\frac{B}{2}\sin\frac{C}{2} - \cos A\cos\frac{B+C}{2}$$

$$= \tfrac{1}{2}\left[\cos\frac{B-C}{2} - \sin\frac{A}{2} - \cos\left(A - \frac{B+C}{2}\right) + \sin\frac{A}{2}\right]$$

$$= \sin\frac{A-B}{2}\sin\frac{A-C}{2}.$$

Hence the area has the value stated.

6. From the ordinary expression for $\cos 7\theta$ in terms of $\cos\theta$ and $\sin\theta$, we find

$$2\cos 7\theta = x^7 - 7x^5 + 14x^3 - 7x, \quad \text{where} \quad x = 2\cos\theta.$$

Now if $\cos 7\theta = 0$, $\theta = \dfrac{(2r+1)\pi}{14}$, and excluding the value $r = 3$, for which $x = 0$, it follows that the roots of the equation

$$x^6 - 7x^4 + 14x^2 - 7 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

are $$2\cos\frac{(2r+1)\pi}{14} \quad (r = 0, 1, 2, 4, 5, 6),$$

$$\text{i.e. } \pm 2\cos\frac{\pi}{14}, \ \pm 2\cos\frac{3\pi}{14}, \ \pm 2\cos\frac{5\pi}{14} \quad \ldots\ldots\ldots\text{(ii)}.$$

But the equation (i) may be written

$$x^6 - 7(x^2-1)^2 = 0,$$

or $$[x^3 - \sqrt{7}(x^2-1)][x^3 + \sqrt{7}(x^2-1)] = 0.$$

Considering the equation $x^3 - \sqrt{7}(x^2-1) = 0$, we see that the sum of its roots is positive and their product negative. The latter fact shews that from (ii) we have to choose two positive signs and one negative. (We cannot choose three negative signs, for then the sum would be negative.)

Also the expression $\cos\frac{\pi}{14}+\cos\frac{3\pi}{14}-\cos\frac{5\pi}{14}$ is clearly positive, since the cosine decreases as the angle increases in the first quadrant. Hence the three roots of the equation are

$$2\cos\frac{\pi}{14},\quad 2\cos\frac{3\pi}{14},\quad -2\cos\frac{5\pi}{14},$$

and these are the same as

$$2\sin\frac{4\pi}{7},\quad 2\sin\frac{2\pi}{7}\ \text{ and }\ 2\sin\frac{8\pi}{7}.$$

Hence the sum of these quantities is $\sqrt{7}$, and the result follows.

7. If $P(x, y)$ is a point on a common tangent whose points of contact are T and T', then

$$PT \pm PT' = TT',$$

$$\text{i.e. } \sqrt{x^2+y^2-2ax} \pm \sqrt{x^2+y^2-2by} = \sqrt{a^2+b^2-(a-b)^2} = \sqrt{2ab},$$

$$\therefore\ 2ab+2ax-2by = \pm 2\sqrt{2ab}\,\sqrt{x^2+y^2-2by}.$$

$$\text{i.e. } (ax-by+ab)^2 = 2ab\,(x^2+y^2-2by).$$

8. Let P be (X, Y) and the confocal $\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1$, so that $a'^2+b'^2=a^2-b^2$.

Solving

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1,\quad \frac{X^2}{a'^2}-\frac{Y^2}{b'^2}=1,$$

and using this condition, we find

$$\frac{X^2}{a^2a'^2}=\frac{Y^2}{b^2b'^2}=\frac{1}{a^2-b^2}=\frac{b'^2X^2+a'^2Y^2}{a'^2b'^2(a^2+b^2)}\cdots\cdots\cdots\text{(i)}.$$

The lines through (X, Y) parallel to the asymptotes are

$$\frac{x-X}{a'}=\pm\frac{y-Y}{b'}.$$

Hence Q is $\left(X-\frac{a'}{b'}Y,\ 0\right)$ and R' is $\left(0,\ Y+\frac{b'}{a'}X\right)$. The equation to QR' is therefore

$$\frac{x}{X-\frac{a'}{b'}Y}+\frac{y}{Y+\frac{b'}{a'}X}=1,$$

or
$$\frac{b'x}{b'X-a'Y}+\frac{a'y}{b'X+a'Y}=1,$$

and remembering that $b'^2X^2-a'^2Y^2=a'^2b'^2$, this may be written in the form

$$b'x\,(b'X+a'Y)+a'y\,(b'X-a'Y)=a'^2b'^2.$$

This will pass through $(kX,-kY)$ provided

$$k\,(b'^2X^2+a'^2Y^2)=a'^2b'^2,$$

i.e. from (i), if
$$k=\frac{a^2-b^2}{a^2+b^2}.$$

Thus QR' passes through the point

$$\left(\frac{a^2-b^2}{a^2+b^2}X,\ -\frac{a^2-b^2}{a^2+b^2}Y\right),$$

and similarly $Q'R$ passes through the same point.

Further these co-ordinates satisfy $a^2xY-b^2yX=(a^2-b^2)\,XY$, and the point is $\therefore$ on the normal at P. Also since $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, its locus is evidently

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

9. If the conic and circle touch at the point whose vectorial angle is α, the common tangent at that point is

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha),$$

or, in Cartesians,

$$(e+\cos\alpha)\,x+\sin\alpha\,.\,y-l=0.$$

Hence the equation to the circle must be of the form

$$(1-e^2)\,x^2+y^2+2elx-l^2$$
$$+\lambda\left[(e+\cos\alpha)\,x+\sin\alpha\,.\,y-l\right]\left[(e+\cos\alpha)\,x-\sin\alpha\,.\,y-k\right]=0,$$

with the condition

$$1-e^2+\lambda\,(e+\cos\alpha)^2=1-\lambda\sin^2\alpha,$$

$$\text{i.e. }\lambda=\frac{e^2}{1+2e\cos\alpha+e^2}.$$

Also, since the circle passes through the pole, $k\lambda = l$. Hence the polar equation to the chord of intersection is

$$\frac{l}{r} \cdot \frac{1 + 2e\cos\alpha + e^2}{e^2} = e\cos\theta + \cos(\theta + \alpha).$$

But the envelope of $L\cos\alpha + M\sin\alpha = N$ is $L^2 + M^2 = N^2$. Hence in this case the envelope is

$$\left(\cos\theta - \frac{2l}{er}\right)^2 + \sin^2\theta = \left[e\cos\theta - \frac{l}{r}\left(1 + \frac{1}{e^2}\right)\right]^2,$$

$$\text{i.e. } \frac{l^2}{r^2}\left(1 - \frac{1}{e^2}\right)^2 + \frac{2l}{r}\left(\frac{1}{e} - e\right)\cos\theta + e^2\cos^2\theta = 1.$$

10. The distance of the C. of G. of the three rods from the joint is

$$\frac{2Wa\cos\theta + wb\cot\theta}{2W + w}$$

and therefore its height above the level of the pegs is

$$c\cot\theta - \frac{2Wa\cos\theta + wb\cot\theta}{2W + w}.$$

For equilibrium this must be a minimum,

$$\therefore\ -c\operatorname{cosec}^2\theta + \frac{2Wa\sin\theta + wb\operatorname{cosec}^2\theta}{2W + w} = 0,$$

$$\text{i.e. } \sin^3\theta = \frac{(2W + w)c - wb}{2Wa}.$$

11. If the original velocity of the bottom pulley is v, those of the other pulleys are $2v$, $4v$, and that of P is $8v$. If the velocity of the bottom pulley after the jerk is v', that of the other pulley is $2v'$, and that of P is $4v'$. Hence, if T_1, T_2 are the impulsive tensions, we have

$$2T_1 = \frac{W + w}{g}(v' - v),\ 2T_2 - T_1 = \frac{w}{g}(2v' - 4v),\ -T_2 = \frac{P}{g}(4v' - 8v).$$

Multiplying these by 1, 2, 4 and adding, we get

$$(W + w)(v' - v) + w(4v' - 8v) + P(16v' - 32v) = 0,$$

$$\text{i.e. } (W + 5w + 16P)v' = (W + 9w + 32P)v.$$

12. Let α, β, γ be the angles between the strings in equilibrium, so that

$$\frac{\sin\alpha}{m_1} = \frac{\sin\beta}{m_2} = \frac{\sin\gamma}{m_3} = k, \text{ suppose.}$$

Now $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are connected by the identical relation

$$2\Sigma \sin^2\beta \sin^2\gamma - \Sigma \sin^4\alpha \equiv 4\Pi \sin^2\alpha$$

(as is easily obtained from $1 - \Sigma\cos^2\alpha + 2\Pi\cos\alpha \equiv 0$).

Hence

$$k^4\Delta^2 = 4k^6 m_1^2 m_2^2 m_3^2; \therefore k = \frac{\Delta}{2m_1m_2m_3}, \text{ i.e. } \sin\alpha = \frac{\Delta}{2m_2m_3} \text{ etc.}$$

Now let the accelerations of M parallel and perpendicular to CP be f and f', and let T, T' be the tensions of the strings. Then

$$T'\sin\alpha - T\sin\beta = Mf' \text{..................(i).}$$

Also the accelerations of m_1 and m_2 are the same as those of M along the respective strings. Hence

$$m_1 g - T = m_1(-f\cos\beta - f'\sin\beta) \text{...........(ii),}$$

$$m_2 g - T' = m_2(-f\cos\alpha + f'\sin\alpha) \text{...........(iii).}$$

From (i), (ii) and (iii), we have

$$m_1\sin\beta(-f\cos\beta - f'\sin\beta) - m_2\sin\alpha(-f\cos\alpha + f'\sin\alpha) = Mf'.$$

Hence $$\frac{f'}{f} = \frac{m_2\sin\alpha\cos\alpha - m_1\sin\beta\cos\beta}{M + m_1\sin^2\beta + m_2\sin^2\alpha}.$$

Now $$\cos\alpha = \frac{m_2^2 + m_3^2 - m_1^2}{2m_2m_3}, \quad \cos\beta = \frac{m_3^2 + m_1^2 - m_2^2}{2m_3m_1}.$$

Substituting these, and the values of $\sin\alpha$, $\sin\beta$ and reducing, we obtain the required expression.

XXXV.

1. Suppose both circles touch AD at E. Then, from equality of tangents,

$$2AE = AB + AD - BD = AC + AD - CD,$$

$$\therefore \quad AB - AC = BD - CD.$$

Hence the construction :—On AB take $AG = AC$, and on BC take $BF = BG$. Bisect FC in D.

2. P and Q must lie on opposite branches. Let V be the middle point of PQ, $-d^2$ the square of the imaginary semi-diameter in direction CV. Draw the diameter CP' parallel to PQ. Then since CP', CV are the directions of conjugate diameters, $\therefore$ $CP' = d$. Also CP' is perpendicular to the diameter conjugate to CP, $\therefore$ $CP = d$.

Now $$PV^2 = CV^2 + d^2 = CV^2 + CP^2.$$

But $$CP^2 + CQ^2 = 2CV^2 + 2PV^2 = 2PV^2 - 2CP^2 + 2PV^2$$

$$= PQ^2 - 2CP^2,$$

$$\therefore \quad 3CP^2 + CQ^2 = PQ^2.$$

3. Let 4Δ, $4\Delta'$ be the discriminants, so that

$$\Delta = (b+u)^2 - (a+u)(c+u) = b^2 - ac - (a+c-2b)\,u$$

$$= (a+c-2b)\left[\frac{b^2-ac}{a+c-2b} - u\right],$$

and similarly for Δ'.

The necessary condition is that Δ and Δ' shall be both positive. Now if $a+c-2b$, $a'+c'-2b'$ have the same sign, it is evident that u can be chosen so that the quantities in the square brackets have both the same sign, positive or negative, as may be required to make Δ, Δ' both positive.

Again $\dfrac{\Delta}{a+c-2b} - \dfrac{\Delta'}{a'+c'-2b'} = \dfrac{b^2-ac}{a+c-2b} - \dfrac{b'^2-a'c'}{a'+c'-2b'}$.

Now if $a+c-2b>0>a'+c'-2b'$, and Δ, Δ' are both positive, then the left side is positive and therefore the right side must be positive. The other set of conditions is clearly an alternative to this.

4. We have

$$\frac{(2m-1)!}{m!\,(m-1)!} = \frac{1 \,.\, 3 \,.\, 5 \,\ldots\, (2m-1)}{m!} \,.\, 2^{m-1}.$$

We have thus to shew that the index of the highest power of 2 contained in $m!$ is less than $m-1$.

Suppose that m is not a power of 2, and that the highest power of 2 which is less than m is 2^p. Then the index in question is

$$I\left(\frac{m}{2}\right) + I\left(\frac{m}{2^2}\right) + \ldots\ldots\ldots + I\left(\frac{m}{2^p}\right),$$

where $I\,(x)$ denotes the greatest integer in x.

Now if m contains 2^r as a factor, then $I\left(\dfrac{m}{2^r}\right) = \dfrac{m}{2^r}$, but if this is not the case, then $I\left(\dfrac{m}{2^r}\right) < \dfrac{m}{2^r}$.

Hence there must be *some* values of r between 1 and p (although not necessarily *all*) for which this last inequality is true. Hence we must have

$$I\left(\frac{m}{2}\right) + I\left(\frac{m}{2^2}\right) + \ldots\ldots + I\left(\frac{m}{2^p}\right) < \frac{m}{2} + \frac{m}{2^2} + \ldots\ldots + \frac{m}{2^p},$$

$$\text{i.e.} < m - \frac{m}{2^p},$$

and therefore $< m-1$, since $\dfrac{m}{2^p}$ lies between 1 and 2.

If $m = 2^p$, all the above inequalities become equalities, so that in this case the number in question is odd.

5. Let A, B, C, D be the points of contact, and $QRST$ the quadrilateral, Q and O lying in the angle APB.

Let $O\hat{A}P = \lambda$, $O\hat{B}P = \mu$. Then

$$r \sin \lambda = d \sin \alpha, \quad r \sin \mu = d \cos \alpha \quad \text{..............(i).}$$

Also $$A\hat{O}B = \frac{\pi}{2} + \lambda + \mu, \quad B\hat{O}C = \frac{\pi}{2} + \lambda - \mu,$$

$$C\hat{O}D = \frac{\pi}{2} - \lambda - \mu, \quad D\hat{O}A = \frac{\pi}{2} - \lambda + \mu;$$

$$\therefore \quad QT = QA + AT = r(\tan \tfrac{1}{2} AOB + \tan \tfrac{1}{2} AOD)$$

$$= \frac{r \sin \frac{1}{2}(AOB + AOD)}{\cos \frac{1}{2} AOB \cos \frac{1}{2} AOD}$$

$$= \frac{2r \cos \mu}{\cos \lambda - \sin \mu}.$$

Similarly $$SR = \frac{2r \cos \mu}{\cos \lambda + \sin \mu}.$$

But $$\Delta = r(QT + SR) = \frac{4r^2 \cos \lambda \cos \mu}{\cos^2 \lambda - \sin^2 \mu}.$$

Hence, from (i),

$$\Delta^2 (r^2 - d^2)^2 = 16r^4 (r^2 - d^2 \sin^2 \alpha)(r^2 - d^2 \cos^2 \alpha)$$

$$= 16r^6 (r^2 - d^2) + 4r^4 d^4 \sin^2 2\alpha.$$

6. When n is odd, the equation $\dfrac{\tan n\theta}{\tan \theta} = 0$ is equivalent to

$${}^nC_1 - {}^nC_3 \tan^2 \theta + \ldots\ldots + (-1)^{\frac{n-1}{2}} \tan^{n-1} \theta = 0,$$

and its roots are $\tan \dfrac{r\pi}{n}$ $(r = 1, 2, \ldots\ldots n - 1)$.

Hence putting $\cot^2 \theta = x$, the equation becomes

$${}^nC_1 x^{\frac{n-1}{2}} - {}^nC_3 x^{\frac{n-1}{2} - 1} + \ldots\ldots + (-1)^{\frac{n-1}{2}} = 0,$$

its roots being $\cot^2\dfrac{r\pi}{n}\left(r=1, 2, \ldots\ldots \dfrac{n-1}{2}\right)$, and their sum is

$$\frac{{}^nC_3}{{}^nC_1} = \tfrac{1}{6}(n-1)(n-2).$$

7. Take P as the origin, and the tangent at P as the axis of x. The equations to the parabolas can then be put in the forms

$$(\alpha x + \beta y)^2 = 2fy, \qquad (\alpha' x + \beta' y)^2 = 2f'y,$$

and the equation to QP, PR is $f'(\alpha x + \beta y)^2 = f(\alpha' x + \beta' y)^2$,

i.e. $\quad X^2 = \dfrac{f}{f'}\,.\,X'^2$, where $X \equiv \alpha x + \beta y$, $X' \equiv \alpha' x + \beta' y$.

Hence these lines, being of the form $X \pm kX' = 0$, are harmonic conjugates to $X = 0$ and $X' = 0$, i.e. to the diameters at P.

8. The chord of curvature at the point whose eccentric angle is α is

$$\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha - \cos 2\alpha = 0,$$

and the equation to the circle of curvature is

$$\frac{a^2\sin^2\alpha + b^2\cos^2\alpha}{a^2-b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) + \left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - 1\right)$$

$$\left(\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha - \cos 2\alpha\right) = 0.$$

Writing down the corresponding equation for the point $\pi + \alpha$, and subtracting, the radical axis must be

$$\left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - 1\right)\left(\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha - \cos 2\alpha\right)$$

$$-\left(\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha + 1\right)\left(\frac{x}{a}\cos\alpha - \frac{y}{b}\sin\alpha + \cos 2\alpha\right) = 0,$$

reducing to

$$\frac{x}{a}\cos^3 a - \frac{y}{b}\sin^3 a = 0,$$

and since $\tan\theta = \frac{b}{a}\tan a$, this is $\frac{x}{a^4}\cos^3\theta - \frac{y}{b^4}\sin^3\theta = 0$.

9. Any conic through the four points is of the form $l\alpha^2 + m\beta^2 + n\gamma^2 = 0$, with the condition $\Sigma l\xi^2 = 0$. If the conic is a rectangular hyperbola, there is the additional condition $\Sigma l = 0$, so that

$$\frac{l}{\eta^2 - \zeta^2} = \ldots = \ldots,$$

and the centre is given by $\frac{l\alpha}{a} = \frac{m\beta}{b} = \frac{n\gamma}{c}$,

$$\text{i.e.}\quad \frac{\alpha(\eta^2 - \zeta^2)}{a} = \ldots = \ldots.$$

10. If a is the length of each rod, the distance between the point of suspension and the opposite corner is $3a\cos\theta$, ∴ the depth of the centre of the sphere is $3a\cos\theta - r\operatorname{cosec}\theta$.

Also three of the rods have their centres of gravity at depth $\frac{a}{2}\cos\theta$, six of them at depth $\frac{3a}{2}\cos\theta$, and the other three at depth $\frac{5a}{2}\cos\theta$. Hence the equation of virtual work is

$$\tfrac{1}{4}W\,.\,\delta\left(\frac{a}{2}\cos\theta\right) + \tfrac{1}{2}W\,.\,\delta\left(\frac{3a}{2}\cos\theta\right) + \tfrac{1}{4}W\,.\,\delta\left(\frac{5a}{2}\cos\theta\right)$$
$$+ w\,.\,\delta(3a\cos\theta - r\operatorname{cosec}\theta) = 0,$$

whence $\quad -(\tfrac{3}{2}W + 3w)\sin\theta + nw\operatorname{cosec}\theta\cot\theta = 0,$

$$\text{i.e.}\quad \left(\frac{W}{2w} + 1\right)\sin^3\theta = \tfrac{1}{3}n\cos\theta.$$

11. The parameters of the feet of the normals from the point μ to the parabola $y^2 = 4ax$ are given by the equation $m^2 + \mu m + 2 = 0$. They are ∴ real or imaginary according as $\mu^2 \gtrless 8$. Now taking

μ as the point of projection, we have $\mu = \tan \alpha$, and this condition becomes

$$\cos^2 \alpha \lesseqgtr \tfrac{1}{9}, \quad \text{i.e. } \alpha \gtreqless \cos^{-1} \tfrac{1}{3},$$

and if the normals are imaginary, the distance from the point of projection is always increasing: if they are real, then while moving between their feet the particle is approaching the point of projection.

If T be the period of approach, then

$$u \cos \alpha \,.\, T = \text{horizontal distance described} = 2a\,(m_1 - m_2),$$

$$\therefore\ T = \frac{2a}{u \cos \alpha} \sqrt{\mu^2 - 8} = \frac{2a}{u \cos^2 \alpha} \sqrt{1 - 9 \cos^2 \alpha}.$$

But $$a = \frac{u^2 \cos^2 \alpha}{2g}, \quad \therefore\ T = \frac{u}{g} \sqrt{1 - 9 \cos^2 \alpha}.$$

12. Suppose the particle leaves the curve at P, at an angle θ to the horizontal. Then, if v is the velocity at P, and ρ the radius of curvature, $\dfrac{v^2}{\rho} = g \cos \theta$. Hence, since $\rho = 2a \operatorname{cosec}^3 \theta$, and the velocity of projection is $\sqrt{8ga}$, we have

$$8ga - 2g\,(2a \cot \theta) = 2ga \,.\, \frac{\cos \theta}{\sin^3 \theta}$$

leading to

$$\cot^3 \theta + 3 \cot \theta - 4 = 0,$$

$$\text{i.e. } (\cot \theta - 1)(\cot^2 \theta + \cot \theta + 4) = 0.$$

Hence $\cot \theta = 1$, so that P is the extremity of the latus-rectum. The equation to the subsequent path, with P as origin, is

$$y = x \tan \theta - \tfrac{1}{2} g \,.\, \frac{x^2}{v^2 \cos^2 \theta},$$

or, in this case

$$y = x - \frac{x^2}{4a},$$

and referred to the vertex of the original parabola, this is

$$y = x + a - \frac{(x-a)^2}{4a}.$$

The remaining intersection of this parabola with $y^2=4ax$ is the point $(9a, -6a)$. Call this point Q.

The vertex of the trajectory is the point $(3a, 3a)$. Hence, from a figure, the tangent at Q makes with the vertical an angle $\tan^{-1}\left(\dfrac{6a}{18a}\right)=\tan^{-1}(\frac{1}{3})$, and this is also the angle which the tangent at Q to $y^2=4ax$ makes with the horizontal. Calling each of these angles α, we have $\cot 2\alpha=\frac{4}{3}$,

$$\therefore\ \tfrac{1}{4}\cot 2\alpha=\tan\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

But 2α is the inclination of the direction of motion before impact to the normal at Q, and if θ is the inclination after impact, we have

$$\cot\theta=e\cot 2\alpha.$$

Comparing this with (i), we see that if $e=\frac{1}{4}$, then $\cot\theta=\tan\alpha$, i.e. $\theta=\dfrac{\pi}{2}-\alpha$. Hence after impact the direction of motion will be horizontal.

XXXVI.

1. Denote the four circles (in the order mentioned) by S_1, S_2, S_3, S_4. Then if we invert with respect to S_2, S_3 inverts into S_1. Hence S_1, S_2, S_3 are coaxal. Also if we invert with respect to S_1, S_4 inverts into S_3. Hence S_1, S_3, S_4 are coaxal. Thus all four circles are coaxal.

2. If Pascal's Theorem be applied to a hexagon in which successive pairs of vertices coincide, we obtain the theorem that 'The tangents at the vertices of a triangle inscribed in a conic meet the opposite sides in three collinear points.'

Hence the following construction :—Let A, B, C be the three given points, the tangents at A and B being given. Let AC and BC meet the tangents at B and A in E and F respectively, and let AB, EF meet in G. Then CG is the tangent at C.

3. Suppose that $c\,^\circ/_\circ$ per annum payable r times a year is equivalent to $x\,^\circ/_\circ$ per annum payable yearly. Then

$$\left(1+\frac{c}{100r}\right)^r = 1+\frac{x}{100}\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)};$$

and if r is indefinitely great, this becomes

$$e^{\frac{c}{100}} = 1+\frac{x}{100},$$

$$\therefore\ \ x = 100\,(e^{\frac{c}{100}}-1).$$

In general the first three terms in the expansion on the left of (i) are

$$1+\frac{c}{100}+\frac{r\,(r-1)}{2\,!}\cdot\frac{c^2}{(100r)^2} = 1+\frac{c}{100}+\frac{1}{2\,!}\left(1-\frac{1}{r}\right)\cdot\frac{c^2}{100^2},$$

and this agrees with

$$\left(1-\frac{1}{2\,!}\cdot\frac{c^2}{100^2}\cdot\frac{1}{r}\right)e^{\frac{c}{100}}$$

as far as c^2.

Now if $r = 12$, c will be small compared with $100r$. Hence, approximately, (i) becomes

$$\left(1-\frac{1}{2\,!}\cdot\frac{c^2}{100^2}\cdot\frac{1}{12}\right)e^{\frac{c}{100}} = 1+\frac{x}{100},$$

whence $$x = \left(100-\frac{c^2}{2400}\right)e^{\frac{c}{100}}-100.$$

4. By Wilson's Theorem

$$(n-1)\,!+1 = M\,(n),\quad (n-3)\,!+1 = M\,(n-2).$$

Let $$(n-1)\,! = nr-1,\quad (n-3)\,! = (n-2)\,s-1.$$

Then if $P = 4\,(n-3)\,!+n+2$, we have

$$P = 4\,\{(n-2)\,s-1\}+n+2 = (n-2)\,(4s+1) = M\,(n-2).$$

Also $\quad (n-1)(n-2)P = 4(n-1)! + (n-1)(n^2-4)$

$$= 4(nr-1) + (n-1)(n^2-4)$$

$$= n^2(n-1) + 4n(r-1) = M(n).$$

Hence, since n is prime, $P = M(n)$.

5. The first equation may be written

$$\cos\theta\,(\Sigma a\cos\alpha) - \sin\theta\,(\Sigma a\sin\alpha) = 0,$$

$$\text{or}\quad \frac{\cos\theta}{\Sigma a\sin\alpha} = \frac{\sin\theta}{\Sigma a\cos\alpha}.$$

Each of these ratios is equal to

$$\frac{\sin\theta\cos\alpha + \cos\theta\sin\alpha}{\cos\alpha\,(\Sigma a\cos\alpha) + \sin\alpha\,(\Sigma a\sin\alpha)} = \frac{\sin(\theta+\alpha)}{a + b\cos(\alpha-\beta) + c\cos(\alpha-\gamma)},$$

and also to the two similar expressions having $\sin(\theta+\beta)$ and $\sin(\theta+\gamma)$ for numerators. Substituting the proportional values thus found in the second equation, we obtain the result given.

6. If $\tan 15\theta = 0$, and $\tan 5\theta \neq 0$, then

$$3 - \tan^2 5\theta = 0,$$

and putting $\tan\theta = x$, this equation becomes

$$(5x - 10x^3 + x^5)^2 - 3(1 - 10x^2 + 5x^4)^2 = 0,$$

$$\text{i.e.}\quad x^{10} - 95x^8 + 410x^6 - 430x^4 + 85x^2 - 3 = 0 \;\ldots\ldots\ldots\text{(i)},$$

and its roots are

$$\tan\left(\tfrac{1}{15}r\pi\right),$$

r having all values from 1 to 14, excluding multiples of 3. The roots thus include $\tan\dfrac{\pi}{3}$ and $\tan\dfrac{2\pi}{3}$, (for the values $r=5$ and $r=10$), and these are the roots of $x^2-3=0$. Hence, removing the factor (x^2-3) from (i), we obtain the equation required.

7. Taking the chords in the form

$$l(x-h)+m(y-k)=0, \quad l'(x-h)+m'(y-k)=0,$$

the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+[l(x-h)+m(y-k)][l'(x-h)+m'(y-k)]=0$$

must coincide with

$$\left(\frac{1}{b^2}-\frac{1}{a^2}\right)xy+\frac{xY}{a^2}-\frac{yX}{b^2}=0,$$

where (X, Y) is the point of concurrence of the normals.

Comparing these equations, we have

$$ll'+\frac{1}{a^2}=mm'+\frac{1}{b^2}=0, \quad (lh+mk)(l'h+m'k)=1.$$

These give

$$\frac{h}{a^2l}+\frac{k}{b^2m}+\frac{1}{lh+mk}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Combining this with

$$\frac{l}{y-k}=\frac{m}{-(x-h)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

we get the form given, which clearly includes both lines, since the equations (i) and (ii) are both true when l', m' are substituted for l, m.

8. Any one of the conics is of the form

$$S \equiv ax^2+by^2-1+2\lambda(hxy-1)=0.$$

The equation to the tangents from (x', y') is

$$SS'=\{x(ax'+\lambda hy')+y(\lambda hx'+by')-1-2\lambda\}^2.$$

If (x', y') is a focus, this equation satisfies the conditions for a circle. Hence

$$aS'-(ax'+\lambda hy')^2=bS'-(\lambda hx'+by')^2,$$

$$\lambda hS'=(ax'+\lambda hy')(\lambda hx'+by'),$$

whence
$$\left.\begin{aligned}(ab-\lambda^2h^2)(y'^2-x'^2)&=(a-b)(1+2\lambda)\\(ab-\lambda^2h^2)x'y'+\lambda h(1+2\lambda)&=0\end{aligned}\right\} \quad \ldots\ldots\ldots\text{(i)}.$$

From these

$$\lambda = -\frac{x'y'(a-b)}{h(y'^2 - x'^2)},$$

whence $$ab - \lambda^2 h^2 = \frac{(ax'^2 - by'^2)(bx'^2 - ay'^2)}{h^2(y'^2 - x'^2)^2(y'^2 - x'^2)^2}.$$

Substituting these values in either of the equations (i), we obtain the locus required for (x', y').

9. The equation to the parabola is $\Sigma\sqrt{\lambda\alpha} = 0$...(i), with the conditions $\Sigma\frac{\lambda}{l} = 0$...(ii) and $\Sigma\frac{\lambda}{a} = 0$...(iii). The co-ordinates of the focus at infinity are $\left(\frac{\lambda}{a^2}, \frac{\mu}{b^2}, \frac{\nu}{c^2}\right)$ since these satisfy both (i) and $\Sigma a\alpha = 0$, by virtue of (iii). Hence the co-ordinates of the finite focus are $\left(\frac{a^2}{\lambda}, \frac{b^2}{\mu}, \frac{c^2}{\nu}\right)$, and its polar is

$$\Sigma\lambda\alpha(a^2 - b^2 - c^2) = 0 \quad \text{or} \quad \Sigma\lambda \,.\, bc\cos A \,.\, \alpha = 0.$$

Eliminating λ, μ, ν by means of (ii) and (iii), the equation to the directrix is thus

$$\begin{vmatrix} bc\cos A \,.\, \alpha, & ca\cos B \,.\, \beta, & ab\cos C \,.\, \gamma \\ \frac{1}{l}, & \frac{1}{m}, & \frac{1}{n} \\ \frac{1}{a}, & \frac{1}{b}, & \frac{1}{c} \end{vmatrix} = 0,$$

$$\text{i.e.} \quad \Sigma\cos A\left(\frac{b}{m} - \frac{c}{n}\right)\alpha = 0.$$

10. If the radius to the weight w makes an angle θ with the vertical, we have, taking moments about the point of contact,

$$w(a\sin\theta - a\sin\alpha) = W \,.\, a\sin\alpha,$$

$$\text{i.e.} \quad \sin\theta = (W + w)\sin\alpha/w \text{..................(i)}.$$

Hence equilibrium is possible provided

$$w > (W + w)\sin\alpha, \quad \text{i.e. } w > \frac{W\sin\alpha}{1 - \sin\alpha}.$$

The height of the C. of G. of the sphere and weight above the point of contact is

$$a \cos \alpha - \frac{aw}{W+w} \cos \theta.$$

Calling this h, the condition for stability is

$$\frac{\cos \alpha}{h} > \frac{1}{a} + \frac{1}{b},$$

$$\text{i.e.} \quad h < \frac{ab}{a+b} \cos \alpha,$$

whence $\quad (W+w) \cos \alpha - w \cos \theta < (W+w) \dfrac{b}{a+b} \cos \alpha,$

$$\text{i.e.} \quad w \cos \theta > (W+w) \frac{a}{a+b} \cos \alpha,$$

i.e. from (i),

$$w^2 \left[1 - \left(\frac{W+w}{w}\right)^2 \sin^2 \alpha\right] > (W+w)^2 . \frac{a^2}{(a+b)^2} \cos^2 \alpha,$$

$$\text{i.e.} \quad w^2 > (W+w)^2 \left[\sin^2 \alpha + \frac{a^2}{(a+b)^2} \cos^2 \alpha\right],$$

$$\text{i.e.} \quad w > (W+w) \lambda$$

$$\text{or} \quad w > \frac{W\lambda}{1-\lambda}.$$

11. When the rings are at a distance x from the intersection of the rods, let the tension of the string be T. Then $T = \dfrac{x \sin \alpha - a}{a} . \lambda$, and if this is the lowest limiting position, we have

$$\frac{W}{\sin (\alpha - \beta)} = \frac{T}{\cos (\alpha - \beta)},$$

$$\text{i.e.} \quad \frac{x \sin \alpha - a}{a} . \lambda = W \cot (\alpha - \beta)$$

$$\text{or} \quad x = a + W\lambda^{-1} a \operatorname{cosec} \alpha \cot (\alpha - \beta).$$

For the other limiting position, changing the sign of β, we get

$$x' = a + W\lambda^{-1} a \operatorname{cosec} \alpha \cot (\alpha + \beta),$$

and the rings may be anywhere in the distance $x - x'$.

12. The steam-pressure produces an acceleration

$$\frac{\frac{1}{4}\pi d^2 p}{w} \cdot g.$$

Hence the velocity immediately before the blow is given by

$$u^2 = 2l\left(g + \frac{\frac{1}{4}\pi d^2 p}{w} \cdot g\right).$$

If V_1 is the velocity just after the blow, then

$$wu = (W + w) V_1.$$

Also, if F be the resistance, the upward force is $F - (W + w)$, and the velocity V_1 is destroyed by this force in a distance $\frac{a}{n}$.

$$\therefore \ \{F - (W + w)\} \frac{a}{n} = \tfrac{1}{2} \cdot \frac{W + w}{g} \cdot V_1{}^2 = \tfrac{1}{2} \cdot \frac{w^2 u^2}{(W + w) g}$$

$$= \frac{w^2}{W + w} \cdot l\left(1 + \frac{\frac{1}{4}\pi d^2 p}{w}\right),$$

$$\therefore \ F = W + w + \frac{nw}{W + w}(w + \tfrac{1}{4}\pi d^2 p)\frac{l}{a}.$$

XXXVII.

1. Let the letters denote the centres of the various circles, and let a, b, c be the radii of the three given circles, p the perpendicular from A on the radical axis of B, C and P. Then the difference of the squares of the tangents from A to B and P is $2p \cdot BP$, i.e.

$$(AB^2 - b^2) - a^2 = 2p \cdot BP,$$

and similarly for C and P. Hence

$$BP : CP = AB^2 - a^2 - b^2 : AC^2 - c^2 - a^2,$$

and writing down the corresponding ratios for $CQ : AQ$ and $AR : BR$, we see that P, Q, R are collinear. Also the three circles are cut orthogonally by the same circle, viz. the orthogonal circle of A, B, C. Hence the three circles are coaxal.

2. Let p, q, r be the corresponding points on the auxiliary circle. Let the tangents at P, p meet in T, and let QR, qr meet in K. Then T and K are both on the major axis. Then, since pqr is equilateral, the tangent at p is parallel to qr. Hence, if PL, QM are the ordinates of P and Q, we have

$$pL : LT = qM : MK, \quad \text{i.e. } pL : qM = LT : MK.$$

But $$pL : qM = PL : QM, \quad \therefore \; PL : QM = LT : MK.$$

Hence PT is parallel to QR, and similarly for the other tangents.

Thus the normals at P, Q, R are the perpendiculars of the triangle PQR and are therefore concurrent.

3. We have

$$\frac{1-x-y}{(1-x)(1-y)} \equiv \frac{1}{1-x} + \frac{1}{1-y} - \frac{1}{(1-x)(1-y)}$$
$$= (1-x)^{-1} + (1-y)^{-1} - (1-x)^{-1}(1-y)^{-1}.$$

The term $x^n y^n$ will not occur in either of the first two expansions.

The third expansion is

$$-(1+x+x^2+\ldots+x^n+\ldots)(1+y+y^2+\ldots+y^n+\ldots),$$

in which the coefficient of $x^n y^n$ is -1, since the term only occurs once.

4. (i) In the identity

$$1-a_1+a_1(1-a_2)+a_1a_2(1-a_3)+\ldots+a_1a_2\ldots a_{n-1}(1-a_n)$$
$$\equiv 1-a_1a_2\ldots a_n$$

put $$a_1=\frac{n+2}{2}, \quad a_2=\frac{n+4}{4} \ldots, \quad a_{n-1}=\frac{3n-2}{2n-2}, \quad a_n=\frac{3}{2}.$$

(ii) In the identity

$$1+a_1-a_1(1+a_2)+a_1a_2(1+a_3)-\ldots$$
$$+(-1)^{n-1}a_1a_2\ldots a_{n-1}(1+a_n)+(-1)^n a_1a_2\ldots a_n \equiv 1$$

put $$a_1=\frac{n}{x}, \quad a_2=\frac{n-1}{x+1}, \quad a_3=\frac{n-2}{x+2} \ldots, \quad a_n=\frac{1}{x+n-1}.$$

5. Denote the perpendiculars from P on the sides by α, β, γ, those from O by α', β', γ'. Then

$$(\alpha-\alpha')^2+(\beta-\beta')^2+2(\alpha-\alpha')(\beta-\beta')\cos C = OP^2\sin^2 C.$$

Now $\quad \Sigma a\alpha = \Sigma a\alpha' = 2\Delta, \quad \therefore \ \Sigma a(\alpha-\alpha')=0,$

i.e. $2ab(\alpha-\alpha')(\beta-\beta') = c^2(\gamma-\gamma')^2 - a^2(\alpha-\alpha')^2 - b^2(\beta-\beta')^2.$

Substituting and reducing, we find

$$\Sigma a\cos A\,(\alpha-\alpha')^2 = \frac{abc}{4R^2}\,.\,OP^2 = \frac{\Delta}{R}\,.\,OP^2.$$

Now

$$\Sigma a\cos A\,.\,\alpha\alpha' = \Sigma a\cos A\,.\,2R\cos B\cos C\,.\,\alpha$$
$$= 4R\Delta\cos A\cos B\cos C,$$

and

$$\Sigma a\cos A\,.\,\alpha'^2 = 4R^2\cos A\cos B\cos C\,.\,\Sigma a\cos B\cos C$$
$$= 4R^2\cos A\cos B\cos C\,.\,2R\sin A\sin B\sin C$$
$$= 4R\Delta\cos A\cos B\cos C.$$

Hence

$$\Sigma a\cos A\,.\,\alpha^2 = 4R\Delta\cos A\cos B\cos C + \frac{\Delta}{R}\,.\,OP^2.$$

6. We have $\qquad d_r = 2a\sin\frac{r\pi}{2n},$

$$\therefore\ \Pi d_r = 2^{n-1}a^{n-1}\,.\,\prod_1^{n-1}\sin\frac{r\pi}{2n}.$$

Now $\qquad \sin 2n\theta = 2^{2n-1}\,.\,\prod_0^{2n-1}\sin\left(\theta+\frac{r\pi}{2n}\right).$

Also

$$\operatorname*{Lt}_{\theta=0}\frac{\sin 2n\theta}{\sin\theta} = 2n, \quad \text{and} \quad \sin\frac{r\pi}{2n} = \sin\frac{(2n-r)\pi}{2n}.$$

Hence, putting $\theta = 0$,

$$n = 2^{2n-2}\,.\,\prod_1^{n-1}\sin^2\frac{r\pi}{2n},$$

the remaining factor being $\sin\frac{n\pi}{2n} = 1$.

Hence $\qquad \Pi d_r = \sqrt{n}\,.\,a^{n-1}.$

7. If one focus is (x, y), the other is $\left(\frac{b^2}{x}, \frac{b^2}{y}\right)$, and the centre is therefore $\left(\frac{b^2+x^2}{2x}, \frac{b^2+y^2}{2y}\right)$.

But the square of the distance of the centre from the origin
= square of the radius of the director circle $= a^2 + b^2$.

$$\therefore \left(\frac{b^2+x^2}{2x}\right)^2 + \left(\frac{b^2+y^2}{2y}\right)^2 = a^2+b^2,$$

i.e. $y^2(b^2+x^2)^2 + x^2(b^2+y^2)^2 = 4(a^2+b^2)x^2y^2$,

i.e. $(x^2+y^2)(x^2y^2+b^4) = 4a^2x^2y^2$.

8. Let $lx+my=1$ be the directrix. Then the line joining the origin to its intersection with the tangent at α is

$$\left(l-\frac{\cos\alpha}{a}\right)x + \left(m-\frac{\sin\alpha}{b}\right)y = 0 \text{ ....(i)},$$

and the line joining the origin to α is

$$\frac{x}{a\cos\alpha} - \frac{y}{b\sin\alpha} = 0 \text{ ....(ii)}.$$

If the origin is a focus, (i) and (ii) are perpendicular;

$$\therefore \frac{l}{a\cos\alpha} - \frac{1}{a^2} - \frac{m}{b\sin\alpha} + \frac{1}{b^2} = 0,$$

$$\text{i.e. } \frac{l}{a\cos\alpha} - \frac{m}{b\sin\alpha} = \frac{1}{a^2} - \frac{1}{b^2} \text{ ....(iii)}.$$

If the conics osculate at α, this relation must also be true for the consecutive tangent;

$$\therefore \frac{l}{a}\cdot\frac{\sin\alpha}{\cos^2\alpha} + \frac{m}{b}\cdot\frac{\cos\alpha}{\sin^2\alpha} = 0 \text{ ....(iv)}.$$

From (iii) and (iv),

$$\frac{l}{a\cos^3\alpha} = \frac{m}{-b\sin^3\alpha} = \frac{1}{a^2} - \frac{1}{b^2}.$$

9. Let X, Y be the horizontal and vertical components of the pressure at the joint, R the pressure on the fulcrum.

Then, resolving horizontally and vertically, and taking moments about the joint, we have

$$X = R \sin \alpha, \quad Y + R \cos \alpha = P,$$

$$R \cdot \frac{1}{n} = P \cos \alpha.$$

From these

$$X = Pn \sin \alpha \cos \alpha, \quad Y = P(1 - n \cos^2 \alpha),$$

$$\therefore \; X^2 + Y^2 = P^2 [n^2 \sin^2 \alpha \cos^2 \alpha + (1 - n \cos^2 \alpha)^2]$$

$$= P^2 (1 - 2n \cos^2 \alpha + n^2 \cos^2 \alpha)$$

$$= P^2 \{(n-1)^2 \cos^2 \alpha + \sin^2 \alpha\}.$$

10. Let A, A' be the upper and lower ends of the rod, and let the directions of limiting friction meet in O. Then O must be vertically above G, the centre of the rod.

Hence

$$\angle OAG = \frac{\pi}{2} - (\alpha - \theta - \lambda), \quad AOG = \alpha - \lambda,$$

$$OA'G = \frac{\pi}{2} - (\alpha' + \theta + \lambda'), \quad A'OG = \alpha' + \lambda'.$$

But

$$\frac{OG}{GA'} = \frac{OG}{GA}, \quad \therefore \; \frac{\cos(\alpha' + \theta + \lambda')}{\sin(\alpha' + \lambda')} = \frac{\cos(\alpha - \theta - \lambda)}{\sin(\alpha - \lambda)},$$

i.e. $\cot(\alpha' + \lambda') \cos\theta - \sin\theta = \cot(\alpha - \lambda)\cos\theta + \sin\theta$,

whence $\qquad 2 \tan\theta = \cot(\alpha' + \lambda') - \cot(\alpha - \lambda).$

Also the limiting frictions, being the resultants of R, μR and R', $\mu' R'$, are $R \sec\lambda$ and $R' \sec\lambda'$. These are in equilibrium with W. Hence

$$\frac{R \sec\lambda}{\sin A'OG} = \frac{R' \sec\lambda'}{\sin AOG} = \frac{W}{\sin AOA'}.$$

11. Let V_1, V_2 be the velocities of one of the impinging balls parallel to the sides of the square at the end of the period of

compression, I and I' the impulses, u the striking velocity. Then

$$m(u\cos\alpha - V_2) = I = mV_2 \quad\text{..................(i)},$$

since the velocities of the spheres along the line of centres are the same.

Let V_1', V_2' be the final velocities. Then

$$m(V_2 - V_2') = eI,$$

$$\therefore\; m(u\cos\alpha - V_2') = (1+e)\,I = (1+e)\,.\,\tfrac{1}{2}mu\cos\alpha, \text{ by (i)},$$

$$\therefore\; V_2' = \tfrac{1}{2}(1-e)\,u\cos\alpha.$$

Also, by Newton's Law, $V_1' = eu\sin\alpha$,

$$\therefore\; \frac{V_1'}{V_2'} = \frac{2e\tan\alpha}{1-e}.$$

12. Suppose the bead in limiting equilibrium at an angular distance θ from the highest point. Then

$$mg\cos\theta - R = ma\omega^2, \quad mg\sin\theta = \mu R,$$

$$\therefore\; mg(\cos\theta - \sin\theta\cot\lambda) = ma\omega^2,$$

$$\text{i.e. } \sin(\lambda-\theta) = \frac{\omega^2 a}{g}\sin\lambda,$$

or

$$\lambda - \theta = \sin^{-1}\left(\frac{\omega^2 a}{g}\sin\lambda\right),$$

$$\text{i.e. } \theta = \lambda - \sin^{-1}\left(\frac{\omega^2 a}{g}\sin\lambda\right).$$

If the bead slip at all, it will be at this point.

XXXVIII.

1. Denote the two circles by S, T and their inverses with respect to a circle, centre O, by S', T'. Let α, β, γ, δ be the inverse points of A, B, C, D. Then, since A, B, β, α are concyclic, the triangles OAB, $O\beta\alpha$ are similar. Hence

$$\frac{AB}{\alpha\beta} = \frac{OA}{O\beta} = \frac{OB}{O\alpha}, \quad \therefore\; \frac{AB^2}{\alpha\beta^2} = \frac{OA\,.\,OB}{O\alpha\,.\,O\beta}.$$

From this and similar results it is clear that

$$(ABCD) = \frac{AB \cdot CD}{AD \cdot CB} = \frac{\alpha\beta \cdot \gamma\delta}{\alpha\delta \cdot \gamma\beta} \quad \text{..............(i).}$$

Now α, β, γ, δ lie on a circle passing through O and cutting S', T' orthogonally. Call this circle U. Let the radical axis of S', T' meet U in O'. Then if we invert with respect to a circle, centre O', and cutting S', T' orthogonally, the inverse of $A'B'C'D'$ will be the circle U. Hence $A'\alpha$, $B'\beta$, etc. meet in O', and we have, as in (i),

$$(A'B'C'D') = \frac{\alpha\beta \cdot \gamma\delta}{\alpha\delta \cdot \gamma\beta}.$$

Hence $$(ABCD) = (A'B'C'D').$$

2. Let S' be the second focus of the fixed ellipse, S'' that of the revolving ellipse, and let P and Q be the common points. Then, since the curves have equal major axes,

$$\therefore \; SP + S'P = SP + S''P, \quad \text{i.e. } S'P = S''P,$$

and similarly $S'Q = S''Q$. Hence PQ must bisect $S'S''$ at right angles. Let Y' be the point of bisection and draw SY perpendicular to PQ, and SZ parallel to PQ, meeting $S'S''$ in Z. Then

$$SS''^2 - SS'^2 = S''Z^2 - S'Z^2 = 4Y'S' \cdot YZ = 4SY \cdot S'Y',$$

i.e. $SY \cdot S'Y'$ is constant, and therefore the envelope of PQ is a conic with foci S, S'.

3. If ω be an imaginary cube root of unity, we have at once

$$A + B + C = (1 + x)^n, \quad A + \omega B + \omega^2 C = (1 + \omega x)^n,$$
$$A + \omega^2 B + \omega C = (1 + \omega^2 x)^n.$$

Multiplying these together, the result follows.

4. If u_n denote either p_n or q_n, then

$$u_n = (4n - 1)\, u_{n-1} - (2n - 2)(2n - 1)\, u_{n-2},$$
$$\text{i.e. } u_n - 2n \cdot u_{n-1} = (2n - 1)\left[u_{n-1} - 2(n - 1)\, u_{n-2}\right],$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$
$$u_3 - 6u_2 = 5(u_2 - 4u_1),$$
$$\therefore \; u_n - 2n \cdot u_{n-1} = (2n - 1)(2n - 3) \dots 5(u_2 - 4u_1).$$

This gives $$p_n - 2n \, . \, p_{n-1} = 3 \, . \, 5 \, . \, 7 \, \ldots \, (2n-1),$$
$$q_n - 2n \, . \, q_{n-1} = 0.$$

The second equation gives

$$q_n = 2 \, . \, 4 \, . \, 6 \, \ldots \, 2n,$$

and the first is satisfied by

$$3 \, . \, 5 \, . \, 7 \, \ldots \, (2n+1) + k \, . \, 2 \, . \, 4 \, . \, 6 \, \ldots \, 2n,$$

where k is any constant. But here

$$p_2 = 7, \quad \therefore \; 15 + 8k = 7, \quad \text{i.e. } k = -1,$$
$$\therefore \; p_n = 1 \, . \, 3 \, . \, 5 \, \ldots \, (2n+1) - 2 \, . \, 4 \, . \, 6 \, \ldots \, 2n.$$

5. If $\tan\theta = t$, the equation becomes

$$a \, . \, \frac{3t - t^3}{1 - 3t^2} + b \, . \, \frac{2t}{1 - t^2} + ct = d,$$

or, on reduction,

$$(a + 3c)\, t^5 - 3dt^4 - (4a + 6b + 4c)\, t^3 + 4dt^2 + (3a + 2b + c)\, t - d = 0,$$

an equation of the fifth degree in t.

If the roots are $\tan\alpha$, etc., then

$$\tan(\Sigma\alpha) = \frac{s_1 - s_3 + s_5}{1 - s_2 + s_4},$$

where s_r is the sum of the products of the roots, r together. Hence

$$\tan(\Sigma\alpha) = \frac{3d + 4d + d}{(a + 3c) + (4a + 6b + 4c) + (3a + 2b + c)}$$
$$= \frac{d}{a + b + c}.$$

6. If $11\theta = n\pi$, then $\tan 8\theta + \tan 3\theta = 0$, and, putting $\tan^2\theta = x$, we have

$$\tan 2\theta = \frac{2\tan\theta}{1 - x},$$

whence $$\tan 4\theta = \frac{4(1 - x)\tan\theta}{(1 - x)^2 - 4x},$$

$$\tan 8\theta = \frac{8(1 - x)(1 - 6x + x^2)\tan\theta}{(1 - 6x + x^2)^2 - 16x(1 - x)^2},$$

and the equation becomes

$$\frac{8(1-x)(1-6x+x^2)}{1-28x+70x^2-28x^3+x^4}+\frac{3-x}{1-3x}=0,$$

whence $$11-165x+\ldots-x^5=0.$$

The roots of this are $\tan^2\frac{r\pi}{11}$ $(r=1, 2, \ldots, 5)$, and the sum of their reciprocals is $\frac{165}{11}=15$.

7. Let m and m' be the extremities of the chord. Then θ is the angle between the lines

$$x-my+am^2=0 \text{ and } 2x-(m+m')\,y+2amm'=0,$$

$$\therefore\ \tan\theta=\frac{m-m'}{2+m\,(m+m')}.$$

So also $$\tan\phi=\frac{m-m'}{2+m'\,(m+m')},$$

$$\therefore\ \cot\theta+\cot\phi=\frac{4+(m+m')^2}{m-m'},$$

$$\cot\theta-\cot\phi=m+m'.$$

Now, if l be the length of the chord,

$$l^2=a^2\,(m^2-m'^2)^2+4a^2\,(m-m')^2=a^2\,(m-m')^2\left[(m+m')^2+4\right],$$

$$\therefore\ \frac{l^2}{a^2}=\left[4+(\cot\theta-\cot\phi)^2\right]^3/(\cot\theta+\cot\phi)^2.$$

8. The equation to PQ is $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=\cos\beta$, and therefore the co-ordinates of T are

$$\left(\frac{a\cos\alpha}{\cos\beta},\ \frac{b\sin\alpha}{\cos\beta}\right),$$

and these are in the ratio $a\cos\alpha : b\sin\alpha$. But the co-ordinates of R are clearly in the same ratio as those of T. Hence, since R is on the ellipse, its co-ordinates must be $(a\cos\alpha,\ b\sin\alpha)$.

We also have

$$p^2 = \frac{\cos^2 \beta}{\dfrac{\cos^2 \alpha}{a^2} + \dfrac{\sin^2 \alpha}{b^2}} \quad \text{and} \quad d^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha,$$

$$\therefore \quad \frac{\cos^2 \beta}{p^2} = \frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2}$$

$$= \frac{1}{a^2} + \frac{1}{b^2} - \left(\frac{\sin^2 \alpha}{a^2} + \frac{\cos^2 \alpha}{b^2}\right)$$

$$= \frac{1}{a^2} + \frac{1}{b^2} - \frac{d^2}{a^2 b^2}.$$

9. The tangential equation of the conic is

$$-bl^2 - am^2 + (ab - h^2)n^2 + 2hlm = 0,$$

and therefore that of any confocal is

$$(\lambda + b)l^2 + (\lambda + a)m^2 - (ab - h^2)n^2 - 2hlm = 0,$$

and the Cartesian equation corresponding to this is

$$(a + \lambda)(ab - h^2)x^2 + (b + \lambda)(ab - h^2)y^2 + 2h(ab - h^2)xy$$
$$= (a + \lambda)(b + \lambda) - h^2,$$

i.e. $\lambda(ab - h^2)(x^2 + y^2) + (ab - h^2)(ax^2 + 2hxy + by^2)$
$$= (a + \lambda)(b + \lambda) - h^2.$$

To obtain the form given, put $\lambda = \dfrac{ab - h^2}{\lambda'}$.

10. On a vertical line take equal lengths A_1A_2, A_2A_3, etc., each representing the weight W. Then if lines be drawn from A_1, A_2, A_3, ... parallel to the successive strings, it follows from the equilibrium of the successive weights that they will all meet in a point O. Let OK be the perpendicular from O on the vertical line. Then K will be the middle point of $A_{n+1}A_{n+2}$, so that

$$A_rK = (n + 1 - r)W + \tfrac{1}{2}W,$$

and OK represents the horizontal tension.

But if the strings make angles θ_1, θ_2, ... with the horizontal, then projecting horizontally

$$\frac{a}{l} = \cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_{n+1}.$$

Now $$\cos\theta_r = \frac{OK}{OA_r} = \frac{T}{\{T^2 + (n+1-r+\frac{1}{2})^2 W^2\}^{\frac{1}{2}}}$$

$$= \left\{1 + (n+1-r+\tfrac{1}{2})^2 \frac{W^2}{T^2}\right\}^{-\frac{1}{2}},$$

whence the given result follows, since r ranges from 1 to $n+1$.

11. Let P be the impulse of the blow, T, T' the impulsive tensions of the strings, V_1 the final velocity of the impinging sphere, and therefore also the velocity of the sphere struck along the line of centres, since the system is inelastic. We have then

$$P - T\cos\theta - T'\sin\theta = mV_1, \quad P = M(V - V_1),$$

θ being the inclination of the direction of M to one of the strings.

Also the velocities of the other two spheres along the respective strings are $V_1\cos\theta$ and $V_1\sin\theta$.

$$\therefore\quad T = mV_1\cos\theta, \quad T' = mV_1\sin\theta.$$

Hence

$$M(V - V_1) - mV_1(\cos^2\theta + \sin^2\theta) = mV_1,$$

whence $$V_1 = \frac{MV}{M + 2m}.$$

12. Let the ball strike with velocity u, and let v, v' be the velocities after the first impact. Then

$$v - v' = -\frac{m}{M}\cdot u, \quad mv + Mv' = mu,$$

whence $$v = 0, \quad v' = \frac{m}{M}\cdot u.$$

They will impinge the second time with these velocities, v' being in the opposite direction. Let V, V' be the velocities in this direction after the second impact. Then

$$V - V' = -\frac{m}{M}(v - v') = \frac{m^2}{M^2} \cdot u,$$

$$mV + MV' = mv + Mv' = mu,$$

whence $$V = \frac{m}{M} \cdot u, \quad V' = \frac{m(M-m)}{M^2} \cdot u,$$

and the heights to which the balls rise are in the ratio $V^2 : V'^2$.

XXXIX.

1. Let A and B be the centres of the circles, a and b their radii, and let AX $(=p)$ and BY $(=q)$ be the perpendiculars from A and B on the line.

Then since (PP', QQ') is harmonic,

$$XP^2 = XQ \, . \, XQ',$$

i.e. $$a^2 - p^2 = BX^2 - b^2 = XY^2 + q^2 - b^2.$$

$$\therefore \quad a^2 + b^2 = p^2 + q^2 + XY^2 \quad \text{.........................(i).}$$

Also clearly $$AB^2 = (p-q)^2 + XY^2.$$

$$\therefore \quad a^2 + b^2 - AB^2 = 2pq.$$

Hence, from (i),

$$2(a^2 + b^2) - AB^2 = (p+q)^2 + XY^2 \quad \text{............(ii).}$$

But if O, L be the middle points of AB, XY, then

$$4OX^2 = 4OL^2 + 4XL^2 = (p+q)^2 + XY^2.$$

Hence, from (ii),

$$4OX^2 = 2(a^2 + b^2) - AB^2,$$

i.e. OX is constant, and clearly $OX = OY$, so that X and Y both lie on a circle with centre O.

2. Let BE, CF be medians of the triangle ABC, and let BC touch the parabola at H. Draw BO parallel to AQ to cut PQ in O. Then since two tangents are cut proportionally by any other tangent,

$$PO : OQ = PB : BA = AC : CQ,$$

$$\therefore \ CO \text{ is parallel to } AP.$$

Now considering the two triads of points POQ and BHC, the three intersections of cross-joins are collinear, i.e. the intersections of BO, PH and CO, QH are collinear with R. Hence the polars of these three points are concurrent. But the polar of the intersection of BO, PH is the line through B conjugate to BO, and since the range $(AEC\infty)$ is harmonic, this line is BE. Similarly the polar of the intersection of CO, QH is CF. Hence the polar of R passes through the intersection of BE and CF, i.e. the centroid of ABC.

3. Let $x^{\frac{1}{5}}$, $y^{\frac{1}{5}}$, $z^{\frac{1}{5}}$ be the roots of

$$t^3 + qt - r = 0 \quad \text{.........................(i).}$$

Then if we eliminate t between this equation and $t^5 = u$, we shall obtain an equation in u whose roots are x, y, z.

Now, from (i), $\qquad u^3 = -(qt - r)^5.$

Also

$$(qt - r)^5 \equiv q^5u - r^5 - 5qrt\,[(qt - r)^3 + qtr\,(qt - r)]$$

$$= q^5u - r^5 - 5qrt\,(-t^9 - qrt^4) \qquad \text{by (i)}$$

$$= q^5u - r^5 + 5qru^2 + 5q^2r^2u.$$

Hence $\qquad u^3 + 5qru^2 + (5q^2r^2 + q^5)\,u - r^5 = 0.$

Thus $\qquad \alpha = -5qr, \quad \beta = 5q^2r^2 + q^5, \quad \gamma = r^5,$

whence $5\beta - \alpha^2 = 5q^5$, and the result easily follows.

4. The given series is $3\,!$ times the coefficient of x^3 in

$$\sum_1^n (-1)^{r-1} \,.\, {}^nC_r \,.\, 2^{n-r} \,.\, e^{rx} \equiv 2^n - (2 - e^x)^n.$$

Now $$(2-e^x)^n = \left\{1-\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ \dots\right)\right\}^n,$$

and the coefficient of x^3 in this is

$$-\frac{n}{6}+\frac{n(n-1)}{2}-\frac{n(n-1)(n-2)}{6}=\frac{-n(n^2-6n+6)}{6}.$$

Hence the value of the given series is $n(n^2-6n+6)$.

5. Draw the perpendiculars OL, OM, ON on the sides of ABC, and let these be α, β, γ. Then

$$MN = OA \sin A = \frac{x}{a} \cdot \sqrt{2R} \cdot \frac{a}{2R} = \frac{x}{\sqrt{2R}}.$$

Hence $$\triangle LMN = \frac{\sqrt{\sigma(\sigma-x)(\sigma-y)(\sigma-z)}}{2R},$$

i.e. $$R \,.\, \Sigma\beta\gamma \sin A = \sqrt{\sigma(\sigma-x)(\sigma-y)(\sigma-z)} \;\dots\dots\dots\text{(i)}.$$

Again

$$\begin{aligned}
\Sigma a^2(y^2+z^2-x^2) &= \Sigma x^2(b^2+c^2-a^2) = 2\Sigma x^2 bc\cos A \\
&= 4R \,.\, \Sigma(\beta^2+\gamma^2+2\beta\gamma\cos A)\, bc\cos A \\
&= 4R\,(\Sigma a^2\alpha^2 + 2\Sigma bc\beta\gamma\cos^2 A) \\
&= 4R\,\{(\Sigma a\alpha)^2 - 2\Sigma bc\beta\gamma\sin^2 A\} \\
&= 4R\,(4\Delta^2 - 4\Delta \,.\, \Sigma\beta\gamma\sin A).
\end{aligned}$$

Hence

$$\begin{aligned}
R \,.\, \Sigma a^2(y^2+z^2-x^2) - a^2b^2c^2 &= -16R^2\Delta \,.\, \Sigma\beta\gamma\sin A \\
&= -4abc \,.\, \sqrt{\sigma(\sigma-x)(\sigma-y)(\sigma-z)} \quad \text{by (i)}.
\end{aligned}$$

6. If O is the centre, and $P\hat{O}A_n = \alpha$, we have

$$PA_r^2 = a^2 + c^2 - 2ac\cos\left(\alpha + \frac{2r\pi}{n}\right).$$

Now $$\sum_1^n \cos\left(\alpha + \frac{2r\pi}{n}\right) = 0,$$

since the expression for the sum contains the factor $\sin\left(n\,.\,\frac{\pi}{n}\right)$.

$$\therefore \quad \Sigma PA_r^2 = n\,(a^2 + c^2).$$

Again, $PA_r^2\,.\,PA_s^2$

$$= (a^2 + c^2)^2 - 2ac\,(a^2 + c^2)\left[\cos\left(\alpha + \frac{2r\pi}{n}\right) + \cos\left(\alpha + \frac{2s\pi}{n}\right)\right]$$

$$+ 4a^2c^2\cos\left(\alpha + \frac{2r\pi}{n}\right)\cos\left(\alpha + \frac{2s\pi}{n}\right).$$

In the sum, the middle term will vanish, since each cosine evidently enters the same number of times.

Also in

$$\Sigma\Sigma\cos\left(\alpha + \frac{2r\pi}{n}\right)\cos\left(\alpha + \frac{2s\pi}{n}\right),$$

the coefficient of

$$\cos\left(\alpha + \frac{2r\pi}{n}\right) \quad \text{is} \quad \sum_1^n \cos\left(\alpha + \frac{2r\pi}{n}\right) - \cos\left(\alpha + \frac{2r\pi}{n}\right),$$

$$\text{i.e.} \quad -\cos\left(\alpha + \frac{2r\pi}{n}\right).$$

Hence the corresponding term is $-\cos^2\left(\alpha + \frac{2r\pi}{n}\right)$, and each term is now counted twice.

Thus the sum is

$$-\tfrac{1}{2}\Sigma\cos^2\left(\alpha + \frac{2r\pi}{n}\right) = -\tfrac{1}{4}\left[n + \sum_1^n \cos\left(2\alpha + \frac{4r\pi}{n}\right)\right] = -\tfrac{1}{4}n.$$

Also the number of terms in

$$\Sigma\Sigma PA_r^2\,.\,PA_s^2 \quad \text{is} \quad {}^nC_2 = \frac{n\,(n-1)}{2},$$

and hence its value is

$$\frac{n\,(n-1)}{2}\,(a^2 + c^2)^2 - na^2c^2.$$

7. The centre of curvature at ϕ is

$$\left(\frac{a^2-b^2}{a}\cos^3\phi, \quad -\frac{a^2-b^2}{b}\sin^3\phi\right),$$

and at $\left(\frac{\pi}{2}+\phi\right)$ is $\left(-\frac{a^2-b^2}{a}\sin^3\phi, \quad -\frac{a^2-b^2}{b}\cos^3\phi\right)$.

Hence if (x, y) is the middle point of the line joining them, we have

$$2ax = (a^2-b^2)(\cos^3\phi - \sin^3\phi),$$
$$2by = -(a^2-b^2)(\cos^3\phi + \sin^3\phi),$$

i.e. $ax + by = -(a^2-b^2)\sin^3\phi, \qquad ax - by = (a^2-b^2)\cos^3\phi.$

$$\therefore \quad (ax+by)^{\frac{2}{3}} + (ax-by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}.$$

8. The line

$$\frac{x-x'}{\cos\theta} = \frac{y-y'}{\sin\theta} = r$$

meets the conic $u = 1$ where

$$a(r\cos\theta + x')^2 + 2h(r\cos\theta + x')(r\sin\theta + y') + b(r\sin\theta + y')^2 = 1,$$

i.e. $r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$

$$+ 2r\left[(ax'+hy')\cos\theta + (hx'+by')\sin\theta\right] + u' - 1 = 0.$$

If the values of r are equal, we have, suppressing the accents,

$$\left[(ax+hy)\cos\theta + (hx+by)\sin\theta\right]^2$$
$$= (u-1)(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta),$$

i.e. $(Cy^2-a)\cos^2\theta - 2(Cxy+h)\sin\theta\cos\theta$

$$+ (Cx^2-b)\sin^2\theta = 0 \;\ldots\ldots\ldots\ldots\text{(i)}.$$

If t be the value of r corresponding to either value of θ, then

$$t^2 = \frac{u-1}{a\cos^2\theta + \ldots + \ldots} = \frac{u-1}{C(y\cos\theta - x\sin\theta)^2}$$
$$= \frac{(u-1)(1+\cot^2\theta)}{C(y\cot\theta - x)^2} \;\ldots\ldots\ldots\text{(ii)}.$$

Now let θ_1, θ_2 be the roots of (i). Then

$$(1+\cot^2\theta_1)(1+\cot^2\theta_2) = \frac{\{(Cx^2-b)-(Cy^2-a)\}^2 + 4(Cxy+h)^2}{(Cy^2-a)^2},$$

and
$$(y\cot\theta_1 - x)(y\cot\theta_2 - x) = -\frac{u}{Cy^2 - a}.$$

Substituting these in the product $t_1 t_2$ given by (ii), we obtain the required expression.

9. Let $(\alpha', \beta', \gamma')$ be any point on the axis. Then the line joining this to the infinitely distant centre whose co-ordinates are $\left(\frac{l}{a^2}, \frac{m}{b^2}, \frac{n}{c^2}\right)$ must be perpendicular to the polar.

But these lines are respectively

$$\Sigma\alpha\left(\beta' \cdot \frac{n}{c^2} - \gamma' \cdot \frac{m}{b^2}\right) = 0,$$

and
$$\Sigma l\alpha\,(l\alpha' - m\beta' - n\gamma') = 0.$$

Writing down the condition of perpendicularity, the coefficient of α'^2 is

$$lmn\left(\frac{1}{c^2} - \frac{1}{b^2}\right) - \left(\frac{ln^2}{c^2} - \frac{lm^2}{b^2}\right)\cos A - \frac{l^2 m}{b^2}\cos B + \frac{l^2 n}{c^2}\cos C,$$

and, using $\Sigma\frac{l}{a} = 0$, this reduces to

$$l\left(\frac{c}{b^3} \cdot m^2 - \frac{b}{c^3} \cdot n^2\right),$$

and similarly for β'^2, γ'^2.

Now the equation satisfied by $(\alpha', \beta', \gamma')$ gives two straight lines, one of which is the line at infinity, viz. $\Sigma a\alpha = 0$. Hence the other must be

$$\Sigma\frac{l}{a}\left(\frac{c}{b^3} \cdot m^2 - \frac{b}{c^3} \cdot n^2\right)\alpha = 0,$$

or
$$\Sigma\frac{a^2\alpha}{l}\left(\frac{b^4}{m^2} - \frac{c^4}{n^2}\right) = 0.$$

10. Suppose the two upper sides make an angle θ with the vertical. Let a be the length of a rod, and let $BE = x$, so that

$$x^2 = a^2 + 4a^2\sin^2\theta \quad\text{.......................(i).}$$

For a small symmetrical vertical displacement, the equation of virtual work is

$$2T\,.\,\delta x + 2W\,.\,\delta\left(\tfrac{1}{2}a\cos\theta\right) + 2W\,.\,\delta\left(\tfrac{1}{2}a + a\cos\theta\right) + 2W\,.\,\delta\left(a + \tfrac{3}{2}a\cos\theta\right) = 0,$$

$$\text{i.e.}\quad 2T\,.\,\delta x = 6W\,.\,a\sin\theta\,.\,\delta\theta.$$

But, from (i), $\quad x\,\delta x = 4a^2\sin\theta\cos\theta\,.\,\delta\theta,$

$$\therefore\quad T = \frac{3x}{4a\cos\theta}\,.\,W.$$

But, if the hexagon is regular,

$$x = 2a \quad\text{and}\quad \cos\theta = \tfrac{1}{2}. \qquad \therefore\quad T = 3W.$$

11. Since m_1 is at rest, the tension of the lower string is $m_1 g$, and therefore the acceleration of m_2 downwards is

$$(m_2 - m_1)\,g/m_2.$$

If f is the acceleration of M upwards, T' the tension of the upper string, then

$$T' - Mg = Mf,$$

$$T' = 2m_1 g, \quad \text{since the pulley is weightless.}$$

Hence $$f = \frac{2m_1 - M}{M}\,.\,g.$$

But evidently the acceleration of m_2 is twice that of the pulley, since m_1 is at rest.

$$\therefore\quad \frac{m_2 - m_1}{m_2} = \frac{2\,(2m_1 - M)}{M},$$

whence $$M = \frac{4m_1 m_2}{3m_2 - m_1}.$$

The velocity of M after T seconds is fT upwards, and therefore in the next t seconds it will rise a distance

$$fTt - \tfrac{1}{2}gt^2,$$

since its acceleration is g downwards. Hence the pulley falls this distance, and in the same time m_1 falls a distance $\frac{1}{2}gt^2$. But when the string is again tight, m_1 will have fallen twice as far as the pulley.

$$\therefore\quad 2\left(fTt - \tfrac{1}{2}gt^2\right) = \tfrac{1}{2}gt^2,$$

$$\text{i.e.}\quad t = \tfrac{4}{3}\frac{fT}{g} = \tfrac{2}{3}\frac{m_2 - m_1}{m_2}\,.\,T.$$

12. Suppose the plane of projection makes an angle α with AB. Then the initial velocities are $V\cos\theta\cos\alpha$ parallel to AB, $V\cos\theta\sin\alpha$ perpendicular to AB, and $V\sin\theta$ vertically, where $V^2 = 2gh$.

The first of these becomes $eV\cos\theta\cos\alpha$ after the first impact, and $e^2V\cos\theta\cos\alpha$ after the third.

Therefore the whole time of flight is

$$\frac{c}{V\cos\theta\cos\alpha} + \frac{a}{eV\cos\theta\cos\alpha} + \frac{a-c}{e^2V\cos\theta\cos\alpha} = t.$$

Similarly, considering the motion perpendicular to AB,

$$\frac{a}{V\cos\theta\sin\alpha} + \frac{a}{eV\cos\theta\sin\alpha} = t.$$

Eliminating α from these, we get

$$t^2 = \frac{1}{V^2\cos^2\theta}\left[\left(c + \frac{a}{e} + \frac{a-c}{e^2}\right)^2 + \left(a + \frac{a}{e}\right)^2\right] \text{.......(i).}$$

But since the vertical velocity is unaltered by any impact, t is also given by

$$V\sin\theta\,.\,t - \tfrac{1}{2}gt^2 = 0 \text{ ....(ii).}$$

Equating the values of t given by (i) and (ii), the result follows.

XL.

1. The triangle EFG is self-conjugate for the circle, and therefore its orthocentre O must be the centre. Hence, OFN being the perpendicular on EG, we have $NF\,.\,ON = EN\,.\,NG$, i.e. if R is the radius, $EN\,.\,NG = ON^2 - R^2$.

$$\therefore\quad EN\,.\,EG = EN^2 + ON^2 - R^2 = OE^2 - R^2 = EA\,.\,ED.$$

Hence N is on the circle ADG, and similarly it is on the circle ABE.

2. Let the straight line cut AB, CD in P and P' and AC, BD in Q, Q'. Then all conics through the four points cut the straight line in pairs of points belonging to the involution determined by the pairs PP', QQ'. Hence if a conic through $ABCD$ touches the line, the point of contact must be one of the double points of the involution. Hence two such conics (real or imaginary) can be drawn.

If the line cuts all the sides of ABC externally and D is inside ABC, then the segments PP', QQ' must overlap, and therefore the centre O of the involution must be between P and P'. Hence OP, OP' have opposite signs, and therefore the double points are imaginary, i.e. the conics are not real in this case. In the other case, they may be either real or imaginary.

3. (i) The equation is

$$(1+x)^2(1+x^2)^2+3(1+x^2)(1+x^4)=0,$$

so that either

$$1+x^2=0 \quad \text{or} \quad (1+x)^2(1+x^2)+3(1+x^4)=0.$$

Putting $x+\dfrac{1}{x}=y$ in the latter equation, we obtain

$$(y+2)y+3(y^2-2)=0,$$

$$\text{i.e.}\ \ 2y^2+y-3=0,$$

whence $$y=1 \quad \text{or} \quad -\tfrac{3}{2}.$$

Hence either $\quad x^2-x+1=0 \quad \text{or} \quad 2x^2+3x+2=0.$

The six roots of the equation are thus

$$\pm i,\quad \tfrac{1}{2}(1\pm\sqrt{3}\,.\,i),\quad \tfrac{1}{4}(-3\pm\sqrt{7}\,.\,i).$$

(ii) We have

$$(\Sigma x)^2=21-20=1. \qquad \therefore\ \Sigma x=\pm 1.$$

Also $\quad (\Sigma yz)^2-2xyz(\Sigma x)=84, \qquad$ whence $\quad xyz=\pm 8.$

Hence x, y, z are the roots of

$$t^3\mp t^2-10t\mp 8=0,$$

and these are $4,\ -1,\ -2$ and $-4,\ 1,\ 2.$

4. The equations giving the numerators of the convergents are

$$\begin{aligned}
-p_n + np_{n-1} + p_{n-2} &= 0,\\
-p_{n-1} + (n-1)\,p_{n-2} + p_{n-3} &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots &\\
-p_3 + 3p_2 + p_1 &= 0,\\
-p_2 + 2p_1 &= 0,\\
p_1 &= 1.
\end{aligned}$$

Solving these as simultaneous equations, the value of p_n is

$$\begin{vmatrix}
0, & n, & 1, & \cdots\cdots & & & \\
0, & -1, & (n-1), & 1, & \cdots & & \\
0, & 0, & -1, & (n-2), & \cdots & & \\
\cdots & \cdots & \cdots & \cdots & & & \\
0, & \cdots & \cdots & \cdots & -1, & 3, & 1\\
0, & \cdots & \cdots & \cdots & & -1, & 2\\
1, & \cdots & \cdots & \cdots & & 0, & 1
\end{vmatrix} \div \Delta,$$

where Δ is the determinant formed by the coefficients.

Similarly, since $-q_2 + 2q_1 = -1$, the value of q_n is

$$\begin{vmatrix}
0, & n, & 1, & \cdots\cdots & & & \\
0, & -1, & (n-1), & 1, & \cdots & & \\
0, & 0, & -1, & (n-2), & & & \\
\cdots & \cdots & \cdots & \cdots & & & \\
0, & \cdots & \cdots & \cdots & -1, & 3, & 1\\
-1, & \cdots & \cdots & \cdots & & -1, & 2\\
1, & \cdots & \cdots & \cdots & & 0, & 1
\end{vmatrix} \div \Delta.$$

The two determinants written above are easily identified with those in the question by placing the first column last (in the second case changing its sign) and then reading rows for columns. The first determinant is equal to that obtained by omitting its first row and column, and so is of order $(n-1)$; the second is of order n.

5. Calling the product on the left P, we have

$$P \sin \alpha = \sin 2^n \alpha,$$

since

$$2 \sin \alpha \cos \alpha = \sin 2\alpha, \quad 2 \sin 2\alpha \cos 2\alpha = \sin 2^2\alpha, \text{ etc.}$$

Hence

$$\begin{aligned} P \sin \alpha &= \sin 2\alpha + (\sin 4\alpha - \sin 2\alpha) + (\sin 6\alpha - \sin 4\alpha) \\ &\qquad + \dots + [\sin 2^n\alpha - \sin (2^n - 2)\alpha] \\ &= \sin \alpha [2 \cos \alpha + 2 \cos 3\alpha + 2 \cos 5\alpha + \dots + 2 \cos (2^n - 1)\alpha], \end{aligned}$$

so that P is equal to the sum on the right in question.

6. Calling the given series S_1, S_2, S_3, and putting $x = 1$, ω, ω^2 in succession in the expansion of $(1 - x)^m$, we find

$$S_1 - S_2 + S_3 = 0, \qquad S_1 - \omega S_2 + \omega^2 S_3 = (1 - \omega)^m,$$
$$S_1 - \omega^2 S_2 + \omega S_3 = (1 - \omega^2)^m.$$

From these, if $1 - \omega = \alpha$, $1 - \omega^2 = \beta$, we get

$$\alpha S_2 - \beta S_3 = \alpha^m, \qquad \beta S_2 - \alpha S_3 = \beta^m,$$

whence $$S_2 = \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha^2 - \beta^2}, \qquad S_3 = \frac{\alpha\beta (\alpha^{m-1} - \beta^{m-1})}{\alpha^2 - \beta^2}.$$

Now the values of α, β are $\dfrac{3 \pm \sqrt{-3}}{2}$, so we may take

$$\alpha = \sqrt{3} \,.\, e^{\frac{i\pi}{6}}, \quad \beta = \sqrt{3} \,.\, e^{-\frac{i\pi}{6}},$$

whence $$\alpha^3 = 3\sqrt{3}i = -\beta^3 = \alpha^2 - \beta^2 \text{ and } \alpha\beta = 3.$$

Hence $$S_2 = \alpha^{m-2} + \beta^{m-2} = 3^{\frac{m-2}{2}} \,.\, 2 \cos (m - 2)\frac{\pi}{6},$$

$$S_3 = 3 (\alpha^{m-4} + \beta^{m-4}) = 3^{\frac{m-2}{2}} \,.\, 2 \cos (m - 4)\frac{\pi}{6},$$

and $$S_1 = S_2 - S_3.$$

7. Denote the angular points by $(a\lambda_1, b\mu_1)$, etc. Then $(a\lambda_1, b\mu_1)$ is the pole of

$$\begin{vmatrix} x, & y, & 1 \\ a\lambda_2, & b\mu_2, & 1 \\ a\lambda_3, & b\mu_3, & 1 \end{vmatrix} = 0,$$

which must therefore coincide with

$$bx\lambda_1 + ay\mu_1 - ab = 0.$$

Hence $$\frac{\mu_2-\mu_3}{\lambda_1}=-\frac{\lambda_2-\lambda_3}{\mu_1}=-(\lambda_2\mu_3-\lambda_3\mu_2),$$

whence $\mu_3(1-\lambda_1\lambda_2)=\mu_2(1-\lambda_3\lambda_1)=\mu_1(1-\lambda_2\lambda_3)$ similarly,

and also $\lambda_3(1-\mu_1\mu_2)=\lambda_2(1-\mu_3\mu_1)=\lambda_1(1-\mu_2\mu_3)$ similarly.

These equations are satisfied by

$$\lambda_1=\cos\beta\cos\gamma\sec\alpha, \text{ etc.,}\quad \mu_1=\sin\beta\sin\gamma\operatorname{cosec}\alpha, \text{ etc.,}$$

each member of the first set being then equal to $\sin\alpha\sin\beta\sin\gamma$, and each member of the second to $\cos\alpha\cos\beta\cos\gamma$.

8. The lines

$$mx+y=am^3+2am \qquad \text{(i)}$$

and $$\mu(x+c)+y=(a+b)\mu^3+2(a+b)\mu$$

are normals to the respective parabolas. If these coincide, we must have $\mu=m$, and equating the absolute terms, we then get

$$am^3+2am=(a+b)m^3+[2(a+b)-c]m,$$

whence $$m=0 \text{ or } \pm\sqrt{\frac{c-2b}{b}}.$$

Substituting the latter values in (i), we obtain the equations given.

9. The tangent at (x_1, y_1) is $xy_1+yx_1=2c^2$. Let the chord of curvature be $lx+my+n=0$. Then the circle of curvature is

$$xy-c^2+(xy_1+yx_1-2c^2)(lx+my+n)=0.$$

The conditions for a circle give

$$ly_1=mx_1, \quad 1+lx_1+my_1=0.$$

Hence $n=1$ and

$$\frac{l}{x_1}=\frac{m}{y_1}=-\frac{1}{x_1^2+y_1^2};$$

so that the chord of curvature is

$$xx_1+yy_1=x_1^2+y_1^2 \qquad \text{(i).}$$

Hence if the four chords pass through (X, Y), we have on adding

$$X.\Sigma x_1+Y.\Sigma y_1=\Sigma CQ_1^2 \qquad \text{(ii).}$$

But from (i), since $x_1 y_1 = c^2$, we have

$$X x_1^3 + Y x_1 c^2 = x_1^4 + c^4,$$

i.e. x_1, x_2, x_3, x_4 are the roots of

$$z^4 - Xz^3 - Yc^2 z + c^4 = 0.$$

Hence $\Sigma x_1 = X$, and similarly $\Sigma y_1 = Y$.

Therefore, from (ii), we have

$$\Sigma CQ_1^2 = X^2 + Y^2 = CP^2.$$

10. Let AB, BC be the two rods (A being the point of suspension) and let BC make an angle θ with the vertical. The C. of G. of the system must be vertically below A. Now the weights of the rods are equivalent to $\frac{1}{2}W$ at A, W at B, and $\frac{1}{2}W$ at C. Hence taking moments about A, we obtain

$$W \,.\, 2a \sin(\theta - \alpha) = \tfrac{1}{2} W \left[2a \sin\theta - 2a \sin(\theta - \alpha)\right];$$

$$\therefore \quad 3 \sin(\theta - \alpha) = \sin\theta,$$

whence
$$\frac{\sin\theta}{3} = \frac{\sin(\theta - \alpha)}{1} = \frac{\cos\theta \sin\alpha}{3\cos\alpha - 1}$$

$$= \frac{\sin\alpha}{\sqrt{(3\sin\alpha)^2 + (3\cos\alpha - 1)^2}}$$

$$= \frac{\sin\alpha}{\sqrt{10 - 6\cos\alpha}}.$$

Now, considering the equilibrium of BC, the action at B balances the weight. Hence the latter consists of a force W vertically, and a couple whose moment is the moment of the weight of BC about B, which moment is $W \,.\, a \sin\theta$. Substituting for $\sin\theta$, we obtain the expression given.

11. Let P_1, P_2 be the impulses between the upper sphere and the two lower ones; V_1, V_2 the horizontal and vertical velocities of the upper sphere after impact; v_1, v_2 the velocities of the two lower spheres. Then we have

$$(P_1 + P_2)\cos 30° = m(V - V_2), \quad (P_2 - P_1)\cos 60° = mV_1 \ldots\ldots\text{(i)},$$

$$P_1 \cos 60° = m\lambda_1 v_1, \qquad\qquad P_2 \cos 60° = m\lambda_2 v_2 \ldots\text{(ii)},$$

and for the motion of the points of contact,

$$V_1 \cos 60° + V_2 \cos 30° = v_1 \cos 60°,$$

$$V_2 \cos 30° - V_1 \cos 60° = v_2 \cos 60°.$$

From these $\quad V_1 = \dfrac{v_1 - v_2}{2}, \quad V_2 = \dfrac{v_1 + v_2}{2\sqrt{3}}.$

Also, from (i) and (ii),

$$\lambda_1 v_1 + \lambda_2 v_2 = \frac{1}{\sqrt{3}}(V - V_2), \quad \lambda_2 v_2 - \lambda_1 v_1 = V_1,$$

whence substituting V_1, V_2, we find

$$(1 + 6\lambda_1) v_1 + (1 + 6\lambda_2) v_2 = 2\sqrt{3}V, \quad (1 + 2\lambda_1) v_1 = (1 + 2\lambda_2) v_2,$$

and these give the required value of v_1.

12. The velocity of an angular distance θ is given by

$$v^2 = 2gr(\cos\alpha - \cos\theta),$$

and the pressure by

$$mg\cos\theta - R = \frac{mv^2}{r}.$$

Hence the pressure vanishes when

$$mg\cos\theta = 2mg(\cos\alpha - \cos\theta),$$

$$\text{i.e.} \quad \cos\theta = \tfrac{2}{3}\cos\alpha \quad \text{and} \quad v^2 = \tfrac{2}{3}gr\cos\alpha.$$

The latus-rectum of the subsequent path is

$$\frac{2v^2\cos^2\theta}{g} = \frac{2}{g}(\tfrac{2}{3}gr\cos\alpha)(\tfrac{2}{3}\cos\alpha)^2 = \tfrac{16}{27}r\cos^3\alpha.$$

XLI.

1. AD bisects BC at right angles in O. Hence

$$PB^2 + PC^2 = 2PO^2 + 2OB^2,$$

$$PA^2 + PD^2 = 2PO^2 + 2OD^2.$$

Now $\quad OB = \tfrac{1}{2}AB; \quad \therefore \; OD^2 = \tfrac{3}{4}AB^2.$

Hence $\quad PA^2 + PD^2 - PB^2 - PC^2 = \tfrac{3}{2}AB^2 - \tfrac{1}{2}AB^2 = AB^2.$

But $\quad PD = BD = AB; \quad \therefore \; PA^2 = PB^2 + PC^2.$

2. Project the conic orthogonally into a circle. Then the equi-conjugate diameters become diameters of the circle inclined at 45° to the diameter which is the projection of the major axis, and are therefore at right angles, while the other diameters in question project into diameters making complementary angles with the projection of the major axis, and on the same side of it, and therefore the equi-conjugates bisect the angles between them. Hence the four projections form a harmonic pencil, and therefore so also do the four original diameters.

3. Consider a selection of r things containing p of the pairs. These can be permuted among themselves in

$$\frac{r\,!}{(2\,!)^p} \text{ ways.}$$

To obtain such a selection we have to take p pairs out of the n, and since the remaining things must be all different, there are really only $n-p$ from which to select them, i.e. we have to select $r-2p$ things from $n-p$.

The combined operation can be performed in

$${}^nC_p \,.\, {}^{n-p}C_{r-2p} \text{ ways.}$$

Hence the number of permutations containing p pairs is

$$\frac{r\,!}{2^p} \,.\, {}^nC_p \,.\, {}^{n-p}C_{r-2p},$$

and therefore the total number of permutations is

$$r\,!\,.\,\sum_0 \frac{1}{2^p} \,.\, {}^nC_p \,.\, {}^{n-p}C_{r-2p},$$

the series being continued as long as possible. But this sum is the coefficient of x^r in

$$\sum_0 {}^nC_p \left(\tfrac{1}{2}x^2\right)^p (1+x)^{n-p},$$

i.e. in $$(1 + x + \tfrac{1}{2}x^2)^n.$$

4. By Fermat's Theorem, $a^{2^n}-1$ is divisible by 2^n+1. Also

$$\begin{aligned} a^{2^n}-1 &= (a^{2^{n-1}})^2-1 \\ &= (a^{2^{n-1}}-1)(a^{2^{n-1}}+1) \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= (a-1)(a+1)(a^2+1)(a^{2^2}+1)\ldots(a^{2^{n-1}}+1), \end{aligned}$$

and since a is odd, these factors are all even, and therefore the expression is divisible by 2^n.

Again, $$c^{20}-1=(c^5-1)(c^5+1)(c^{10}+1),$$

and since c is odd, the first two factors are consecutive even numbers and therefore the product is divisible by 8. Further, c is of one of the forms $5p\pm 1$, $5p\pm 2$.

In the first case,

$$c^5 = M(25)\pm 1, \text{ by the Binomial Theorem,}$$

and in the second,

$$c^5 = M(25)\pm 7\,; \quad \therefore \; c^{10}=M(25)+49,$$

$$\text{i.e. } c^{10}=M(25)-1.$$

Hence in either case, one of the three factors of $c^{20}-1$ is a multiple of 25.

5. Since $$r = R(\Sigma\cos A - 1)$$

and $$\rho^2 = -4R^2\cos A\cos B\cos C,$$

we have

$$(r+2R)^2-\rho^2 = R^2\left[(\Sigma\cos A+1)^2+4\cos A\cos B\cos C\right].$$

But since $\Sigma\cos^2 A + 2\cos A\cos B\cos C = 1$,

the expression in the square bracket easily reduces to

$$2(1+\cos A)(1+\cos B)(1+\cos C),$$

so that $$(r+2R)^2-\rho^2 = 16R^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}$$

$$= R^2(\Sigma\sin A)^2 = s^2\,;$$

$$\therefore\; \Delta = rs = r\sqrt{(r+2R)^2-\rho^2}.$$

6. Denote the two expressions by p and q. The value of the sum

$$\cos\theta + \cos 4\theta + \cos 7\theta + \ldots + \cos 28\theta + \cos 31\theta$$

(the angles being in A.P.) is zero, since it contains a factor

$$\sin\left(11.\frac{3\theta}{2}\right) = \sin\pi = 0.$$

But this sum is $$p + q + \cos 22\theta,$$

since $$\cos 19\theta = \cos 14\theta, \text{ etc.};$$

$$\therefore \quad p + q = -\cos 22\theta = -\cos\frac{4\pi}{3} = \tfrac{1}{2}.$$

If we form the product $2pq$, replacing each product of cosines by a sum, and remembering that $33\theta = 2\pi$, its value is found to be

$$2(\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos 16\theta)$$
$$+ 3(\cos 3\theta + \cos 6\theta + \ldots + \cos 15\theta) + 3\cos 11\theta.$$

The value of the first sum is

$$\frac{\cos\frac{17\theta}{2}\sin\frac{16\theta}{2}}{\sin\frac{\theta}{2}} = -\frac{\cos\frac{16\theta}{2}\sin\frac{16\theta}{2}}{\sin\frac{\theta}{2}} = -\frac{\sin\frac{32\theta}{2}}{2\sin\frac{\theta}{2}} = -\tfrac{1}{2},$$

and the value of the second is

$$\frac{\cos 9\theta\sin\frac{15\theta}{2}}{\sin\frac{3\theta}{2}} = -\frac{\cos\frac{15\theta}{2}\sin\frac{15\theta}{2}}{\sin\frac{3\theta}{2}} = -\frac{\sin\frac{30\theta}{2}}{2\sin\frac{3\theta}{2}} = -\tfrac{1}{2}.$$

Hence $$2pq = -1 - \tfrac{3}{2} - \tfrac{3}{2} = -4;$$

$$\therefore \quad pq = -2.$$

Thus p and q are the roots of

$$z^2 - \tfrac{1}{2}z - 2 = 0,$$

and therefore have the given values, p being positive, since the angles involved in it evidently make its numerical value greater than that of q.

7. The centre of curvature of the point on

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

whose eccentric angle is α, is

$$x=\frac{a^2-b^2}{\sqrt{a^2+\lambda}}\,.\cos^3\alpha, \qquad y=\frac{b^2-a^2}{\sqrt{b^2+\lambda}}\,.\sin^3\alpha,$$

and if the point α lies on $y=mx$, we have

$$\sqrt{b^2+\lambda}\,.\sin\alpha=m\sqrt{a^2+\lambda}\,.\cos\alpha,$$

whence
$$\frac{a^2+\lambda}{\sin^2\alpha}=\frac{b^2+\lambda}{m^2\cos^2\alpha}=\frac{a^2-b^2}{k^2},$$

where
$$k^2\equiv\sin^2\alpha-m^2\cos^2\alpha.$$

Hence
$$x=k\sqrt{a^2-b^2}\,.\frac{\cos^3\alpha}{\sin\alpha},$$

$$y=-k\sqrt{a^2-b^2}\,.\frac{\sin^3\alpha}{m\cos\alpha}.$$

Now
$$k^2(\cos^2\alpha+\sin^2\alpha)^3-\sin^2\alpha+m^2\cos^2\alpha\equiv 0,$$

i.e.
$$\sin^2\alpha\left[k^2\left(\frac{\cos^6\alpha}{\sin^2\alpha}+3\sin^2\alpha\cos^2\alpha\right)-1\right]$$
$$+\cos^2\alpha\left[k^2\left(\frac{\sin^6\alpha}{\cos^2\alpha}+3\sin^2\alpha\cos^2\alpha\right)+m^2\right]=0;$$

so that

$$\sin^2\alpha\left(\frac{x^2-3mxy}{a^2-b^2}-1\right)+\cos^2\alpha\left(\frac{m^2y^2-3mxy}{a^2-b^2}+m^2\right)=0.$$

Also
$$\frac{y}{x}=-\frac{\sin^4\alpha}{m\cos^4\alpha},$$

so that finally

$$y\left(\frac{x^2-3mxy}{a^2-b^2}-1\right)^2=-mx\left(\frac{my^2-3xy}{a^2-b^2}+m\right)^2.$$

8. Reciprocating from the common focus, we get two equal circles with four real common tangents. Hence the reciprocal circles do not intersect in real points and therefore the origin of

reciprocation cannot be inside both of them, i.e. one of the conics must be a hyperbola.

Now let S be the origin, C and C' the centres of the reciprocal circles, S being inside the circle C. Draw SN perpendicular to CC'. Then if r be the radius of either circle,

$$SC = er, \quad SC' = e'r;$$

$$\therefore \quad CC'^2 = (e'^2 + e^2 - 2ee' \cos\gamma)\, r^2.$$

Now if k is the constant of reciprocation, the sum required is $\frac{1}{k^2} \times$ sum of the perpendiculars from S on the common tangents. Now the sum of two of these perpendiculars is $2r$, and the sum of the other two is $2NL \sin\alpha$, where L is the point of intersection of the transverse common tangents, and 2α the angle between these tangents. But

$$SC'^2 - SC^2 = 2CC' \,.\, NL;$$

$$\therefore \quad 2NL \sin\alpha = \frac{SC'^2 - SC^2}{CC'} \cdot \frac{2r}{CC'}$$

$$= 2r \,.\, \frac{e'^2 - e^2}{e'^2 + e^2 - 2ee' \cos\gamma}.$$

Hence the sum of the four perpendiculars is

$$4r \,.\, \frac{e'^2 - ee' \cos\gamma}{e'^2 + e^2 - 2ee' \cos\gamma},$$

whence the given result since

$$l = k^2/r.$$

9. If ϕ be the angle between the tangents from (x', y') to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and ρ, ρ' the focal distances of (x', y'), then

$$\rho\rho' \sin\phi = 2ab \sqrt{\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1}. \quad \text{(See XXXII. 8.)}$$

Now suppose the triangle ABC formed by the tangents at α, β, γ. Then if A is (x', y'), we have

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=\frac{\cos^2\frac{\beta+\gamma}{2}}{\cos^2\frac{\beta-\gamma}{2}}+\frac{\sin^2\frac{\beta+\gamma}{2}}{\cos^2\frac{\beta-\gamma}{2}}-1=\tan^2\frac{\beta-\gamma}{2};$$

$$\therefore\ AS\,.\,AS'\sin A=2ab\tan\frac{\beta-\gamma}{2},$$

so that the identity to be proved is

$$\Sigma\,\frac{1}{p}\tan\frac{\beta-\gamma}{2}=0.$$

Now if the tangent at θ is the variable tangent, then p, q, r are proportional to

$$\cos\theta\,.\,\frac{\cos\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}}+\sin\theta\,.\,\frac{\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}}-1,\ \text{etc.},$$

and the expression last written is

$$\frac{\cos\left(\theta-\frac{\beta+\gamma}{2}\right)}{\cos\frac{\beta-\gamma}{2}}-1=-\frac{2\sin\frac{\theta-\beta}{2}\sin\frac{\theta-\gamma}{2}}{\cos\frac{\beta-\gamma}{2}}.$$

Hence the identity to be proved is

$$\Sigma\sin\frac{\beta-\gamma}{2}\sin\frac{\theta-\alpha}{2}=0,$$

which is obviously true.

10. Let h be the depth of the rings below O, and suppose each string makes an angle θ with the vertical. Then evidently

$$a\sin\theta=h\tan\alpha,$$

and the equation of virtual work is

$$3w\,.\,\delta h+W\,.\,\delta\,(a\cos\theta+h)=0,$$

$$\text{i.e.}\ \ (3w+W)\frac{a\cos\theta}{\tan\alpha}\,.\,\delta\theta-W\,.\,a\sin\theta\,.\,\delta\theta=0,$$

whence
$$\tan\theta=\frac{(3w+W)\cot\alpha}{W},$$

and the required depth is $a\cos\theta$.

11. Let A be the vertex, PN the radius of the string in the initial position, $P'N'$ $(=x)$ the radius in the final position. Then the work done in stretching the string is

$$(\text{extension}) \times (\text{mean tension}) = (2\pi x - 2\pi r) \times \tfrac{1}{2}\lambda \, . \, \frac{x-r}{r}$$

$$= \pi\lambda \, . \, \frac{(x-r)^2}{r}.$$

The work done by gravity is

$$W . NN' = W . \frac{x^2 - r^2}{4a}.$$

Equating these, we get

$$W . \frac{x+r}{4a} = \pi\lambda \, . \, \frac{x-r}{r},$$

whence
$$x = \frac{4\lambda\pi a + Wr}{4\lambda\pi a - Wr} \, . \, r.$$

12. Let α be the angle of the plane. Then the velocity on reaching the top is given by

$$V^2 = \tfrac{5}{2}gl - 2gl \sin\alpha = \tfrac{1}{2}(5 - 4\sin\alpha)\, gl,$$

and the range on the platform is

$$\frac{V^2 \sin 2\alpha}{g} = \tfrac{1}{2}l\,(5 - 4\sin\alpha)\sin 2\alpha$$

$$= \tfrac{1}{2}l\,(5\sin 2\alpha - 2\cos\alpha + 2\cos 3\alpha).$$

If we write $\alpha + \theta$ for α, where θ is small, the expression in the bracket is increased by

$$2\theta\,(5\cos 2\alpha + \sin\alpha - 3\sin 3\alpha),$$

and therefore the range will be continually increasing between 0 and α, provided

$$5\cos 2\alpha + \sin\alpha - 3\sin 3\alpha$$

remains positive in this interval. But putting $\sin\alpha = x$, this expression is

$$5\,(1 - 2x^2) + x - 3\,(3x - 4x^3),$$

$$\text{i.e.}\quad (2x - 1)\,(6x^2 - 2x - 5).$$

This is positive if $x < \frac{1}{2}$, since both factors are negative, it vanishes when $x = \frac{1}{2}$, i.e. when $\alpha = 30°$, and becomes negative when $\frac{1}{2} < x < 1$, whence the results stated follow.

XLII.

1. Let O be the given point. Then inverting from O the circles become two straight lines cutting at a given angle and touching a given circle. Hence the locus of the intersection of the two lines is a concentric circle, and therefore the locus required is the inverse of this concentric circle, i e. a circle coaxal with O and the given circle.

2. Let PF be the perpendicular from P on CD, the diameter conjugate to CP. Then the radius of curvature is CD^2/PF. But, in the present case, since the circle touches the tangents at the extremities of the diameter through P, its radius must be PF, and therefore $PF = CD$. But $PF \cdot CD = ab$.

$$\therefore \quad CD^2 = ab.$$

3. Denoting the expression on the right by $f(n)$, we have

$$f(n-1) = \frac{1}{n-1} + \tfrac{1}{2}\left(\frac{1}{n-1} + \frac{1}{n-2}\right)$$
$$+ \ldots + \frac{1}{n-1}\left(\frac{1}{n-1} + \frac{1}{n-2} + \ldots + \tfrac{1}{2} + 1\right).$$

Hence

$$f(n-1) - f(n) = \left(\frac{1}{n-1} - \frac{1}{n}\right) + \tfrac{1}{2}\left(\frac{1}{n-2} - \frac{1}{n}\right) + \tfrac{1}{3}\left(\frac{1}{n-3} - \frac{1}{n}\right)$$
$$+ \ldots + \frac{1}{n-1}\left(1 - \frac{1}{n}\right) - \frac{1}{n}\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + 1\right),$$

i.e. $$f(n-1) - f(n) = \frac{1}{n(n-1)} + \frac{1}{n(n-2)} + \ldots + \frac{1}{n \cdot 1}$$
$$- \frac{1}{n}\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \tfrac{1}{2} + 1\right)$$
$$= -\frac{1}{n^2}.$$

$$\therefore \quad f(n) - f(n-1) = \frac{1}{n^2}, \quad \text{whence} \quad f(n) = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2}.$$

4. Let A, B, C, D be the four men cutting in this order. At his first cut A's chance is $\frac{1}{13}$, at his second $(\frac{12}{13})^4 \times \frac{1}{13}$, since each player must have failed once before A can have a second cut. At his third cut his chance is $(\frac{12}{13})^8 \times \frac{1}{13}$, and so on. Hence A's chance is the infinite series

$$\tfrac{1}{13}\{1 + (\tfrac{12}{13})^4 + (\tfrac{12}{13})^8 + \dots\}.$$

Denoting the series in the bracket by S, the respective chances of A, B, C, D are

$$\tfrac{1}{13}S, \quad \tfrac{12}{13}\cdot\tfrac{1}{13}S, \quad (\tfrac{12}{13})^2\cdot\tfrac{1}{13}S, \quad (\tfrac{12}{13})^3\cdot\tfrac{1}{13}S,$$

and these are in the ratio

$$1 : \tfrac{12}{13} : (\tfrac{12}{13})^2 : (\tfrac{12}{13})^3.$$

Hence A's expectation (in pounds) is

$$\frac{1}{1 + (\frac{12}{13}) + (\frac{12}{13})^2 + (\frac{12}{13})^3} \times 4 = \tfrac{2197}{7825} \times 4 = 1{\cdot}123\dots.$$

5. Denoting the areas of the triangles ABC, $A'B'C'$ by Δ, Δ', we have

$$\begin{aligned} &b^2c'^2 + c^2b'^2 - 2bb'cc' \cos(A + A') \\ &\quad = b^2c'^2 + c^2b'^2 - 2bb'cc' \cos A \cos A' + 8\Delta\Delta' \\ &\quad = b^2c'^2 + c^2b'^2 - \tfrac{1}{2}(b^2 + c^2 - a^2)(b'^2 + c'^2 - a'^2) + 8\Delta\Delta' \\ &\quad = \tfrac{1}{2}(\Sigma a^2)(\Sigma a'^2) - \Sigma a^2a'^2 + 8\Delta\Delta' \quad\dots\dots\dots\dots(\text{i}),\end{aligned}$$

and, by symmetry, this is equal to the other two similar expressions.

Further $\qquad \Sigma a^2 = 8R^2(1 + \cos A \cos B \cos C) \qquad$ (XI. 6).

$$\begin{aligned}\therefore\quad (\Sigma a^2)(\Sigma a'^2) = 32R^2R'^2(&4 - \Sigma\cos^2 A - \Sigma\cos^2 A' \\ &+ 2\cos A\cos B\cos C\cos A'\cos B'\cos C').\end{aligned}$$

Also

$$\begin{aligned}\Sigma a^2a'^2 &= 16R^2R'^2 . \Sigma(1 - \cos^2 A)(1 - \cos^2 A') \\ &= 16R^2R'^2(3 - \Sigma\cos^2 A - \Sigma\cos^2 A' + \Sigma\cos^2 A\cos^2 A'),\end{aligned}$$

and $\qquad \Delta = 2R^2 \sin A \sin B \sin C.$

Substituting these values in (i), the result follows.

6. Replacing the first element $\cos\theta$ by $2\cos\theta$, and denoting the resulting determinant by Δ_n, we evidently have

$$\Delta_n = 2\cos\theta \,.\, \Delta_{n-1} - \Delta_{n-2} \quad \text{.........................(i).}$$

Now $\quad \Delta_1 = 2\cos\theta = \dfrac{\sin 2\theta}{\sin\theta},$

$$\Delta_2 = 4\cos^2\theta - 1 = 2\cos 2\theta + 1 = \frac{\sin 3\theta}{\sin\theta}.$$

This suggests $\Delta_n = \dfrac{\sin(n+1)\theta}{\sin\theta}$, and assuming this true for all suffixes up to $(n-1)$, we have, by (i),

$$\Delta_n = \frac{2\cos\theta\sin n\theta - \sin(n-1)\theta}{\sin\theta} = \frac{\sin(n+1)\theta}{\sin\theta},$$

and hence the result is true generally.

Now, denoting the given determinant by D, we have

$$\Delta_n = D + \cos\theta \,.\, \Delta_{n-1}.$$

$$\therefore \quad D = \frac{\sin(n+1)\theta - \cos\theta\sin n\theta}{\sin\theta} = \cos n\theta.$$

7. Suppose the equation has a root $X + iY$. Then

$$\cot(X + iY) = a(X + iY).$$

$$\therefore \quad \cot(X - iY) = a(X - iY),$$

whence

$$aX = \frac{\sin 2X}{\cosh 2Y - \cos 2X}, \quad aY = -\frac{\sinh 2Y}{\cosh 2Y - \cos 2X},$$

$$\text{i.e.} \quad \frac{\sin 2X}{2X} = -\frac{\sinh 2Y}{2Y} = \frac{\cosh 2Y - \cos 2X}{2} \,.\, a.$$

Now $\dfrac{\sinh 2Y}{2Y}$ is always positive, so that the second of these three quantities is always negative. But since

$$\cosh 2Y > 1 \geqslant \cos 2X,$$

the third quantity is always positive. Hence the impossibility.

8. Each of the given fractions is equal to $\dfrac{Ax - A'y + A''}{\Delta t^2}$, (where Δ is the given determinant), as we see by multiplying the

numerators and denominators of successive fractions by $A, -A', A''$ and adding. Similarly, each fraction is equal to

$$\frac{-Bx+B'y-B''}{\Delta t} \quad \text{and} \quad \frac{Cx-C'y+C''}{\Delta}.$$

Hence the equation to the conic is

$$(Ax-A'y+A'')(Cx-C'y+C'')-(Bx-B'y+B'')^2=0,$$

in which the coefficients of x^2, y^2 and xy are

$$AC-B^2, \quad A'C'-B'^2, \quad -(AC'+A'C-2BB').$$

But for the general conic

$$\frac{e^4}{1-e^2}+4=\frac{(a+b)^2}{ab-h^2}.$$

Hence the result.

9. The equation to the parabola must be of the form

$$x^2-y^2-a^2+\lambda(xx'-yy'-a^2)^2=0,$$

with the condition

$$(1+\lambda x'^2)(-1+\lambda y'^2)=\lambda^2 x'^2 y'^2,$$

whence $$\lambda=-\frac{1}{a^2},$$

and the equation becomes

$$(xy'-x'y)^2-2a^2(xx'-yy')+2a^4=0 \text{............(i)}.$$

The equation to the directrix is therefore of the form

$$xx'+yy'=k,$$

and if (X, Y) is the focus, the equation to the parabola is

$$(x-X)^2+(y-Y)^2=\frac{(xx'+yy'-k)^2}{x'^2+y'^2},$$

i.e. $$(xy'-x'y)^2-2x[X(x'^2+y'^2)-x'k]-2y[Y(x'^2+y'^2)-y'k]$$
$$+(X^2+Y^2)(x'^2+y'^2)-k^2=0 \text{..............(ii)}.$$

Comparing (i) and (ii), we have

$$X(x'^2+y'^2)=(a^2+k)x', \qquad (X^2+Y^2)(x'^2+y'^2)=k^2+2a^4.$$
$$Y(x'^2+y'^2)=(-a^2+k)y',$$

Hence, squaring and adding,

$$(x'^2 + y'^2)(k^2 + 2a^4) = (a^2 + k)^2 x'^2 + (a^2 - k)^2 y'^2,$$

$$\text{i.e.}\quad a^4 (x'^2 + y'^2) = 2ka^2 (x'^2 - y'^2) = 2ka^4.$$

$$\therefore\quad k = \frac{x'^2 + y'^2}{2}.$$

10. Since the rods are smooth, the strings must be perpendicular to them in the position of equilibrium, and O, the intersection of the directions of the strings, must be vertically below the centre of the rod. Let G be the centre of the rod AB, and suppose the rod makes an angle θ with the vertical. Then

$$\frac{GO}{AG} = \frac{\sin(\pi - \theta - \alpha)}{\sin\alpha}, \quad \frac{GO}{BG} = \frac{\sin(\theta - \beta)}{\sin\beta}.$$

$$\therefore\quad \sin(\theta + \alpha)\sin\beta = \sin(\theta - \beta)\sin\alpha,$$

whence
$$\tan\theta = \frac{2\sin\alpha\sin\beta}{\sin(\alpha - \beta)}.$$

Also, projecting on the vertical,

$$l\cos\alpha - AB\cos\theta - l'\cos\beta = 0.$$

$$\therefore\quad AB = (l\cos\alpha - l'\cos\beta)\sec\theta.$$

11. The infinitely rapid series of impacts is equivalent to a continuous finite force (cf. the fall of a chain on a table, etc.). Let this force, equivalent to the action of the man's feet against the plank, be Q downwards on the man in the direction of the plank, and upwards on the plank in the direction of its length.

Then, for the plank,

$$Q = Mg\cos\alpha.$$

For the man, if f is his acceleration downwards in the direction of the plank (which will also be his acceleration in space, since the plank does not move),

$$mf = mg\cos\alpha + Q.$$

$$\therefore\quad f = \frac{M + m}{m} \cdot g\cos\alpha.$$

The velocity at the end of the plank is given by

$$v^2 = 2fa = 2ga\cos\alpha \cdot \frac{M + m}{m}.$$

12. The acceleration of the train towards the centre is $f = \frac{V^2}{a}$, and the pendulum oscillates as though the acceleration due to gravity were g_1, where $g_1 = \sqrt{f^2 + g^2}$.

Hence, if l' be the length of the ordinary equivalent pendulum, we have

$$2\pi\sqrt{\frac{l'}{g}} = 2\pi\sqrt{\frac{l}{g_1}}.$$

$$\therefore \quad l' = l \cdot \frac{g}{g_1} = l\left(1 + \frac{f^2}{g^2}\right)^{-\frac{1}{2}} = l\left(1 + \frac{V^4}{a^2 g^2}\right)^{-\frac{1}{2}} = l\left(1 - \tfrac{1}{2} \cdot \frac{V^4}{a^2 g^2}\right)$$

approximately, since a is large.

XLIII.

1. Invert the system from O, denoting the inverse points by the corresponding small letters. Then the triangles OPO', Opo' are similar.

$$\therefore \quad O'P : o'p = OP : Oo'.$$

Similarly for Q. Hence

$$O'P \cdot O'Q : OP \cdot OQ = o'p \cdot o'q : Oo'^2,$$

and similarly,

$$O'R \cdot O'S : OR \cdot OS = o'r \cdot o's : Oo'^2.$$

But the points p, q, r, s are on a circle,

$$\therefore \quad o'p \cdot o'q = o'r \cdot o's,$$

whence the result follows.

2. Let V be the middle point of PR, and PF the perpendicular on CD, the diameter conjugate to CP.

Then $$CR^2 + CP^2 = 2CV^2 + 2VP^2.$$

Also

$$\begin{aligned} CR^2 &= CP^2 + PR^2 + 2PR \cdot PF \\ &= CP^2 + \rho^2 + 2CD^2, \text{ where } \rho \text{ is the radius of curvature,} \\ &= a^2 + b^2 + \rho^2 + CD^2. \end{aligned}$$

$$\therefore \quad 2CV^2 + 2VP^2 = 2(a^2 + b^2) + \rho^2.$$

But $\rho = 2VP$. $\therefore\ CV^2 = VP^2 + a^2 + b^2$.

Hence the circle on PR as diameter cuts the director circle orthogonally and therefore P and R are conjugate points.

3. We have

$$x^3(z-x)(x-y) = -x^5 - x^3yz + x^4(y+z)$$
$$= -2x^5 - x^3yz, \quad \text{since} \quad y+z=-x.$$
$$\therefore\ \Sigma x^3(z-x)(x-y) = -2\Sigma x^5 - xyz\,.\,\Sigma x^2.$$

Also $\quad \Sigma y^3z^3(y-z) = -(y-z)(z-x)(x-y)\,.\,(\Sigma y^2z^2).$

Hence the first product is

$$(2\Sigma x^5 + xyz\,.\,\Sigma x^2)(\Sigma y^2z^2)/x^3y^3z^3.$$

Now let x, y, z be the roots of $t^3 + qt - r = 0$. Then

$$s_5 + qs_3 - rs_2 = 0, \quad \text{where} \quad s_n = \Sigma x^n.$$

But $\quad s_3 = 3r, \quad s_2 = -2q. \quad \therefore\ s_5 = -5qr.$

Hence the above expression is

$$\frac{(-10qr - 2qr)q^2}{r^3} = -12r\frac{q^3}{r^3} = -12xyz\left(\Sigma\frac{1}{x}\right)^3.$$

4. In the n trials, suppose that each success is represented by a white counter, and each failure by a black one. Arrange the n counters in a row, and suppose that there are p black ones. Now before the p black counters, between each successive pair, and at the end of them, there are $(p+1)$ spaces, each of which may be occupied by 0, 1 or 2 white counters. Hence the number of admissible arrangements, with p black counters, is the coefficient of x^{n-p} in the expansion of $(1+x+x^2)^{p+1}$, i.e. the coefficient of x^{n+1} in the expansion of $(x+x^2+x^3)^{p+1}$. Also the probability of any one arrangement with p black counters is

$$(\tfrac{3}{4})^{n-p}(\tfrac{1}{4})^p.$$

Hence the chance required when there are p black counters is

$$\frac{1}{4^n} \times \text{coefficient of } x^{n+1} \text{ in } 3^{n-p}(x+x^2+x^3)^{p+1}.$$

Hence the total chance is

$$\frac{1}{4^n} \times \text{coefficient of } x^{n+1} \text{ in } \sum_{p=0}^{p=n} 3^{n-p}(x+x^2+x^3)^{p+1}.$$

Now putting $x+x^2+x^3=t$, we have

$$\sum_{p=0}^{p=n} . \, 3^{n-p} . \, t^{p+1} = t . \frac{3^{n+1}-t^{n+1}}{3-t}.$$

Omitting t^{n+1}, which gives powers of x all higher than x^{n+1}, the required chance is thus

$$\frac{3^{n+1}}{4^n} \times \text{coefficient of } x^{n+1} \text{ in } \frac{t}{3-t}.$$

Also

$$\frac{t}{3-t} = \frac{3}{3-t} - 1 = \frac{3}{3-x-x^2-x^3} - 1 = \frac{3}{(1-x)(3+2x+x^2)} - 1,$$

and since $3+2x+x^2 = 3(1-\alpha x)(1-\beta x)$, we find

$$\frac{3}{(1-x)(3+2x+x^2)} = \tfrac{1}{2} \cdot \frac{1}{1-x} + \tfrac{1}{4}\left(\frac{1}{1-\alpha x} + \frac{1}{1-\beta x}\right).$$

Thus the required chance is

$$\frac{3^{n+1}}{4^n} \cdot \left\{\tfrac{1}{2} + \tfrac{1}{4}(\alpha^{n+1}+\beta^{n+1})\right\} = \left(\tfrac{3}{4}\right)^{n+1}(2+\alpha^{n+1}+\beta^{n+1}).$$

5. Subtracting the third row from the first, and the fourth from the second, then adding the first column to the second, and the third to the fourth, we get

$$\begin{vmatrix} \sin\alpha - \sin\beta, & 0 \quad , & \sin\gamma - \sin\delta, & 0 \\ \cos\alpha - \cos\beta, & 0 \quad , & \cos\gamma - \cos\delta, & 0 \\ \sin\beta \quad , & \sin\alpha + \sin\beta, & \sin\delta \quad , & \sin\gamma + \sin\delta \\ \cos\beta \quad , & \cos\alpha + \cos\beta, & \cos\delta \quad , & \cos\gamma + \cos\delta \end{vmatrix} = 0,$$

and the determinant is the product of the factors

$$(\sin\alpha - \sin\beta)(\cos\gamma - \cos\delta) - (\cos\alpha - \cos\beta)(\sin\gamma - \sin\delta)$$

and

$$(\sin\alpha + \sin\beta)(\cos\gamma + \cos\delta) - (\cos\alpha + \cos\beta)(\sin\gamma + \sin\delta).$$

The first factor is

$$2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\,.\,2\sin\frac{\gamma+\delta}{2}\sin\frac{\delta-\gamma}{2}$$

$$-2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\,.\,2\cos\frac{\gamma+\delta}{2}\sin\frac{\gamma-\delta}{2}$$

$$=4\sin\frac{\alpha+\beta-\gamma-\delta}{2}\sin\frac{\beta-\alpha}{2}\sin\frac{\delta-\gamma}{2}.$$

In the same way, the second factor is

$$4\sin\frac{\alpha+\beta-\gamma-\delta}{2}\cos\frac{\beta-\alpha}{2}\cos\frac{\delta-\gamma}{2}.$$

Hence $$\sin^2\frac{\alpha+\beta-\gamma-\delta}{2}\sin(\alpha-\beta)\sin(\gamma-\delta)=0,$$

leading to the conclusions stated.

6. We have

$$\log\frac{1+x}{1-x}=2\left(x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\dots\right)\quad(x<1).$$

Now let $\theta=e^{\frac{i\pi}{4}}$, $\phi=e^{-\frac{i\pi}{4}}$, so that $\theta^4=\phi^4=-1$, $\theta\phi=1$.

Then

$$\log\frac{1+\theta x}{1-\theta x}=2\left(\theta x-\tfrac{1}{3}\phi x^3-\tfrac{1}{5}\theta x^5+\tfrac{1}{7}\phi x^7+\tfrac{1}{9}\theta x^9-\dots\right),$$

$$\log\frac{1+\phi x}{1-\phi x}=2\left(\phi x-\tfrac{1}{3}\theta x^3-\tfrac{1}{5}\phi x^5+\tfrac{1}{7}\theta x^7+\dots\right).$$

Subtracting,

$$\log\frac{(1+\theta x)(1-\phi x)}{(1-\theta x)(1+\phi x)}=2(\theta-\phi)\left(x+\tfrac{1}{3}x^3-\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\dots\right),$$

whence, putting $x=1$, we have

$$1+\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\dots=\frac{1}{2\sqrt{2}i}\log(-1)$$

$$=\frac{1}{2\sqrt{2}i}(\pi i)=\frac{\pi}{2\sqrt{2}}.$$

Also the denominator is $\frac{\pi}{4}$, therefore the value of the fraction is $\sqrt{2}$.

7. Any line parallel to the tangent at α is

$$\frac{k}{r} = e\cos\theta + \cos(\theta - \alpha),$$

as is evident from the Cartesian form.

If this coincides with $\frac{l}{r} = e\cos\theta + \cos(\theta - \beta)$, we have

$$\frac{e + \cos\beta}{e + \cos\alpha} = \frac{\sin\beta}{\sin\alpha} = \frac{l}{k} \ (\equiv \lambda, \text{ suppose}).$$

$$\therefore \quad \{\lambda(e + \cos\alpha) - e\}^2 + \lambda^2 \sin^2\alpha = 1,$$

i.e. $$\lambda^2(1 + 2e\cos\alpha + e^2) - 2\lambda e(e + \cos\alpha) + e^2 - 1 = 0,$$

or $$\{\lambda(1 + 2e\cos\alpha + e^2) - (e^2 - 1)\}(\lambda - 1) = 0,$$

whence $$\lambda = \frac{e^2 - 1}{1 + 2e\cos\alpha + e^2} \text{ or } 1.$$

The latter value gives the original tangent.

8. The line $lx + my + n = 0$ will be a normal if

$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2},$$

as we easily find by comparing with the eccentric angle form.

Hence to find the condition that $\frac{x - x_0}{\lambda} = \frac{y - y_0}{\mu}$ is a normal, we must write μ, $-\lambda$, $-\mu x_0 + \lambda y_0$ for l, m, n in the above, and we then get

$$(a^2\lambda^2 + b^2\mu^2)(-\mu x_0 + \lambda y_0)^2 = (a^2 - b^2)^2 \mu^2 \lambda^2,$$

and writing $x - x_0$, $y - y_0$ for λ, μ, this gives the equation to the four normals.

9. The line $\frac{x - x'}{\cos\theta} = \frac{y - y'}{\sin\theta} = r$ meets the general conic where

$$a(x' + r\cos\theta)^2 + 2h(x' + r\cos\theta)(y' + r\sin\theta) + \ldots + \ldots = 0,$$

i.e. $r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$

$$+ 2r[(ax' + hy' + g)\cos\theta + (hx' + by' + f)\sin\theta] + S' = 0,$$

and the line will be a tangent provided this equation in r has equal roots. This will be the case if

$$[(ax' + hy' + g)\cos\theta + (hx' + by' + f)\sin\theta]^2$$
$$= S'(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta),$$

reducing to

$$(Cy'^2 - 2Fy' + B)\cos^2\theta - 2(Cx'y' - Fx' - Gy' + H)\cos\theta\sin\theta$$
$$+ (Cx'^2 - 2Gx' + A)\sin^2\theta = 0 \quad \ldots(\text{i}).$$

The values of θ given by this equation are the directions of the two tangents which can be drawn from (x', y') to the curve. Calling these θ_1 and θ_2, the angle between the tangents is $\theta_1 - \theta_2$.

Now writing (i) as an equation in $\tan\theta$, we have

$$\tan\theta_1 + \tan\theta_2 = \frac{2(Cx'y' - Fx' - Gy' + H)}{Cx'^2 - 2Gx' + A},$$

$$\tan\theta_1 \tan\theta_2 = \frac{Cy'^2 - 2Fy' + B}{Cx'^2 - 2Gx' + A},$$

and from these $(\tan\theta_1 - \tan\theta_2)^2$

$$= \frac{4\{(Cx'y' - Fx' - Gy' + H)^2 - (Cx'^2 - 2Gx' + A)(Cy'^2 - 2Fy' + B)\}}{(Cx'^2 - 2Gx' + A)^2}.$$

Now the quantity in the bracket in the numerator reduces to

$$(F^2 - BC)x'^2 + \ldots + \ldots + 2(BG - FH)x + \ldots + \ldots,$$

and remembering that $BC - F^2 \equiv a\Delta$, etc., this is simply $-\Delta S'$, so that

$$\tan\theta_1 - \tan\theta_2 = \frac{2\sqrt{-\Delta S'}}{Cx'^2 - 2Gx' + A}.$$

Hence

$$\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= \frac{2\sqrt{-\Delta S'}}{C(x'^2 + y'^2) - 2Gx' - 2Fy' + A + B}.$$

Again the equation to the asymptotes is $S - \frac{\Delta}{C} = 0$, and if this is equivalent to $(lx + my + n)(l'x + m'y + n') = 0$, we have $ll' = a$, $mm' = b$, $lm' + l'm = 2h$, whence

$$(l^2 + m^2)(l'^2 + m'^2) = (ll' - mm')^2 + (lm' + l'm)^2$$
$$= (a - b)^2 + 4h^2,$$

and the product of the perpendiculars is

$$\frac{lx + my + n}{\sqrt{l^2 + m^2}} \cdot \frac{l'x + m'y + n'}{\sqrt{l'^2 + m'^2}} = \left(S - \frac{\Delta}{C}\right) \Big/ \sqrt{(a - b)^2 + 4h^2}.$$

10. In the position of limiting equilibrium, let AB be the upper rod, $A'B'$ the lower, M, N the projections of A and B on the plane, θ the angle between AB, $A'B'$, and ϕ the angle either string makes with the horizontal. Then

$$2a \cos \phi = A'M = 2a \sin \frac{\theta}{2}. \qquad \therefore \quad \phi = 90^\circ - \frac{\theta}{2}.$$

Now evidently the ends of the rod will begin to move in a direction perpendicular to its length, and we may suppose the weight of the rod replaced by two weights, each $\frac{1}{2}W$, at its extremities. Hence if R be the pressure at either end of the rod, T the tension of either string, we have, resolving horizontally, perpendicular to the rod,

$$\mu R = T \cos \phi \cos \frac{\theta}{2} = T \cos \phi \sin \phi,$$

and vertically $\qquad R + T \sin \phi = \frac{1}{2} W.$

From these we get $\qquad T = \dfrac{\frac{1}{2} W}{\sin \phi \,(\sqrt{3} \cos \phi + 1)}.$

Now, $\sin \phi\,(\sqrt{3} \cos \phi + 1) = \frac{\sqrt{3}}{2} \sin 2\phi + \sin \phi$, and this increases as long as $\sqrt{3} \cos 2\phi + \cos \phi$ remains positive (as in XLI. 12).

But putting $\cos \phi = x$, this is

$$\sqrt{3}\,(2x^2 - 1) + x = (2x + \sqrt{3})(\sqrt{3}x - 1).$$

This is positive as long as $x > \frac{1}{\sqrt{3}}$, and changes sign when x passes through that value. Hence $\sin\phi\,(\sqrt{3}\cos\phi + 1)$ is a maximum when $\cos\phi = \frac{1}{\sqrt{3}}$, $\sin\phi = \frac{\sqrt{2}}{\sqrt{3}}$, and in this case

$$T = \frac{\sqrt{3}\,W}{4\sqrt{2}}.$$

11. Let T, T' be the tensions of the upper and lower parts of the string. Since there is no initial velocity, m has no initial acceleration along BA. Let its acceleration perpendicular to AB be f. Then that of m' along CB will be $f\sin\beta$, since BC is inextensible.

Hence we have the equations

$$T = mg\cos\alpha + T'\cos\beta \quad \text{...............(i)},$$

$$mf = mg\sin\alpha - T'\sin\beta \quad \text{...............(ii)},$$

$$m'f\sin\beta = T' - mg\cos(\alpha+\beta) \quad \text{.............(iii)}.$$

Eliminating f from (ii) and (iii), we find

$$T' = \frac{mm'g\cos\alpha\cos\beta}{m + m'\sin^2\beta},$$

whence from (i) $$T = \frac{m\,(m+m')\,g\cos\alpha}{m + m'\sin^2\beta}.$$

12. Let v be the velocity with which the stone strikes the train, so that $v^2 = 2gh$. Then

impulse due to friction $= \mu \times$ normal impulse

$$= \mu \times mv\,(1+e) = \tfrac{2}{3}\,mv\,(1 + \tfrac{1}{2}) = mv.$$

Hence the horizontal velocity communicated to the particle is v. Let V be the velocity of the train, so that $V^2 = 2gk$. Then the relative velocity of the train and particle is $V - v$. The time of flight of the particle is $\frac{2ev}{g} = \frac{v}{g}$, and in this time the train will describe relatively to the particle a distance

$$\frac{v}{g}\,(V - v) = 2h^{\frac{1}{2}}\,(k^{\frac{1}{2}} - h^{\frac{1}{2}}).$$

Hence if the length of the train is less than this distance, the particle will not hit the train again.

XLIV.

1. It is known that if $AP + PB$ is a minimum, AP and PB must make equal angles with CP, i.e. CP is the external bisector of the angle APB. Now let D divide AB internally in the same ratio as C divides it externally, so that D is a fixed point, and PD the internal bisector of APB, i.e. CPD is a right angle. Hence drawing a sphere on CD as diameter the locus of P is the section of this sphere by the given plane, i.e. a circle.

2. Let the normals at Q, Q', Q'' all pass through P and meet the parabola again in q, q', q''. Then since the centre of the circle $qq'q''$ is on the axis, one of the sides of $qq'q''$ must be perpendicular to the axis. Let this be qq' meeting the axis in n, and draw QN, $Q'N'$, PL perpendicular to the axis. Let the normals at Q, Q' meet the axis in G, G'. Then

$$AN \,.\, An = AG^2, \quad AN' \,.\, An = AG'^2 \qquad \text{(as in III. 2)}.$$

Hence
$$\frac{AG^2}{AN} = \frac{AG'^2}{AN'} = \frac{AG^2 - AG'^2}{NN'} = AG + AG',$$

since
$$AG = 2a + AN, \quad AG' = 2a + AN'.$$

Hence
$$AG^2 = AN(AG + AG'),$$

$$\text{i.e. } AG \,.\, 2a = AN \,.\, AG',$$

$$\text{i.e. } (AN + 2a)\, 2a = AN(AN' + 2a);$$

$$\therefore\ AN \,.\, AN' = 4a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Now, from similar triangles,

$$PL : LG = QN : 2a;$$

$$\therefore\ PL^2 : (AG - AL)^2 = AN : a = 4a : AN',$$

and
$$PL^2 : (AL - AG')^2 = 4a : AN.$$

Hence
$$PL^2 : (AL - AG')^2 - (AG - AL)^2 = 4a : NN',$$

$$\therefore\ PL^2 = 4a\,(2AL - AG - AG') \ldots\ldots\ldots\ldots\text{(ii)}.$$

But $\dfrac{(AG-AL)^2}{AN'}=\dfrac{(AL-AG')^2}{AN}=\dfrac{(AG-AL)(AL-AG')}{2a}$ by (i);

$$\therefore\ \frac{AG-AL}{AN'}=\frac{AL-AG'}{2a}=\frac{AG-AG'}{AG'};$$

$$\therefore\ AL\,.\,AG'=AG'^2+2a\,(AG-AG')$$

$$=AG'^2+AN\,.\,AG'-2a\,.\,AG';$$

$$\therefore\ AL=AG'+AN-2a=AG'+AG-4a.$$

Hence by (ii) $\qquad PL^2=4a\,(AL-4a),$

i.e. the locus of P is an equal coaxal parabola.

3. The first part easily follows from the Binomial Expansions of

$$\left(1+\frac{1}{m}\right)^m \text{ and } \left(1+\frac{1}{n}\right)^n.$$

To prove the second part, we observe that the A.M. of the quantities

$$\left(\frac{1}{n}\right)^3,\ \left(\frac{2}{n}\right)^3,\ \left(\frac{3}{n}\right)^3 \cdots \left(\frac{n}{n}\right)^3$$

is greater than their G.M.;

$$\text{i.e.}\quad \frac{\left(\frac{1}{n}\right)^3+\left(\frac{2}{n}\right)^3+\ldots+\left(\frac{n}{n}\right)^3}{n}>\left\{\left(\frac{n\,!}{n^n}\right)^3\right\}^{\frac{1}{n}}$$

$$\text{i.e.}\quad \frac{(n+1)^2}{4n^2}>\left(\frac{n\,!}{n^n}\right)^{\frac{3}{n}},$$

$$\text{i.e.}\quad \frac{n+1}{n}>2\left(\frac{n\,!}{n^n}\right)^{\frac{3}{2n}},$$

$$\text{i.e.}\quad \left(1+\frac{1}{n}\right)^n>2^n\left(\frac{n\,!}{n^n}\right)^{\frac{3}{2}}.$$

4. Let N be the number, and suppose that

$$N=M\,(91)+r.$$

By hypothesis $\qquad N^5=M\,(91)+5\,;$

$$\therefore\ N^6=M\,(91)+5r.$$

But by Fermat's theorem, since N must be prime to 13, N^6 has one of the forms $M(13) \pm 1$, and also has the form $M(7)+1$, since N is prime to 7. Hence N^6 is of one of the forms

$$M(91)+1 \quad \text{or} \quad M(91)+64.$$

We therefore have

$$5r = M(91)+1 \quad \text{or} \quad M(91)+64,$$

and remembering that $r < 91$, we find numerically that the only possible values are $r=73$ in the first case, and $r=31$ in the second. But we must also have $r^5 = M(91)+5$, and this is satisfied by $r=31$ and not by $r=73$. Hence $r=31$.

5. Since $\sin(\beta+\gamma)\sin(\beta-\gamma) \equiv \sin^2\beta - \sin^2\gamma$; therefore if we put $\sin^2\alpha = x$, etc., the first expression becomes

$$\Sigma x^n (y-z) \text{ ....(i)},$$

and the second

$$\Sigma x^2 (y-z) = -(y-z)(z-x)(x-y).$$

But the first expression vanishes when $y=z$ and is therefore divisible by $y-z$. Similarly for $z-x$ and $x-y$.

If $n=3$, (i) becomes

$$\Sigma x^3 (y-z) \equiv -(y-z)(z-x)(x-y)(x+y+z),$$

so that the quotient in this case is $\Sigma \sin^2\alpha$.

If $n=4$, (i) becomes

$$\Sigma x^4 (y-z) \equiv -(y-z)(z-x)(x-y)(\Sigma x^2 + \Sigma yz),$$

and the quotient is

$$\Sigma \sin^4\alpha + \Sigma \sin^2\beta \sin^2\gamma.$$

6. We have

$$b\sin(\theta+C) + c\sin(\theta-B) \equiv a\sin\theta.$$

Hence if

$$\kappa a \sin\theta = x, \quad \kappa b \sin(\theta+C) = y, \quad \kappa c \sin(\theta-B) = z,$$

we have $y+z=x$, and the squares of the sides are a^2-x^2, etc.;

$$\begin{aligned} \therefore\ 16\Delta'^2 &= 2\Sigma(b^2-y^2)(c^2-z^2) - \Sigma(a^2-x^2)^2 \\ &= 16\Delta^2 - 2\Sigma(b^2+c^2-a^2)x^2, \end{aligned}$$

since $2\Sigma y^2z^2 - \Sigma x^4 = 0$, one factor being $y+z-x$;

$$\therefore\ 16\Delta'^2 = 16\Delta^2 - 4\Sigma bc \cos A \,.\, x^2$$

$$= 16\Delta^2 - 4\kappa^2 abc$$

$$[a \sin^2\theta \cos A + b \sin^2(\theta + C) \cos B + c \sin^2(\theta - B) \cos C].$$

The expression in the bracket is

$$\sin^2\theta\,(a \cos A + b \cos^2 C \cos B + c \cos^2 B \cos C)$$

$$+ \cos^2\theta\,(b \sin^2 C \cos B + c \sin^2 B \cos C)$$

$$= \sin^2\theta\,(a \cos A + a \cos B \cos C) + \cos^2\theta \,.\, b \sin C \sin A$$

$$= a \sin B \sin C\,(\sin^2\theta + \cos^2\theta);$$

$$\therefore\ 16\Delta'^2 = 16\Delta^2 - 4\kappa^2 a^2 bc \sin B \sin C$$

$$= 16\Delta^2 - 16\kappa^2\Delta^2,$$

i.e. $\Delta' = (1 - \kappa^2)^{\frac{1}{2}}\,\Delta.$

7. The equation to the conic having double contact with

$$ax^2 + by^2 = 1 \quad \text{along} \quad lx + my = 1$$

and passing through the origin is

$$ax^2 + by^2 - 1 + (lx + my - 1)^2 = 0,$$

and its centre is given by

$$\left.\begin{aligned} ax + l\,(lx + my - 1) = 0 \\ by + m\,(lx + my - 1) = 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

whence $\dfrac{l}{ax} = \dfrac{m}{by} = k$, suppose,

and from (i) we then have

$$1 + k\,\{k\,(ax^2 + by^2) - 1\} = 0,$$

i.e. $\{1 + k^2\,(ax^2 + by^2)\}^2 = k^2$ ....(ii).

Now the condition that $lx + my = 1$ should be a normal is easily found to be

$$\frac{ab^2}{l^2} + \frac{a^2 b}{m^2} = (a - b)^2,$$

whence $\dfrac{b^2}{ax^2} + \dfrac{a^2}{by^2} = k^2\,(a - b)^2$ ....(iii).

Eliminating k^2 from (ii) and (iii), we find

$$\{ab\,(a-b)^2\,x^2y^2 + (a^3x^2+b^3y^2)\,(ax^2+by^2)\}^2$$
$$= ab\,(a-b)^2\,x^2y^2\,(a^3x^2+b^3y^2).$$

8. Let the point ϕ be one of the extremities of the conjugate semi-diameters, so that

$$\rho = \frac{(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{2}}}{ab}.$$

The equation to the normal at ϕ is

$$\frac{x-a\cos\phi}{p\,.\,b\cos\phi} = \frac{y-b\sin\phi}{p\,.\,a\sin\phi} = r,$$

where $$p^2\,(a^2\sin^2\phi + b^2\cos^2\phi) = 1.$$

This meets the ellipse where

$$\frac{(a+pbr)^2\cos^2\phi}{a^2} + \frac{(b+par)^2\sin^2\phi}{b^2} = 1,$$

giving $r=0$, or

$$2\left(\frac{b}{a}\cos^2\phi + \frac{a}{b}\sin^2\phi\right) + pr\left(\frac{b^2}{a^2}\cos^2\phi + \frac{a^2}{b^2}\sin^2\phi\right) = 0;$$

so that $$n = \frac{2ab\,(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{2}}}{a^4\sin^2\phi + b^4\cos^2\phi};$$

$$\therefore\ \frac{\rho}{n} = \frac{a^4\sin^2\phi + b^4\cos^2\phi}{2a^2b^2},$$

and writing $\frac{\pi}{2}+\phi$, for ϕ, we have also

$$\frac{\rho'}{n'} = \frac{a^4\cos^2\phi + b^4\sin^2\phi}{2a^2b^2};$$

$$\therefore\ \frac{\rho}{n} + \frac{\rho'}{n'} = \frac{a^4+b^4}{2a^2b^2}.$$

9. The tangent at A is $n\beta + m\gamma = 0$. If the normal is $p\beta + q\gamma = 0$, the condition of perpendicularity is

$$np + mq - (nq + mp)\cos A = 0,$$

or $$p\,(n - m\cos A) + q\,(m - n\cos A) = 0,$$

whence the normal is

$$(m - n\cos A)\,\beta - (n - m\cos A)\,\gamma = 0,$$

and similarly the normals at B and C are

$$(n - l\cos B)\,\gamma - (l - n\cos B)\,\alpha = 0,$$

and $$(l - m\cos C)\,\alpha - (m - l\cos C)\,\beta = 0.$$

If these meet in a point, we must have

$$(m - n\cos A)(n - l\cos B)(l - m\cos C)$$
$$= (n - m\cos A)(l - n\cos B)(m - l\cos C),$$

whence $$\Sigma l\,(m^2 - n^2)(\cos A + \cos B\cos C) = 0,$$

$$\text{i.e. } \Sigma l\,(m^2 - n^2)\sin B\sin C = 0.$$

10. Let a be the length of the plank, h the distance of the C. of G. from the top end, r the radius of the cylinder, ϕ the inclination of the plank when unloaded, x the distance between w' and the point of contact when the plank is horizontal. Then taking moments about the point of contact in each case, we have

$$w'x = w\,(h - x) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$W(a - x - r\alpha) = w\,(x + r\alpha - h) + W'(x + r\alpha),$$

$$W'(a - x - r\beta) + w\,(h - x - r\beta) = W(x + r\beta),$$

whence $$\left.\begin{aligned} Wa &= (W + W' + w)(x + r\alpha) - wh \\ W'a &= (W + W' + w)(x + r\beta) - wh \end{aligned}\right\} \ldots\ldots\ldots\text{(ii)}.$$

Now $h - x = r\phi$, and combining this with (i) we get

$$x = \frac{r\phi w}{w'}, \quad h = \frac{r\phi\,(w + w')}{w'},$$

whence from (ii)

$$\frac{W}{W'} = \frac{(W + W' + w)(\phi w + \alpha w') - w\phi\,(w + w')}{(W + W' + w)(\phi w + \beta w') - w\phi\,(w + w')}$$

$$= \frac{\phi w\,(W + W' - w') + \alpha w'\,(W + W' + w)}{\phi w\,(W + W' - w') + \beta w'\,(W + W' + w)},$$

giving the required value of ϕ.

11. Let t be the time from the instant of projection of the second ball to the first impact: the first ball will, at the beginning of this interval, be at a height

$$\frac{v_1^2 - v_2^2}{2g}.$$

$$\therefore \quad (v_1 - v_2)\, t = \frac{v_1^2 - v_2^2}{2g}, \quad \text{i.e. } t = \frac{v_1 + v_2}{2g}.$$

The height of the balls will then be

$$v_1 t - \tfrac{1}{2} g t^2 = \frac{v_1 + v_2}{2g}\left(v_1 - \frac{v_1 + v_2}{4}\right) = \frac{(3v_1 - v_2)(v_1 + v_2)}{8g},$$

and the velocities will be $v_1 - gt$ and $v_2 - gt$, both upwards,

$$\text{i.e. } \frac{v_1 - v_2}{2} \text{ and } -\frac{v_1 - v_2}{2}.$$

At the first impact the balls, being perfectly elastic, interchange velocities, i.e. the velocity of the second ball is merely reversed, and it must therefore reach the ground again after time t with velocity v_1. But the velocity of the first ball upwards at this instant is

$$\frac{v_1 - v_2}{2} - gt = \frac{v_1 - v_2}{2} - \frac{v_1 + v_2}{2} = -v_2.$$

Hence from this instant the previous motion is repeated, the only change being the substitution of $-v_2$ for v_2, thus completing the results (ii) and (iii).

Again, the time between successive impacts of the balls

= time between first impact and second ball reaching the ground

+ time between second ball reaching ground and second impact

$$= \frac{v_1 + v_2}{2g} + \frac{v_1 - v_2}{2g} = \frac{v_1}{g}.$$

12. If the particle is describing a conical pendulum of semi-vertical angle $\alpha + \theta$, we have

$$T = m\omega^2 l, \quad mg = T \cos(\alpha + \theta);$$

$$\therefore \quad \omega^2 = \frac{g}{l \cos(\alpha + \theta)}.$$

If in this case the cone is about to turn about a point O in the base, the moments of T and the weight of the cone about O must be equal, i.e. $T\sin\theta = Mg\sin\alpha$;

$$\therefore \quad \frac{\cos(\alpha+\theta)}{m} = \frac{\sin\theta}{M\sin\alpha} = \frac{\cos\alpha\cos\theta}{m + M\sin^2\alpha}$$

$$= \frac{1}{\sqrt{M^2\sin^2\alpha + \dfrac{1}{\cos^2\alpha}(m + M\sin^2\alpha)^2}}$$

$$= \frac{1}{\sqrt{(M+m)^2\tan^2\alpha + m^2}};$$

$$\therefore \quad \omega^2 = \frac{g}{l}\cdot\frac{\sqrt{(M+m)^2\tan^2\alpha + m^2}}{m}$$

$$= \frac{g}{l}\left[1 + \left(1 + \frac{M}{m}\right)^2\tan^2\alpha\right]^{\frac{1}{2}},$$

giving the required maximum value of ω.

XLV.

1. Let I be the in-centre and let AP, AQ cut the inner circle in E and F. Let PQ touch the inner circle at D. Then if HAK is the common tangent at A,

$$A\hat{D}E = A\hat{Q}P = H\hat{A}P.$$

Also $\qquad EDP = EAD$; $\therefore$ $ADP = ADE + EAD$.

But $\qquad ADP = AQD + DAQ$; $\therefore$ $EAD = DAQ$;

$\therefore$ I lies on AD.

Further, $AE : AP =$ the ratio of the diameters of the circles;

$\therefore$ $PE : PA$ is a constant ratio.

But $\qquad PE \,.\, PA = PD^2$, and $PA : PD = AI : ID$;

$\therefore$ $AI : ID$ is constant.

Hence $AI : AD$ is constant, and therefore since D describes the circle ADF, evidently I describes another circle touching this circle at A.

2. Let PQ be the chord of curvature at P, V its middle point. Let the tangents at P and Q meet in T. Then TV is parallel to the axis, and PV, PT are equally inclined to the axis;

$$\therefore \quad PT = PV.$$

Also if TV meets the parabola in P', then $TP' = P'V$. Hence PP' is perpendicular to TV. Suppose PP' and PQ meet the axis in N and L respectively, and draw VM perpendicular to the axis.

Then $$VM = P'N = PN; \quad \therefore \quad LM = NL.$$

But $$AL = AN + NL;$$

$$\therefore \quad AM = AN + 2NL = AN + 2Nt,$$

if the tangent at P meets the axis in t.

But $$AN = At;$$

$$\therefore \quad Nt = 2AN; \quad \therefore \quad AM = 5AN,$$

and $$VM^2 = P'N^2 = 4AS \,.\, AN = \tfrac{4}{5}AS \,.\, AM,$$

i.e. the locus of V is a parabola of latus-rectum $\frac{4}{5}AS$.

3. Taking a particular value of z, say p, the equation

$$x + y = n - 2p$$

has (including zero) $n - 2p + 1$ solutions.

Hence the number of solutions of the original equation is

$$\sum_{p=0}^{p=m} (n - 2p + 1),$$

where $m = \dfrac{n}{2}$ if n is even, and $= \dfrac{n-1}{2}$ if n is odd.

This sum is

$$(n+1)(m+1) - m(m+1) = (m+1)(n+1-m),$$

and if $m = \dfrac{2n - 1 + (-1)^n}{4}$ (which includes both cases above) this is equal to

$$\frac{2n + 3 + (-1)^n}{4} \cdot \frac{2n + 5 - (-1)^n}{4}$$

$$= \frac{(2n+3)(2n+5) - 1 + 2(-1)^n}{16}$$

$$= \tfrac{1}{8}\{2n^2 + 8n + 7 + (-1)^n\}.$$

4. If u_n denote either the numerator or denominator of the nth convergent, we have

$$u_{n+1} = u_n - x^n (1 - x^{n-1}) u_{n-1};$$

$$\therefore \quad u_{n+1} - (1 - x^n) u_n = x^n \left[u_n - (1 - x^{n-1}) u_{n-1}\right].$$

Similarly

$$u_n - (1 - x^{n-1}) u_{n-1} = x^{n-1} \left[u_{n-1} - (1 - x^{n-2}) u_{n-2}\right],$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_3 - (1 - x^2) u_2 = x^2 \left[u_2 - (1 - x) u_1\right].$$

Multiplying these equations we get

$$u_{n+1} - (1 - x^n) u_n = x^{\frac{n(n+1)}{2} - 1} \left[u_2 - (1 - x) u_1\right].$$

Now for the numerators $u_2 - (1 - x) u_1 = x$, so that this equation becomes

$$u_{n+1} - (1 - x^n) u_n = x^{\frac{n(n+1)}{2}},$$

or
$$\frac{u_{n+1}}{(1 - x)(1 - x^2) \dots (1 - x^n)} - \frac{u_n}{(1 - x)(1 - x^2) \dots (1 - x^{n-1})}$$

$$= \frac{x^{\frac{n(n+1)}{2}}}{(1 - x)(1 - x^2) \dots (1 - x^n)}.$$

Writing down the succession of similar equations and adding, we obtain

$$\frac{u_{n+1}}{(1 - x)(1 - x^2) \dots (1 - x^n)} = 1 + \sum_1^n \frac{x^{\frac{n(n+1)}{2}}}{(1 - x)(1 - x^2) \dots (1 - x^n)}.$$

For the denominators $u_2 - (1 - x) u_1 = 0$, which evidently leads to

$$\frac{u_{n+1}}{(1 - x)(1 - x^2) \dots (1 - x^n)} = 1.$$

Thus the infinite continued fraction is equal to the infinite series

$$1 + \sum_1^\infty \frac{x^{\frac{n(n+1)}{2}}}{(1 - x)(1 - x^2) \dots (1 - x^n)} \quad \dots\dots\dots\dots\dots\text{(i)}.$$

Now consider the infinite product

$$(1+x)(1+ax)(1+a^2x)\ldots,$$

which is convergent if $a<1$. Suppose it equal to

$$1+A_1x+A_2x^2+\ldots.$$

Then, changing x into ax, we have the identity

$$(1+x)(1+A_1ax+\ldots+A_na^nx^n+\ldots)$$
$$=1+A_1x+\ldots+A_nx^n+\ldots.$$

Equating coefficients of x^n, we have

$$A_n=A_na^n+A_{n-1}a^{n-1}; \qquad \therefore\ A_n=\frac{a^{n-1}}{1-a^n}\cdot A_{n-1}.$$

Hence $$A_n=\frac{a^{n-1}}{1-a^n}\cdot\frac{a^{n-2}}{1-a^{n-1}}\cdots\frac{a}{1-a^2}\cdot\frac{1}{1-a}$$

$$=\frac{a^{\frac{n(n-1)}{2}}}{(1-a)(1-a^2)\ldots(1-a^n)}.$$

Now putting $a=x$ (as we may do, since $x<1$), it follows that the infinite product $(1+x)(1+x^2)(1+x^3)\ldots$ is equal to the series (i), which proves the result.

5. We have

$$\cos(\theta+\alpha)\left[\cos(\psi+\theta)\cos(\psi+\alpha)-\cos(\theta+\phi)\cos(\phi+\alpha)\right]$$
$$=a(\cos 2\psi-\cos 2\phi),$$

i.e. $$\tfrac{1}{2}\cos(\theta+\alpha)\left[\cos(2\psi+\theta+\alpha)-\cos(2\phi+\theta+\alpha)\right]$$
$$=a(\cos 2\psi-\cos 2\phi),$$

whence, since $\sin(\phi-\psi)\neq 0$, we get

$$\cos(\theta+\alpha)\sin(\theta+\phi+\psi+\alpha)=2a\sin(\phi+\psi)\ \ldots\ldots\text{(i)},$$

and two similar equations, which are all satisfied if

$$\theta+\phi+\psi+\alpha=n\pi, \qquad a=0.$$

If not, $\cos(\theta+\alpha)\sin(\psi+\theta)=\cos(\phi+\alpha)\sin(\phi+\psi)$,

i.e. $\sin(2\theta+\psi+\alpha)-\sin(2\phi+\psi+\alpha)=0$, whence, since $\sin(\theta-\phi)\neq 0$, we get

$$\cos(\theta+\phi+\psi+\alpha)=0;$$

$$\therefore\quad \theta+\phi+\psi+\alpha=n\pi+\frac{\pi}{2}.$$

In this case $\qquad \sin(\theta+\phi+\psi+\alpha)=(-1)^n,$

and $\qquad \sin(\phi+\psi)=(-1)^n\cos(\theta+\alpha),$

whence, from (i), $\qquad a=\tfrac{1}{2}.$

6. From the equation

$$\cos n\phi-\cos n\theta=2^{n-1}.\prod_0^{n-1}\left\{\cos\phi-\cos\left(\theta+\frac{2r\pi}{n}\right)\right\}$$

we get

$$\log(\cos n\phi-\cos n\theta)=\log 2^{n-1}+\sum_0^{n-1}\log\left\{\cos\phi-\cos\left(\theta+\frac{2r\pi}{n}\right)\right\},$$

whence, changing ϕ into $\phi+h$ and equating coefficients of h (or differentiating with regard to ϕ), we get

$$\frac{n\sin n\phi}{\sin\phi}\cdot\frac{1}{\cos n\phi-\cos n\theta}=\sum_0^{n-1}\frac{1}{\cos\phi-\cos\left(\theta+\frac{2r\pi}{n}\right)}.$$

Hence, if $a<b$, putting $\cos\phi=-\frac{a}{b}$, the required sum is

$$-\frac{n\sin n\phi}{\sqrt{b^2-a^2}}\cdot\frac{1}{\cos n\phi-\cos n\theta}.$$

If $a>b$, the sum is

$$\frac{n\sinh nu}{\sqrt{a^2-b^2}}\cdot\frac{1}{\cosh nu-\cos n\theta},$$

where $\qquad \cosh u=\frac{a}{b}.$

7. Suppose the radius of the circle is ρ. Its centre is $(a, 0)$, the parabola being $y^2=4ax$.

If the chords m_1, m_2 and m_1, m_3 touch the circle, we must have

$$\frac{4a^2(1+m_1m_2)^2}{4+(m_1+m_2)^2}=\frac{4a^2(1+m_3m_1)^2}{4+(m_3+m_1)^2}=\rho^2,$$

i.e. m_2 and m_3 are the roots of

$$\frac{(1+m_1 t)^2}{4+(m_1+t)^2} = \frac{\rho^2}{4a^2} \; (\equiv \lambda, \text{ suppose}),$$

or $$(\lambda - m_1^2)\, t^2 + 2m_1 (\lambda - 1)\, t + \lambda (4 + m_1^2) - 1 = 0.$$

Hence the equation to the chord m_2, m_3 will be

$$x(\lambda - m_1^2) + y m_1 (\lambda - 1) + a\{\lambda (4 + m_1^2) - 1\} = 0,$$

and its envelope for different values of m_1 is

$$y^2 (\lambda - 1)^2 = 4\,(a\lambda - x)(x\lambda + 4a\lambda - a),$$

or $$4\lambda x^2 + (\lambda - 1)^2 y^2 - 4a(\lambda^2 - 4\lambda + 1)\, x - 4a^2 \lambda (4\lambda - 1) = 0,$$

which is a circle coinciding with the original circle if

$$(\lambda - 1)^2 = 4\lambda, \qquad \text{i.e. } \lambda = 3 \pm 2\sqrt{2},$$

whence $$\frac{\rho}{2a} = \sqrt{2} \pm 1.$$

The greater value corresponds to one of the escribed circles.

8. Any conic of closest contact at (x', y') is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right)^2 = 0,$$

and this will be a parabola if

$$\left(\frac{1}{a^2} + \frac{\lambda x'^2}{a^4}\right)\left(\frac{1}{b^2} + \frac{\lambda y'^2}{b^4}\right) = \lambda^2 \cdot \frac{x'^2 y'^2}{a^4 b^4},$$

whence $\lambda = -1$, and the equation is

$$(xy' - x'y)^2 + 2b^2 xx' + 2a^2 yy' - 2a^2 b^2 = 0,$$

and the usual formulae gives the axis and latus-rectum.

[For $(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$ they are

$$\alpha x + \beta y + \frac{\alpha g + \beta f}{\alpha^2 + \beta^2} = 0 \text{ and } \frac{2(f\alpha - g\beta)}{(\alpha^2 + \beta^2)^{\frac{3}{2}}}.]$$

9. P is the pole of $\alpha = 0$, and its co-ordinates are therefore given by

$$h\alpha' + b\beta' + f\gamma' = 0, \qquad g\alpha' + f\beta' + c\gamma' = 0,$$

$$\text{i.e. } \frac{\alpha'}{A} = \frac{\beta'}{H} = \frac{\gamma'}{G},$$

and therefore the equation to AP is

$$\frac{\beta}{H}=\frac{\gamma}{G}, \text{ or } \frac{\beta}{HF}=\frac{\gamma}{FG}.$$

Similarly for BQ, CR, and the lines meet in the point

$$\frac{\alpha}{GH}=\frac{\beta}{HF}=\frac{\gamma}{FG}.$$

10. In the extended position, let the strings make angles $\alpha-\theta$ with the vertical, θ being small. Let $AB=a$. Then if T be the tension of either wire

$$T=E\,.\,\frac{a\dfrac{\sin\alpha}{\sin(\alpha-\theta)}-a}{a}=E\left(\frac{1}{1-\theta\cot\alpha}-1\right)$$

$$=E\,.\,\theta\cot\alpha,$$

neglecting θ^2 and higher powers.

Also $$2T\cos(\alpha-\theta)=W,$$

i.e. $$2E\,.\,\theta\cot\alpha\,(\cos\alpha+\theta\sin\alpha)=W;$$

$$\therefore\quad \theta=\frac{W\sin\alpha}{2E\cos^2\alpha},$$

and B is lowered a distance

$$\frac{a\sin\theta}{\sin(\alpha-\theta)}=\frac{a\theta}{\sin\alpha}=\frac{Wa}{2E\cos^2\alpha}.$$

11. Let u_r, v_r be the velocities of the particle along and perpendicular to the face of the wedge, V_r that of the wedge horizontally just after the rth impact, and suppose the next impact occurs after time t. In this time the wedge will have moved a distance $V_rt\sin\alpha$ perpendicular to its face;

$$\therefore\quad v_rt-\tfrac{1}{2}g\cos\alpha\,.\,t^2=-V_rt\sin\alpha;$$

$$\therefore\quad t=2\,.\,\frac{v_r+V_r\sin\alpha}{g\cos\alpha}.$$

Hence the velocity of the particle perpendicular to the face just before the next impact is

$$v_r-g\cos\alpha\,.\,t=-v_r-2V_r\sin\alpha.$$

Hence, by Newton's Law,

$$-v_{r+1}-V_{r+1}\sin\alpha=-e(V_r\sin\alpha+v_r),$$

$$\text{i.e.}\quad (V_{r+1}-eV_r)\sin\alpha=ev_r-v_{r+1}.$$

Taking this equation for all values of r from $n-1$ to 0, multiplying successive equations by 1, e, e^2, ... e^{n-1} and adding, we have

$$V_n\sin\alpha=e^n v_0-v_n \quad\text{.....................(i)},$$

since $V_0=0$ (the suffix zero referring to the velocities just before the first impact).

Again, the equation of momentum is

$$(v_{r+1}+v_r+2V_r\sin\alpha)\sin\alpha=\tfrac{1}{2}(V_{r+1}-V_r),$$

or, by (i),
$$(v_{r+1}-v_r+2e^r v_0)\sin\alpha=\tfrac{1}{2}(V_{r+1}-V_r),$$

and adding this succession of equations, we get

$$\{v_n-v_0+2(1+e+\dots+e^{n-1})v_0\}\sin\alpha=\tfrac{1}{2}V_n,$$

i.e., by (i),

$$\left\{(e^n-1)v_0-V_n\sin\alpha+2\,.\,\frac{1-e^n}{1-e}v_0\right\}\sin\alpha=\tfrac{1}{2}V_n,$$

whence
$$(1-e^n)\frac{1+e}{1-e}v_0\sin\alpha=(\tfrac{1}{2}+\sin^2\alpha)V_n,$$

and since $v_0=u\cos\alpha$, this gives the required value of V_n.

12. We may suppose the two particles projected from the same point with the same velocity V, the interval between the instants of projection being τ. The equations to the tangents to their paths after times t and t' are

$$y-(V\sin\alpha.t-\tfrac{1}{2}gt^2)=\frac{V\sin\alpha-gt}{V\cos\alpha}.(x-V\cos\alpha\,.\,t)$$

and a similar equation in t'. Solving these, we find

$$x=\tfrac{1}{2}(t+t')V\cos\alpha,$$

$$y=\tfrac{1}{2}(t+t')V\sin\alpha-\tfrac{1}{2}gtt'$$

$$=\tfrac{1}{2}(t+t')V\sin\alpha-\tfrac{1}{8}g\{(t+t')^2-\tau^2\},$$

where
$$t-t'=\tau.$$

Putting $$\tfrac{1}{2}(t+t')=T,$$
these are $$x=V\cos\alpha\,.\,T,$$
$$y-\tfrac{1}{8}g\tau^2=V\sin\alpha\,.\,T-\tfrac{1}{2}gT^2,$$
shewing that the intersection moves in the manner specified.

XLVI.

1. Let $OP=r$, and let p be the perpendicular from O on the tangent at P, the corresponding accented letters referring to the reciprocal curve, and the suffix 1 to the consecutive point on the curve. Then if C is the centre of curvature at P, we have, by Euclid,
$$OC^2=r^2+\rho^2-2\rho p,$$
and OC and ρ are the same for the consecutive point;
$$\therefore\quad \rho=\frac{r_1^2-r^2}{2(p_1-p)}.$$
So $$\rho'=\frac{r_1'^2-r'^2}{2(p_1'-p')}.$$
Now if k is the constant of reciprocation,
$$pr'=p'r=k^2.$$

Using these to eliminate the quantities p, we find
$$4\rho\rho'=\frac{rr_1(r+r_1)}{k^2}\,.\,\frac{r'r_1'(r_1'+r')}{k^2},$$
or, ultimately when the points coincide,
$$\rho\rho'=\frac{r^3r'^3}{k^4}.$$

Now $$PV=2\rho\,.\,\frac{p}{r}=2\rho\,.\,\frac{k^2}{rr'};$$
$$\therefore\quad PV\,.\,P'V'=4\rho\rho'\,.\,\frac{k^4}{r^2r'^2}=4rr'.$$

2. Let the section cut the base in the line PP' and let AB be the diameter of the base perpendicular to this line, these lines intersecting in N. Let V be the vertex of the section, V' its projection on the base. Now the section is greatest when its projection on the base is greatest, since the angle between the two is always the same. But this projection is a portion of a parabola, vertex V', bounded by the double ordinate PNP'. The area of this is $\frac{2}{3}.\triangle PBP'$, since V' bisects BN and therefore the tangents at P and P' meet in B. Now the triangle PBP' is greatest when it is equilateral, and its area is then $\dfrac{3\sqrt{3}}{4}r^2$. Hence the greatest area of the parabolic projection is $\dfrac{\sqrt{3}}{2}r^2$, and therefore that of the original section is $\dfrac{\sqrt{3}}{2}r^2 \sec\alpha$, where α is the angle the section makes with the base. But since the section is parallel to a generator, $\sec\alpha = \dfrac{\sqrt{r^2+h^2}}{r}$. Hence the required maximum area is

$$\frac{\sqrt{3}}{2} r\sqrt{r^2+h^2}.$$

3. Consider the equation

$$\frac{x_1}{a_1+\theta}+\frac{x_2}{a_2+\theta}+\dots+\frac{x_n}{a_n+\theta} = 1 - \frac{(\theta-a_1)(\theta-a_2)\dots(\theta-a_n)}{(a_1+\theta)(a_2+\theta)\dots(a_n+\theta)}.$$

By virtue of the given equations it is satisfied by

$$\theta = a_1,\ a_2,\ \dots\ a_n.$$

But, clearing of fractions, we see that the equation is only of degree $n-1$ in θ. Hence the equation must be an identity, and therefore true for all values of θ. Putting $\theta = 0$, the result in question follows immediately.

4. We have

$$\begin{aligned}(3p)!-6(p!)^3 &= 6p^3.(p-1)!\,[(p+1)\dots(2p-1)(2p+1)\dots\\ &\qquad\qquad (3p-1)-\{(p-1)!\}^2]\\ &= 6p^3.(p-1)!\,[(4p^2-1^2)(4p^2-2^2)\dots\\ &\qquad\qquad (4p^2-\overline{p-1}^2)-\{(p-1)!\}^2].\end{aligned}$$

Now, if p is prime and

$$(x+1)(x+2)\dots(x+p-1)$$
$$= x^{p-1} + A_1 x^{p-2} + \dots + A_{p-2}x + (p-1)!,$$

then, by Lagrange's Theorem, the coefficients A are all multiples of p.

Also, since $(p-1)$ is even, we have

$$(x-1)(x-2)\dots(x-\overline{p-1})$$
$$= x^{p-1} - A_1 x^{p-2} - \dots - A_{p-2}x + (p-1)!.$$

Hence multiplying

$$(x^2-1^2)(x^2-2^2)\dots(x^2-\overline{p-1}^2) - \{(p-1)!\}^2$$
$$= x^{2p-2} + \dots + \{2A_{p-3}\,.\,(p-1)! - A_{p-2}^2\}\, x^2,$$

and the coefficient of x^2 on the right is a multiple of p. Hence, putting $x = 2p$, we see that the expression in the square bracket above is a multiple of p^3. Hence the original expression is a multiple of p^6.

5. We have

$$\sin(\xi + i\eta) = \frac{1}{\sin(x+iy)} = \frac{\sin(x-iy)}{\sin^2 x + \sinh^2 y}.$$

Calling the denominator D, this equation gives

$$\left.\begin{aligned} \sin\xi\cosh\eta &= \frac{\sin x\cosh y}{D} \\ \cos\xi\sinh\eta &= -\frac{\cos x\sinh y}{D} \end{aligned}\right\} \dots\dots\dots\dots\dots\text{(i)}.$$

Eliminating η, and putting $\tan\xi = t$, we obtain

$$\frac{\sin^2 x\cosh^2 y}{D^2}\left(\frac{1}{t^2}+1\right) - \frac{\cos^2 x\sinh^2 y}{D^2}(t^2+1) = 1,$$

or $\quad \cos^2 x\sinh^2 y\,.\,t^4 + (D^2 + \cos^2 x\sinh^2 y - \sin^2 x\cosh^2 y)\,t^2$

$$- \sin^2 x\cosh^2 y = 0 \dots\text{(ii)}.$$

The coefficient of t^2 is

$$(\sin^2 x + \sinh^2 y)^2 + (1-\sin^2 x)\sinh^2 y - \sin^2 x(1+\sinh^2 y)$$
$$= \sin^4 x + \sinh^4 y + \sinh^2 y - \sin^2 x$$
$$= \cosh^2 y\sinh^2 y - \cos^2 x\sin^2 x.$$

Hence the equation (ii) becomes

$$(\sinh^2 y \,.\, t^2 - \sin^2 x)\,(\cos^2 x \,.\, t^2 + \cosh^2 y) = 0,$$

and therefore, since ξ is real,

$$\tan \xi = \pm \frac{\sin x}{\sinh y}.$$

Also from (i)

$$\frac{\tanh \eta}{\tan \xi} = -\frac{\cos x \sinh y}{\sin x \cosh y},$$

from which, using the values for $\tan \xi$, the second result follows.

6. The distances in question are

$$a \sec \alpha - a \cos \theta, \qquad a \sec \alpha - a \cos\left(\theta + \frac{2\pi}{n}\right), \text{ etc.}$$

Now $\quad x^n - 2\cos n\theta + x^{-n} = \prod_{r=0}^{r=n-1} \left\{x - 2\cos\left(\theta + \frac{2r\pi}{n}\right) + x^{-1}\right\}.$

Putting $x = \sec \alpha - \tan \alpha$, we have $x^{-1} = \sec \alpha + \tan \alpha$, and this becomes

$$(\sec \alpha - \tan \alpha)^n + (\sec \alpha + \tan \alpha)^n - 2 \cos n\theta$$
$$= \prod_{r=0}^{r=n-1} \left\{2 \sec \alpha - 2 \cos\left(\theta + \frac{2r\pi}{n}\right)\right\} \quad \text{......(i)}.$$

Now, neglecting h^2, h^3, etc., we have

$$\sec(\alpha + h) = \sec \alpha + h \sec \alpha \tan \alpha, \quad \tan(\alpha + h) = \tan \alpha + h \sec^2 \alpha.$$

Substituting these in (i) and equating coefficients of h (or differentiating with regard to α), we get

$$n(\sec \alpha - \tan \alpha)^{n-1}(\sec \alpha \tan \alpha - \sec^2 \alpha)$$
$$+ n(\sec \alpha + \tan \alpha)^{n-1}(\sec \alpha \tan \alpha + \sec^2 \alpha)$$
$$= 2 \sec \alpha \tan \alpha \,.\, \sum_0^{n-1} P_r \quad \text{............(ii)},$$

where P_r is the product obtained by omitting the rth factor in the product (i).

Now, dividing (ii) by (i), we obtain

$$\frac{n \sec \alpha \{(\sec \alpha + \tan \alpha)^n - (\sec \alpha - \tan \alpha)^n\}}{(\sec \alpha - \tan \alpha)^n + (\sec \alpha + \tan \alpha)^n - 2\cos n\theta}$$

$$= \sec \alpha \tan \alpha \,.\, \sum_{r=0}^{r=n-1} \frac{1}{\sec \alpha - \cos\left(\theta + \dfrac{2r\pi}{n}\right)},$$

whence the result follows.

7. If P be any point on a circle whose centre is O, then

$$\Sigma AP^2 \,.\, \triangle BOC = OP^2 \,.\, \triangle ABC + \Sigma OA^2 \,.\, \triangle BOC \ldots\ldots \text{(i)}.$$

Now if O be the centre of the circle in question, S the circumcentre, and I the incentre, then O, S, I are collinear and $OS = SI$. If x be the perpendicular from O on BC, then

$$x + r = 2R \cos A\,;$$

$$\therefore \quad \triangle BOC = \tfrac{1}{2} xa = \tfrac{1}{2} aR \left(2 \cos A - \frac{r}{R}\right),$$

and
$$\frac{r}{R} = \Sigma \cos A - 1\,;$$

$$\therefore \quad \triangle BOC = \tfrac{1}{2} aR\,(1 + \cos A - \cos B - \cos C),$$

and so for the others.

Further,

$$OA^2 + AI^2 = 2AS^2 + 2SI^2 = 2R^2 + 2\,(R^2 - 2Rr)\,;$$

$$\therefore \quad OA^2 = 4R^2 - 4Rr - r^2 \operatorname{cosec}^2 \frac{A}{2}$$

$$= 4R^2 - bc.$$

Also $OP = 2R$, since the circumcircle of the original triangle is the nine-point circle of the triangle formed by the e-centres.

Hence the right-hand side of (i) is

$$8R^2\Delta - \Sigma bc \,.\, \triangle BOC$$
$$= \tfrac{1}{2} abcR\,[4 - \Sigma\,(1 + \cos A - \cos B - \cos C)]$$
$$= \tfrac{1}{2} abcR\,(1 + \Sigma \cos A).$$

8. If the axes be inclined at an angle ω, the latus-rectum of the parabola

$$(ax + \beta y)^2 + 2gx + 2fy + c = 0$$

is $$2\,(fa - g\beta)\sin^2\omega/(a^2 + \beta^2 - 2a\beta\cos\omega)^{\frac{3}{2}}.$$

Now, taking AB, AC as axes, the parabola in question is

$$\sqrt{\frac{x}{b}} + \sqrt{\frac{y}{c}} = 1,$$

the axes being inclined at an angle A. The rational form of this is

$$\left(\frac{x}{b} - \frac{y}{c}\right)^2 - 2\frac{x}{b} - 2\frac{y}{c} + 1 = 0;$$

and therefore the latus-rectum is

$$2\left(\frac{2}{bc}\right)\sin^2 A\Big/\left(\frac{1}{b^2} + \frac{1}{c^2} + \frac{2}{bc}\cos A\right)^{\frac{3}{2}},$$

i.e. $$\frac{4b^2c^2\sin^2 A}{(a^2 + 4bc\cos A)^{\frac{3}{2}}} = \frac{4b\sin B\sin^2 C\sin^2 A}{(\sin^2 A + 4\cos A\sin B\sin C)^{\frac{3}{2}}},$$

and $$p = b\sin C.$$

9. If P be the point (x, y, z), and E the centre of mean position for multiples l, m, n at A, B, C, we have

$$\Sigma lx^2 = (\Sigma l)\,PE^2 + \Sigma l\,.\,EA^2,$$

so that, in general, the equation $\Sigma lx^2 = k$ represents a circle, centre E. If however $\Sigma l = 0$, then E is at infinity and the equation represents a straight line.

Now let O be the circumcentre, D, E, F the middle points of the sides, and suppose that the line makes an angle θ with BC, and let OD, OF meet the line in L, M. Then the co-ordinates of L are given by $y = z$, $x^2 - y^2 = \dfrac{k}{l}$. Hence if we draw LV perpendicular to AB, and MU perpendicular to BC, we have

$$AV^2 - VB^2 = \frac{k}{l}; \qquad \therefore\ FV = \frac{k}{2cl},$$

and similarly $$DU = \frac{k}{2an}.$$

But
$$\frac{DU}{\cos\theta} = -\frac{FV}{\cos(B+\theta)},$$
$$\text{i.e.}\quad an\cos\theta + cl\cos(B+\theta) = 0,$$
$$\text{i.e.}\quad (an + cl\cos B)\cos\theta = cl\sin B\sin\theta,$$
or
$$\frac{\cos\theta}{cl\sin B} = \frac{\sin\theta}{an + cl\cos B} = \frac{1}{\sqrt{a^2n^2 + 2acnl\cos B + c^2l^2}}$$
$$= \frac{1}{\sqrt{-\Sigma mna^2}}.$$

Hence the perpendicular from O on the line
$$= OL\cos\theta = \frac{FV\cos\theta}{\sin B} = \frac{k}{2\sqrt{-\Sigma mna^2}}.$$

But the line is a tangent to the circumcircle if this perpendicular $= R$, i.e. if
$$k = 2R\sqrt{-\Sigma mna^2}.$$

10. Let the angles BOC, DOE, FOA be 2α, 2β, 2γ, so that $\alpha+\beta+\gamma = 90°$. Let a be the length of a rod.

Then
$$OP = 2a\cos\alpha, \quad OQ = 2a\cos\beta;$$
$$\therefore\ PQ^2 = 4a^2\{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos(60° + \alpha + \beta)\}$$
$$= 4a^2(\cos^2\alpha + \cos^2\beta - \cos\alpha\cos\beta\sin\gamma$$
$$+ \sqrt{3}\cos\alpha\cos\beta\cos\gamma).$$

Similarly
$$QR^2 = 4a^2(\cos^2\beta + \cos^2\gamma - \cos\beta\cos\gamma\sin\alpha$$
$$+ \sqrt{3}\cos\alpha\cos\beta\cos\gamma);$$
$$\therefore\ PQ^2 - QR^2 = 4a^2(\cos^2\alpha - \cos^2\gamma + \cos\beta\sin\overline{\alpha-\gamma})$$
$$= 4a^2\{-\sin(\alpha+\gamma)\sin(\alpha-\gamma) + \cos\beta\sin(\alpha-\gamma)\}$$
$$= 0;$$
$$\therefore\ PQ = QR.$$

Also if T is the tension in PQ, and T' the thrust in QR, the equation of virtual work for a displacement in which the directions of PQ, QR remain unchanged is
$$-T\,.\,\delta(PQ) + T'\,.\,\delta(QR) = 0;$$
$$\therefore\ T = T'.$$

11. The horizontal velocities after the various impacts are

$$eV\cos\alpha,\quad e^2V\cos\alpha,\ \dots\ e^{2n}V\cos\alpha.$$

Hence the time that elapses before the particle returns to the point of projection is

$$T=\frac{a}{V\cos\alpha}+\frac{c}{eV\cos\alpha}+\frac{c}{e^2V\cos\alpha}+\dots$$
$$+\frac{c}{e^{2n-1}V\cos\alpha}+\frac{c-a}{e^{2n}V\cos\alpha}$$
$$=\frac{a}{V\cos\alpha}\left(1-\frac{1}{e^{2n}}\right)+\frac{c}{eV\cos\alpha}\cdot\frac{1-\dfrac{1}{e^{2n}}}{1-\dfrac{1}{e}}$$
$$=\frac{1}{V\cos\alpha}(1-e^{2n})\left\{\frac{c-a(1-e)}{e^{2n}-e^{2n+1}}\right\}.$$

During this time, the vertical velocity remains constant, and the vertical distance described is zero;

$$\therefore\quad V\sin\alpha\,.\,T-\tfrac{1}{2}gT^2=0,$$

$$\text{i.e.}\quad T=\frac{2V\sin\alpha}{g}.$$

Equating these values of T, the result follows.

12. Let ρ, ρ' be the initial radii of curvature, and T the tension of the string. Then

$$T=\frac{pu^2}{\rho}=\frac{qv^2}{\rho'}\ \dots\dots\dots\dots\dots\dots\dots\text{(i)}.$$

If the lengths of the two parts of the string remained constant, the normal accelerations would be $\dfrac{u^2}{a}$ and $\dfrac{v^2}{b}$.

Hence the increase in the normal acceleration due to the lengthening of one string is

$$\frac{u^2}{\rho}-\frac{u^2}{a}.$$

Similarly that due to the lengthening of the other string is

$$\frac{v^2}{\rho'}-\frac{v^2}{b}.$$

But since the string is inextensible, one part lengthens as much as the other shortens. Hence the sum of the two expressions above must be zero, i.e.

$$\frac{u^2}{\rho} + \frac{v^2}{\rho'} = \frac{u^2}{a} + \frac{v^2}{b} \quad \text{.........................(ii)}.$$

Solving (i) and (ii), we obtain the values given.

XLVII.

1. Calling the five given rays a, a'; b, b'; c, we have to construct a sixth ray c', such that $(aa'bc) = (a'ab'c')$. Through a point A' on a' draw $A'ABC$ cutting a, b, c in A, B, C, and also $A'B'K$ cutting a, b' in K, B'. Join CB' cutting a in V, and join VB cutting $A'K$ in C'. Then OC' is the ray required, for we have (if O is the vertex of the pencil)

$$(aa'bc) = O(AA'BC) = AA'BC = V(AA'BC)$$
$$= (KA'C'B') = (aa'c'b') = (a'ab'c').$$

2. Let the parallels through C and D meet the parabola again in points Ω, Ω' on the line at infinity. Then considering the Pascal hexagon $BAD\Omega'\Omega C$, we have that the intersections of BA, $\Omega'\Omega$; AD, ΩC; $D\Omega'$, CB are collinear. But the two latter points are P and Q. Hence PQ is parallel to AB.

3. Let $S \equiv 1 + x + x^2 + \ldots + x^{2m}$. If in the given identity we write $x + h$ for h, the coefficient of h^2 on the right-hand side is

$$a_2 + 3a_3x + 6a_4x^2 + 10a_5x^3 + \ldots.$$

Suppose that by this change S becomes $S + hS' + h^2S'' + \ldots$. We then have

$$(S + hS' + h^2S'' + \ldots)^n = S^n + nS^{n-1}(hS' + h^2S'' + \ldots)$$
$$+ \frac{n(n-1)}{2!} S^{n-2}(hS' + \ldots)^2,$$

and the coefficient of h^2 in this is

$$nS^{n-1}S'' + \frac{n(n-1)}{2!}S^{n-2}S'^2,$$

and we want the value of this when $x = 1$, in which case

$$S = 2m + 1.$$

Now $S' = 1 + 2x + 3x^2 + \ldots + 2mx^{2m-1}$, and when $x = 1$, this is equal to $m(2m+1)$.

Also $S'' = 1 + 3x + 6x^2 + \ldots + \frac{2m(2m-1)}{2} \cdot x^{2m-2}$, and when $x = 1$, this is $\frac{2m(2m-1)(2m+1)}{6}$.

Hence the given series is

$$n(2m+1)^{n-1} \cdot \frac{2m(2m-1)(2m+1)}{6}$$

$$+ \frac{n(n-1)}{2!} \cdot (2m+1)^{n-2} \cdot m^2(2m+1)^2$$

$$= \frac{(2m+1)^n \cdot mn}{6}[2(2m-1) + 3m(n-1)].$$

4. We have

$$(\Sigma a^x)(\Sigma a^y) - \left(\Sigma a^{\frac{x+y}{2}}\right)^2 \equiv \Sigma\left(a^{\frac{x}{2}}b^{\frac{y}{2}} - a^{\frac{y}{2}}b^{\frac{x}{2}}\right)^2.$$

Hence $\qquad \Sigma a^x \cdot \Sigma a^y > \left(\Sigma a^{\frac{x+y}{2}}\right)^2 \qquad$ unless $x = y$,

so that in the continued product

$$\Sigma a^x \cdot \Sigma a^y \cdot \Sigma a^z \ldots,$$

as long as any two indices remain unequal, the product can be diminished by replacing these two indices by their arithmetic mean. Hence

$$\prod_{p=1}^{p=n}(\Sigma a^{r+s+2p-1}) \times (\Sigma a^s)^r$$

is least when each index is replaced by

$$\frac{\sum\limits_{p=1}^{p=n}(r+s+2p-1)+rs}{n+r}=\frac{n(r+s)+n^2+rs}{n+r}$$

$$=n+s,$$

and the product then becomes

$$(\Sigma a^{n+s})^{n+r},$$

since $(n+r)$ is the number of factors.

5. Let $ABCD$ be the quadrilateral, AC and BD making angles θ and ϕ respectively with the sides of the square through their extremities. Then if α is the angle between the diagonals, $\theta+\phi=90^\circ+\alpha$.

Also $\qquad a=x\sin\theta=y\sin\phi,\quad \cos(\theta+\phi)=-\sin\alpha,$

$$\therefore\ \cos\theta\cos\phi=\frac{a^2}{xy}-\sin\alpha\,;$$

$$\therefore\ \left(1-\frac{a^2}{x^2}\right)\left(1-\frac{a^2}{y^2}\right)=\left(\frac{a^2}{xy}-\sin\alpha\right)^2,$$

whence
$$a^2=\frac{x^2y^2\cos^2\alpha}{x^2+y^2-2xy\sin\alpha}.$$

But $xy\sin\alpha=2\Delta$,

$$\therefore\ a^2=\frac{x^2y^2-4\Delta^2}{x^2+y^2-4\Delta}.$$

6. If n is odd, we have

$$\cos n\theta=\cos\theta\,.\prod_{r=1}^{r=\frac{1}{2}(n-1)}\left(1-\frac{\sin^2\theta}{\sin^2\alpha}\right),$$

where $\alpha=\dfrac{(2r-1)\pi}{2n}$, and since $1-\dfrac{\sin^2\theta}{\sin^2\alpha}=\cos^2\theta\left(1-\dfrac{\tan^2\theta}{\tan^2\alpha}\right)$, this may be written

$$\cos n\theta=\cos^n\theta\,.\prod_{1}^{\frac{1}{2}(n-1)}\left(1-\frac{\tan^2\theta}{\tan^2\alpha}\right).$$

Taking logarithms, this becomes

$$\log(\cos n\theta) = \log(\cos^n\theta) + \Sigma\log(\tan^2\alpha - \tan^2\theta) - \Sigma\log(\tan^2\alpha).$$

Writing $\theta + h$ for θ, expanding, and equating coefficients of h (or differentiating with regard to θ), we find

$$-\frac{n\sin n\theta}{\cos n\theta} = -n\tan\theta - 2\Sigma\frac{\tan\theta\sec^2\theta}{\tan^2\alpha - \tan^2\theta},$$

$$\text{i.e. } \frac{n\tan n\theta}{\tan\theta} = n + 2\Sigma\frac{\sec^2\theta}{\tan^2\alpha - \tan^2\theta}.$$

Now $\dfrac{\sec^2\theta}{\tan^2\alpha - \tan^2\theta} + 1 = \dfrac{\sec^2\alpha}{\tan^2\alpha - \tan^2\theta}$, and adding 1 to each term in Σ means adding $(n-1)$ to 2Σ.

$$\therefore\ \frac{n\tan n\theta}{\tan\theta} = 1 + 2\sum_{r=1}^{r=\frac{1}{2}(n-1)}\frac{\sec^2\alpha}{\tan^2\alpha - \tan^2\theta}.$$

7. We have

$$\frac{1}{u_1} + \frac{1}{u_2} + \dots + \frac{1}{u_n} = \frac{1}{u_1 -}\,\frac{u_1^2}{u_1 + u_2 -}\,\frac{u_2^2}{u_2 + u_3 -}\dots\frac{u_{n-1}^2}{u_{n-1} + u_n}.$$

Applying this to the series $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dots$, its equivalent is the continued fraction in question, and the value of the series is $\dfrac{\pi^2}{6}$.

8. Suppose that two of the normals from (x_0, y_0) are at distance r from the centre. Then these are tangents to $x^2 + y^2 = r^2$, so that their equation is

$$(x^2 + y^2 - r^2)(x_0^2 + y_0^2 - r^2) = (xx_0 + yy_0 - r^2)^2,$$

or, when the origin is moved to (x_0, y_0),

$$x^2(y_0^2 - r^2) + y^2(x_0^2 - r^2) - 2x_0y_0xy = 0 \quad \dots\dots\dots\text{(i)}.$$

In the same case the equation to the four normals is (XLIII. 8)

$$(a^2x^2 + b^2y^2)(xy_0 - x_0y)^2 - (a^2 - b^2)^2x^2y^2 = 0 \quad \dots\dots\text{(ii)}.$$

Hence (i) must be a factor of (ii); and so the remaining factor must be

$$\frac{a^2y_0^2}{y_0^2-r^2}\cdot x^2+\frac{b^2x_0^2}{x_0^2-r^2}\cdot y^2+\lambda xy=0 \quad \ldots\ldots\ldots\text{(iii)}.$$

Comparing the remaining coefficients, we get

$$\left.\begin{aligned}\frac{2a^2y_0^2}{y_0^2-r^2}\cdot x_0y_0-\lambda\,(y_0^2-r^2)=2a^2x_0y_0\\ \frac{2b^2x_0^2}{x_0^2-r^2}\cdot x_0y_0-\lambda\,(x_0^2-r^2)=2b^2x_0y_0\end{aligned}\right\}\ \ldots\ldots\ldots\text{(iv)},$$

and $$\frac{b^2x_0^2\,(y_0^2-r^2)}{x_0^2-r^2}+\frac{a^2y_0^2\,(x_0^2-r^2)}{y_0^2-r^2}-2\lambda x_0y_0$$

$$=a^2x_0^2+b^2y_0^2-(a^2-b^2)^2\ldots\ldots\text{(v)}.$$

From (iv) we get

$$\frac{(y_0^2-r^2)^2}{a^2}=\frac{(x_0^2-r^2)^2}{b^2}=\frac{2x_0y_0r^2}{\lambda},$$

whence $$\frac{y_0^2-r^2}{a}=\frac{x_0^2-r^2}{\pm b}=\frac{y_0^2-x_0^2}{a\mp b},\quad r^2=\frac{ax_0^2\mp by_0^2}{a\mp b},$$

$$\lambda=\frac{2x_0y_0\,(a\mp b)\,(ax_0^2\mp by_0^2)}{(y_0^2-x_0^2)^2},$$

whence, substituting in (v), and dividing by $a\mp b$, we find

$$(ax_0^2\mp by_0^2)\cdot\frac{(x_0^2+y_0^2)^2}{(y_0^2-x_0^2)^2}=(a\mp b)\,(a\pm b)^2,$$

which, changing to polars, is the required locus.

9. Let $\Sigma'=0$ be the tangential equation to the circular points at infinity. Then in trilinears

$$\Sigma'=\lambda^2+\mu^2+\nu^2-2\mu\nu\cos A-2\nu\lambda\cos B-2\lambda\mu\cos C.$$

Hence the discriminant of $\Sigma+k\Sigma'$ (where $\Sigma=0$ is the tangential equation to the general conic) is

$$\begin{vmatrix} A+k, & H-k\cos C, & G-k\cos B\\ H-k\cos C, & B+k, & F-k\cos A\\ G-k\cos B, & F-k\cos A, & C+k\end{vmatrix}=0,$$

where $A=bc-f^2$, etc.

On expansion, remembering that $BC - F^2 = a\Delta$, etc., this reduces to

$$\Delta^2 + k\Delta\,(a + b + c - 2f\cos A - 2g\cos B - 2h\cos C)$$
$$+ k^2\,(A\sin^2 A + B\sin^2 B + C\sin^2 C$$
$$+ 2F\sin B\sin C + 2G\sin C\sin A + 2H\sin A\sin B),$$

and the coefficient of k^2 is the determinant given in the question (say D).

In Cartesians $\Sigma' = \lambda^2 + \mu^2$ and the discriminant of $\Sigma + k\Sigma'$ takes the form

$$\Delta^2 + k\Delta\,(a+b) + k^2\,(ab - h^2).$$

Hence, since the two forms are equivalent, we must have $a + b + c - 2f\cos A - 2g\cos B - 2h\cos C$ as the equivalent of $a + b$ in Cartesians, and D as the equivalent of $ab - h^2$. But, in Cartesians,

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}.$$

Hence, in trilinears,

$$\tan\theta = \frac{2\sqrt{D}}{a + b + c - 2f\cos A - 2g\cos B - 2h\cos C}.$$

10. Let O be the centre of the hemisphere, A the point of contact, G the centre of gravity of the remainder.

If x be the side of the cube, then

$$x^2 = a^2 - \left(\frac{x}{\sqrt{2}}\right)^2, \text{ whence } x = \frac{\sqrt{6}}{3}a.$$

The volumes of the hemisphere and cube are $\frac{2}{3}\pi a^3$ and $\dfrac{2\sqrt{6}}{9}a^3$,

and the distances of their centres of gravity from O are $\frac{3}{8}a$, $\dfrac{\sqrt{6}}{6}a$,

$$\therefore OG = \frac{\frac{2}{3}\pi a^3 \cdot \frac{3}{8}a - \dfrac{2\sqrt{6}}{9}a^3 \cdot \dfrac{\sqrt{6}}{6}a}{\frac{2}{3}\pi a^3 - \dfrac{2\sqrt{6}}{9}a^3} = \frac{9\pi - 8}{8\,(3\pi - \sqrt{6})} \cdot a.$$

Now AG is vertical and OA perpendicular to the plane.

Hence if the base makes an angle θ with the horizontal, OG makes an angle θ with the vertical and $\angle OAG = \alpha$. Hence

$$OG \sin\theta = a \sin\alpha, \text{ i.e. } \theta = \sin^{-1}\left(\frac{a \sin\alpha}{OG}\right).$$

11. The acceleration of the system is

$$f = \frac{M'}{M + M' + m} \cdot g,$$

and the velocity after time t is $V = ft$.

After a further time τ, the particle will have moved horizontally a distance

$$x = V\tau + \tfrac{1}{2} f\tau^2,$$

and vertically downwards a distance $y = \frac{1}{2} g\tau^2$.

Hence the equation to the path is

$$\left(x - \frac{yf}{g}\right)^2 = V^2 \cdot \frac{2y}{g},$$

or

$$(gx - fy)^2 - 2gV^2y = 0,$$

and this is a parabola of latus-rectum $\dfrac{2g^2V^2}{(g^2+f^2)^{\frac{3}{2}}}$. Now substitute for V and f.

12. Let O be the fixed end of the tube. Then any drop issuing at P describes a parabola in the vertical plane through P perpendicular to OP. Suppose that at time t after leaving P, this drop is at Q. Draw QN perpendicular to the vertical through P, NM perpendicular to the vertical through O (and therefore parallel to OP). We have then

$$PN = \tfrac{1}{2} gt^2, \quad QN = ut, \quad NM = a, \text{ the length of the tube};$$

$$\therefore QM^2 = QN^2 + NM^2 = u^2t^2 + a^2 = u^2 \cdot \frac{2OM}{g} + a^2$$

$$= \frac{2u^2}{g}\left(OM + \frac{ga^2}{2u^2}\right).$$

Hence Q lies on a paraboloid of revolution about the vertical axis, whose latus-rectum is $\frac{2u^2}{g}$ and whose vertex is at a distance $\frac{ga^2}{2u^2}$ above O.

XLVIII.

1. Let A, B, C be the centres of the circles, $O\alpha$, $O\beta$, $O\gamma$ the radical axes of the pairs B, C, etc. Let P be the moving point, PL, PM the perpendiculars on $O\alpha$, $O\gamma$. Then if T_A is the tangent from P to the circle A, we have

$$T_A^2 - T_B^2 = 2PM \,.\, AB, \quad T_B^2 - T_C^2 = 2PL \,.\, BC.$$

Hence, by hypothesis

$$PM \,.\, AB = PL \,.\, BC, \text{ i.e. } PL : PM = AB : BC.$$

Hence the locus of P is a straight line through O.

Further $\frac{PL}{PM} = \frac{\sin PO\alpha}{\sin PO\gamma}, \quad \frac{AB}{BC} = \frac{\sin C}{\sin A} = \frac{\sin \beta O\alpha}{\sin \beta O\gamma}$.

Hence the pencil $O\,(\alpha\gamma,\, P\beta)$ is harmonic.

2. The director circles of all conics touching four straight lines are coaxal. There is one parabola touching the four lines and therefore its directrix, being the limiting form of a director circle, must be the radical axis of the coaxal system. Also since the double lines joining the extremities of diagonals are limiting forms of conics touching the four lines, it follows that the circles on the diagonals as diameters must be members of the coaxal system, and must therefore cut the radical axis (i.e. the directrix of the parabola) in the same two points.

3. (i) Since ${}^{r+1}C_2 = \frac{(r+1)\,r}{2}$, the given series is evidently the coefficient of $\frac{1}{x}$ in the expansion of

$$-\left(1-\frac{1}{x}\right)^n (1-x)^{-3},$$

i.e. $(-1)^{n+1}\,(1-x)^{n-3}/x^n$, and this coefficient is zero.

(ii) If the expression

$$\frac{(x+2)(x+4)\dots(x+2p)}{(x+1)(x+3)\dots(x+2p+1)}$$

be put into partial fractions, the coefficient corresponding to the denominator $x+2p-2r+1$ is

$$\frac{(-2p+2r+1)(-2p+2r+3)\dots(-1)\,.\,1\,.\,\dots(2r-3)(2r-1)}{(-2p+2r)(-2p+2r+2)\dots(-2)\,.\,2\,.\,\dots(2r-2)\,.\,2r}$$

$$=\frac{(2p-2r-1)(2p-2r-3)\dots 3\,.\,1\,.\,1\,.\,3\dots(2r-1)}{2^p\,.\,(p-r)!\,r!}$$

$$=\frac{1}{2^{2p}}\cdot\frac{(2p-2r)!}{\{(p-r)!\}^2}\cdot\frac{(2r)!}{(r!)^2}$$

$$=\frac{1}{4^p}\cdot{}^{2p-2r}C_{p-r}\,.\,{}^{2r}C_r,$$

and the formula given follows immediately by putting $x=0$.

4. If p_n is the numerator of the nth convergent, we have

$$p_n=(1-x^{2n-1})\,p_{n-1}+x^{2n-3}\,p_{n-2},$$

$$\therefore\ p_n+x^{2n-1}\,p_{n-1}=p_{n-1}+x^{2n-3}\,p_{n-2},$$

$$\dots\dots\dots\dots\dots\dots$$

$$p_3+x^5\,p_2=p_2+x^3\,p_1,$$

$$\therefore\ p_n+x^{2n-1}\,p_{n-1}=p_2+x^3\,p_1.$$

Now $p_2+x^3p_1=1$, and $q_2+x^3q_1=1$. Hence

$$p_n+x^{2n-1}\,p_{n-1}=1,$$

and since $2n-1=n^2-(n-1)^2$, this may be written

$$(-1)^{n-1}x^{-n^2}\,.\,p_n-(-1)^{n-2}\,.\,x^{-(n-1)^2}\,.\,p_{n-1}=(-1)^{n-1}\,.\,x^{-n^2}.$$

Writing down the succession of equations, the last is

$$-p_2x^{-2^2}-p_1x^{-1^2}=-x^{-2^2},$$

whence adding,

$$(-1)^{n-1}\,.\,x^{-n^2}\,.\,p_n=p_1x^{-1^2}-x^{-2^2}+x^{-3^2}-\dots+(-1)^{n-1}\,.\,x^{-n^2},$$

and the same equation is true for q_n.

But $p_1 = 1$, and $q_1 = 1 - x$. Hence

$$(-1)^{n-1} \cdot x^{-n^2} \cdot p_n = \sigma_n,$$
$$(-1)^{n-1} \cdot x^{-n^2} \cdot q_n = \sigma_n - 1,$$

i.e. the nth convergent is $\dfrac{\sigma_n}{\sigma_n - 1}$.

5. Take the case $A > B > C$, and the angles all acute, and denote the areas in question by Δ, Δ_1, Δ_2, Δ_3. Then we have

$$\Delta = API + AOI - APO, \quad \Delta_2 = APO + AOI_2 - API_2,$$
$$\Delta_1 = API_1 + AOI_1 - APO, \quad \Delta_3 = AOI_3 - API_3 - APO,$$

whence

$$\begin{aligned}\Delta + \Delta_1 - \Delta_2 + \Delta_3 &= (API + AOI) + (API_1 + AOI_1) + (API_2 - AOI_2) \\ &\qquad - (API_3 - AOI_3) - 4APO \\ &= \tfrac{1}{2}(AP + AO)(AI + AI_1) \sin \frac{B - C}{2} \\ &\qquad + \tfrac{1}{2}(AP - AO)(AI_2 - AI_3) \cos \frac{B - C}{2} - 4APO,\end{aligned}$$

since each of the angles PAI and OAI is $\dfrac{B - C}{2}$.

But $\quad AI + AI_1 = 4R \cos \dfrac{B - C}{2}, \quad AI_2 - AI_3 = 4R \sin \dfrac{B - C}{2}$;

$$\begin{aligned}\therefore\quad \Delta + \Delta_1 - \Delta_2 + \Delta_3 &= R(AP + AO) \sin (B - C) \\ &\qquad + R(AP - AO) \sin (B - C) - 4APO \\ &= 2R \cdot AP \sin (B - C) - 4APO \\ &= 0,\end{aligned}$$

and similarly for other cases.

6. We have

$$1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \ldots = \tfrac{1}{3}(e^x + e^{\omega x} + e^{\omega^2 x}) \text{ where } \omega = e^{\frac{2i\pi}{3}} \text{ (so that } \omega^3 = 1).$$

Hence the given sum is the real part of

$$\tfrac{1}{3}\left[e^{e^{i\theta}} + e^{\omega e^{i\theta}} + e^{\omega^2 e^{i\theta}}\right],$$

and since $\omega = e^{\frac{2i\pi}{3}}$, this is

$$\frac{1}{3}\left[e^{e^{i\theta}} + e^{e^{i\left(\frac{2\pi}{3}+\theta\right)}} + e^{e^{i\left(\frac{4\pi}{3}+\theta\right)}}\right],$$

and the real part of this is

$$\frac{1}{3}\left[e^{\cos\theta}\cos(\sin\theta) + e^{\cos\left(\frac{2\pi}{3}+\theta\right)}\cos\left\{\sin\left(\frac{2\pi}{3}+\theta\right)\right\}\right.$$
$$\left. + e^{\cos\left(\frac{4\pi}{3}+\theta\right)}\cos\left\{\sin\left(\frac{4\pi}{3}+\theta\right)\right\}\right],$$

which is equal to the given expression, since

$$\frac{2\pi}{3} + \theta = \pi - \left(\frac{\pi}{3} - \theta\right), \quad \frac{4\pi}{3} + \theta = \pi + \left(\frac{\pi}{3} + \theta\right).$$

7. The line $\dfrac{x-h}{\cos\theta} = \dfrac{y-k}{\sin\theta} = r$ meets $y^2 = 4ax$ where

$$r^2\sin^2\theta + 2r(k\sin\theta - 2a\cos\theta) + k^2 - 4ah = 0 \quad \ldots\ldots\text{(i)}.$$

If the line is a tangent, this equation has equal roots,

$$\text{i.e.}\quad (k\sin\theta - 2a\cos\theta)^2 = \sin^2\theta\,(k^2 - 4ah),$$

whence $$k\sin\theta\cos\theta - a\cos^2\theta = h\sin^2\theta \ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and then from (i), if $r^2 = t$, we have

$$t = \frac{k^2 - 4ah}{\sin^2\theta}, \text{ i.e. } \sin^2\theta = \frac{k^2 - 4ah}{t}.$$

Hence, putting $k^2 - 4ah = \lambda$, and substituting in (ii), we get

$$k^2\frac{\lambda}{t}\left(1 - \frac{\lambda}{t}\right) = \left\{a + (h-a)\frac{\lambda}{t}\right\}^2,$$

whence $$a^2t^2 - t\lambda\left[k^2 - 2a(h-a)\right] + \lambda^2\left[(h-a)^2 + k^2\right] = 0.$$

8. The equation to the pair of tangents from x', y' is

$$SS' = \{(ax' + hy' + g)x + (hx' + by' + f)y + (gx' + fy' + c)\}^2,$$

and if the point (x', y') moves off to infinity along the line $\dfrac{x}{l} = \dfrac{y}{m}$, we may put $x' = \dfrac{l}{k}$, $y' = \dfrac{m}{k}$ and afterwards make k zero. The equation then takes the given form.

Further, the imaginary tangents through the foci pass through the circular points, so that if the equation is satisfied by a focus, the result of substituting its co-ordinates must be to give the tangential equation of the circular points, which for oblique axes is

$$l^2 + m^2 + 2lm \cos \omega = 0,$$

and the second set of equations follows by comparing coefficients.

9. The equation of a conic through B and C cannot contain terms in β^2 or γ^2, and if it also passes through I and I_1 it must be satisfied by the co-ordinates (1, 1, 1) and (−1, 1, 1). Hence its equation must be of the form

$$\alpha^2 - \beta\gamma + k\alpha(\beta - \gamma) = 0 \qquad \text{(i)}.$$

Similarly the equation to the other conic is of the form

$$\alpha^2 + \beta\gamma + k'\alpha(\beta + \gamma) = 0 \qquad \text{(ii)}.$$

But if the equations (i) and (ii) are satisfied by any co-ordinates $(\alpha', \beta', \gamma')$, they will also, from their form, be satisfied by the reciprocals $\left(\frac{1}{\alpha'}, \frac{1}{\beta'}, \frac{1}{\gamma'}\right)$. But since U is a conic inscribed in ABC, if one focus is $(\alpha', \beta', \gamma')$, the other will be

$$\left(\frac{1}{\alpha'}, \frac{1}{\beta'}, \frac{1}{\gamma'}\right).$$

Hence if one focus is at an intersection of (i) and (ii), so also is the other.

10. Let G be the centre of gravity of the sphere and weight, I the point of contact, and let $IG = x$. Then

$$x = \frac{W + 2w}{W + w} \cdot a.$$

Now let the sphere be turned through a small angle ϕ. Let J be the new point of contact and let IJ subtend an angle θ at the centre of the shell, so that

$$b\theta = a\phi \qquad \text{(i)}.$$

The height of G above I is now

$$b - (b - a)\cos\theta + (x - a)\cos(\theta - \phi).$$

Retaining squares and fourth powers of the small angles, and using (i), this becomes on reduction

$$x + \left[(b-a) - \frac{(b-a)^2}{a^2}(x-a)\right]\frac{\theta^2}{2} + \left[-b+a+\frac{(b-a)^4}{a^4}(x-a)\right]\frac{\theta^4}{24}.$$

For stability the coefficient of θ^2 must be positive, provided it does not vanish, i.e.

$$(x-a)\,.\,\frac{b-a}{a^2} < 1,$$

whence
$$x < \frac{ab}{b-a},$$

$$\text{i.e. } \frac{W+2w}{W+w} < \frac{b}{b-a}, \text{ i.e. } \frac{W}{w} > \frac{b-2a}{a}.$$

If this is an equality, the coefficient of θ^2 vanishes, and the equilibrium is stable, provided the coefficient of $\dfrac{\theta^4}{24}$ is positive.

But if $x - a = \dfrac{a^2}{b-a}$, this coefficient is

$$\frac{(b-a)^3}{a^2} - (b-a) = \frac{b-a}{a^2}\left[(b-a)^2 - a^2\right] = \frac{b}{a^2}(b-a)(b-2a),$$

which is positive, since $b - 2a > 0$, by virtue of the value of W/w.

11. The equation to the path is

$$y = x\tan\alpha - \tfrac{1}{2}g\,.\,\frac{x^2}{v^2\cos^2\alpha},$$

and since h is greater than the value of y when $x = a$ and $\alpha = 45°$, the maximum range is impossible.

If the shot just grazes the wall, we have

$$h = a\tan\alpha - \tfrac{1}{2}g\,.\,\frac{a^2}{v^2\cos^2\alpha},$$

or
$$ga^2\tan^2\alpha - 2av^2\tan\alpha + ga^2 + 2hv^2 = 0 \;\ldots\ldots\ldots\ldots\text{(i)}.$$

This gives two possible directions of projection, say α_1 and α_2.

If α satisfies (i), the range is

$$\frac{v^2}{g}\,.\,\frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{v^2}{g}\,.\,\frac{2ga^2\tan\alpha}{2av^2\tan\alpha - 2hv^2} = \frac{a^2\tan\alpha}{a\tan\alpha - h}.$$

Hence the distance commanded is

$$a^2\left(\frac{\tan \alpha_1}{a \tan \alpha_1 - h} - \frac{\tan \alpha_2}{a \tan \alpha_2 - h}\right) = a^2 h \cdot \frac{\tan \alpha_2 - \tan \alpha_1}{(a \tan \alpha_1 - h)(a \tan \alpha_2 - h)}$$

$$= \frac{ah \cdot 2\sqrt{v^4 - a^2g^2 - 2hv^2g}/g}{\dfrac{ga^2 + 2hv^2}{g} - h \cdot \dfrac{2v^2}{g} + h^2} = \frac{2ha}{g(h^2 + a^2)} \cdot \sqrt{v^4 - a^2g^2 - 2hv^2g}.$$

12. Let the horizontal and vertical distances from the assigned point of the point from which the particle falls be x, y. Then the striking velocity is given by

$$u^2 = 2g(y - x \tan \alpha).$$

After the first rebound, the velocity perpendicular to the plane is $u \cos \alpha$. Hence the time to the next impact is given by

$$u \cos \alpha \cdot t - \tfrac{1}{2} g \cos \alpha \cdot t^2 = 0,$$

$$\text{i.e. } t = \frac{2u}{g}.$$

The distance described parallel to the plane in this time is

$$u \sin \alpha \cdot t + \tfrac{1}{2} g \sin \alpha \cdot t^2 = \frac{2u^2 \sin \alpha}{g} + \frac{2u^2 \sin \alpha}{g} = \frac{4u^2 \sin \alpha}{g},$$

and by hypothesis this is $x \sec \alpha$;

$$\therefore\ 8 \sin \alpha (y - x \tan \alpha) = x \sec \alpha,$$

$$\text{i.e. } y = x\left(\tan \alpha + \frac{\sec \alpha}{8 \sin \alpha}\right),$$

or

$$y = x(\tan \alpha + \tfrac{1}{4} \operatorname{cosec} 2\alpha).$$

XLIX.

1. Let AA_1 be the given straight line. With centre A_1 and radius AA_1 describe a circle. By marking off the radius three times round the circumference from A we obtain a point A_2, which is clearly on AA_1 produced, and such that $AA_2 = 2AA_1$.

Now with centre A_2 and radius A_1A_2 describe a circle cutting the previous circle in B. With centres A and A_2, and radii AA_2 and AB, describe circles intersecting in C, and with centre C and radius CA_2 describe a circle cutting AA_1 in E. Then E is the middle point of AA_1.

For, taking AA_1 as the unit length, we have

$$AB^2 = AA_2^2 - A_2B^2 = 3.$$

Hence if CN be perpendicular to AA_2, we have

$$AN^2 - A_2N^2 = AC^2 - A_2C^2 = 4 - 3 = 1,$$

$$\text{i.e. } AA_2(AN - A_2N) = 1.$$

But $AN - A_2N = AN - EN = AE$, and $AA_2 = 2$; $\therefore AE = \frac{1}{2}$.

2. Let L, M be the middle points of GT, $G'T'$ and let LM meet the normal at P in V. Draw PN, Pn perpendicular to the axes. Then

$$GT = CT - CG = \frac{a^2}{CN} - e^2 . CN = \frac{a^2 - e^2 . CN^2}{CN}$$

$$= \frac{SP . S'P}{CN} = \frac{CD^2}{CN}.$$

Similarly $G'T' = \dfrac{CD^2}{Cn}$. But $PL = \frac{1}{2}GT$, and $PM = \frac{1}{2}G'T'$;

$$\therefore PL : PM = Cn : CN = PN : Pn,$$

and evidently PL, PM are at right angles; therefore the triangles PLM, PNn are similar.

Now let the line through C parallel to the tangent meet the normal in F. Then P, F, C, n are cyclic,

$$\therefore P\hat{F}n = P\hat{C}n = P\hat{N}n = P\hat{L}M.$$

Also $F\hat{P}n = P\hat{G}N = L\hat{P}V$. Hence the triangles PFn, PLV are similar; therefore $\quad PF : Pn = PL : PV$,

$$\text{i.e. } PF : CN = \tfrac{1}{2}GT : PV;$$

$$\therefore PF . PV = \tfrac{1}{2}GT . CN = \tfrac{1}{2}CD^2.$$

But the radius of curvature is CD^2/PF; therefore PV = half the radius of curvature.

3. If possible, let u_r be of the form $A\lambda^r + Br + C$. Then we have to satisfy the equation

$$A\lambda^{r-2}(3\lambda^2+10\lambda+3)+3(Br+C)+10(B\overline{r-1}+C)+3(B\overline{r-2}+C)$$
$$=16(r-1),$$

which will be possible provided

$$3\lambda^2+10\lambda+3=0, \quad B=1 \text{ and } C=0.$$

Hence λ must be -3 or $-\frac{1}{3}$, and the general form of u_r is

$$u_r = A_1(-3)^r + A_2(-\tfrac{1}{3})^r + r \quad \ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and A_1, A_2 must be chosen to satisfy the two remaining equations.

From (i) we have

$$u_1 = -3A_1 - \tfrac{1}{3}A_2 + 1, \quad u_2 = 9A_1 + \tfrac{1}{9}A_2 + 2;$$
$$\therefore\ 9u_1 + 3u_2 = -\tfrac{8}{3}A_2 + 15; \ \therefore\ A_2 = -\tfrac{3}{8}.$$

Also
$$u_n = (-3)^n A_1 + (-\tfrac{1}{3})^n A_2 + n,$$
$$u_{n-1} = (-3)^{n-1}A_1 + (-\tfrac{1}{3})^{n-1}A_2 + n - 1;$$
$$\therefore\ 3u_{n-1} + 5u_n = 4(-3)^n A_1 - 4(-\tfrac{1}{3})^n A_2 + 8n - 3;$$
$$\therefore\ (-3)^n A_1 = (-\tfrac{1}{3})^n A_2; \quad \therefore\ A_1 = -\tfrac{3}{8}(-\tfrac{1}{3})^{2n}.$$

Substituting these values in (i), we obtain the result.

4. The coefficient of x^n in the product on the left-hand side is

$$\frac{1}{(n-1)!} - \frac{1}{(n-2)!}\cdot\frac{1}{2!\,2} + \frac{1}{(n-3)!}\cdot\frac{1}{3!\,3} - \ldots + (-1)^{n-1}\cdot\frac{1}{n!\,n}$$
$$= \frac{1}{n!}\left[{}^nC_1 - \tfrac{1}{2}\cdot{}^nC_2 + \tfrac{1}{3}\cdot{}^nC_3 - \ldots + (-1)^{n-1}\cdot\frac{1}{n}\cdot{}^nC_n\right].$$

Now by the ordinary rule of partial fractions

$$\frac{n!}{y(y+1)\ldots(y+n)} = \sum_{r=0}^{r=n}(-1)^r\cdot{}^nC_r\cdot\frac{1}{y+r};$$
$$\therefore\ \frac{n!}{(y+1)\ldots(y+n)} - 1 = \sum_{r=1}^{r=n}(-1)^r\cdot{}^nC_r\cdot\frac{y}{y+r},$$

i.e. $(1+y)^{-1}\left(1+\dfrac{y}{2}\right)^{-1}\ldots\left(1+\dfrac{y}{n}\right)^{-1} - 1$

$$= \sum_{r=1}^{r=n}(-1)^r\cdot{}^nC_r\left(1+\frac{y}{r}\right)^{-1}\cdot\frac{y}{r}.$$

Equating coefficients of y, we obtain

$$-\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n}\right)=\sum_{r=1}^{r=n}(-1)^r\cdot\frac{1}{r}\cdot{}^nC_r,$$

i.e. $^nC_1-\frac{1}{2}\,{}^nC_2+\frac{1}{3}\,{}^nC_3-\ldots+(-1)^{n-1}\cdot\frac{1}{n}\cdot{}^nC_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$

Hence the coefficient of x^n in the product is

$$\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}\right)\Big/ n\,!$$

5. The roots of the equation $x^3-21x^2+35x-7=0$ are $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$, $\tan^2\frac{3\pi}{7}$ (see XII. 6). These are equivalent to $\tan^2\frac{6\pi}{7}$, $\tan^2\frac{2\pi}{7}$, $\tan^2\frac{4\pi}{7}$, and also to $\tan^2\frac{\pi}{7}$, $\tan^2\frac{5\pi}{7}$, $\tan^2\frac{3\pi}{7}$.

Hence $\tan^2\frac{2\pi}{7}\tan^2\frac{4\pi}{7}\tan^2\frac{6\pi}{7}=$ product of roots $=7$, and in taking the square root we take the positive sign, since $\tan\frac{2\pi}{7}$ is positive, and the other two are negative.

Again, making the substitution $y=21-x$, we obtain

$$y^3-42y^2+476y-728=0,$$

and the roots of this equation are $\tan^2\frac{\pi}{7}+\tan^2\frac{3\pi}{7}$ and two similar expressions. The sum of the reciprocals of these quantities is therefore $\frac{476}{728}=\frac{17}{26}$.

6. Multiplying the convergent series

$$\sec x=\frac{2^2}{\pi}s_1+\frac{2^4}{\pi^3}x^2s_3+\frac{2^6}{\pi^5}x^4s_5+\ldots\quad\left(x<\frac{\pi}{2}\right),$$

and

$$\cos x=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\ldots,$$

we obtain, by equating the coefficient of x^{2n} in the product to zero,

$$\frac{2^{2n+2}}{\pi^{2n+1}}\cdot s_{2n+1}-\frac{1}{2\,!}\cdot\frac{2^{2n}}{\pi^{2n-1}}\cdot s_{2n-1}+\frac{1}{4\,!}\cdot\frac{2^{2n-2}}{\pi^{2n-3}}\cdot s_{2n-3}-\ldots=0,$$

whence $$s_{2n+1}=\frac{\pi^2}{4}\left(\frac{1}{2\,!}\cdot s_{2n-1}-\frac{1}{4\,!}\cdot\frac{\pi^2}{2^2}s_{2n-3}+\dots\right).$$

Thus $$s_3=\frac{\pi^2}{4}\cdot\frac{1}{2\,!}s_1=\frac{\pi^3}{32},\quad\text{since } s_1=\frac{\pi}{4},$$

$$s_5=\frac{\pi^2}{4}\left(\frac{1}{2\,!}s_3-\frac{1}{4\,!}\cdot\frac{\pi^2}{2^2}s_1\right)$$

$$=\frac{\pi^2}{4}\left(\frac{\pi^3}{64}-\frac{\pi^3}{384}\right)=\frac{5\pi^5}{1536}.$$

7. Take the diagonals as axes of co-ordinates. The parabola

$$(\alpha x+\beta y)^2+2gx+2fy+c=0$$

cuts the axis of x where $\alpha^2x^2+2gx+c=0$, and the product of the intercepts is $\frac{c}{\alpha^2}$. Similarly the product of the intercepts on the axis of y is $\frac{c}{\beta^2}$. Hence if the quadrilateral is cyclic, we must have $\alpha=\pm\beta$, and the parabolas may be taken in the form

$$(x\pm y)^2+2gx+2fy+c=0.$$

Their latera-recta are

$$\frac{2\,(f\pm g)\sin^2\omega}{(2\pm 2\cos\omega)^{\frac{3}{2}}},$$

so that their product is $\frac{1}{2}(f^2-g^2)\sin\omega$.

But d_1 is the algebraical difference of the intercepts on the axis of x, i.e. $d_1^2=4f^2-4c$, and similarly $d_2^2=4g^2-4c$;

$$\therefore\ f^2-g^2=\tfrac{1}{4}(d_1^2-d_2^2),$$

so that the product is $\frac{1}{8}(d_1^2-d_2^2)\sin\omega$.

8. Since the equation contains one variable parameter k, and may be written in the forms

$$4kC=(C-C'+k)^2,$$

$$4kC'=(C-C'-k)^2,$$

and $C-C'$ is of the first degree, it represents the most general form of conic satisfying the given conditions.

Again, if $y = \mu(x - h)$ is an asymptote, and we solve for the intersections of this line with the curve, the resulting quadratic in x has both roots infinite and therefore the coefficients of x^2 and x must both vanish. Equating the coefficient of x to zero, we find

$$-2\mu^2 h + a + b = 0,$$

so that the equation to the line may be written in the form

$$y = \mu x - \frac{a+b}{2\mu},$$

or $$2\mu^2 x - 2\mu y - (a+b) = 0,$$

and its envelope is $$y^2 + 2x(a+b) = 0.$$

9. Transferring the origin to (x', y') the equation becomes

$$ax^2 + 2hxy + by^2 + 2(ax' + hy' + g)x + 2(hx' + by' + f)y + S' = 0.$$

If (x', y') is one of the points at which the tangent is parallel to the axis of x, we have $ax' + hy' + g = 0$, $S' = 0$.

Supposing then $hx' + by' + f = \lambda$, we have

$$\begin{aligned} hx' + by' + f - \lambda &= 0, \\ ax' + hy' + g &= 0, \\ gx' + (f+\lambda)y' + c &= 0, \end{aligned} \qquad \therefore \begin{vmatrix} h, & b, & f-\lambda \\ a, & h, & g \\ g, & f+\lambda, & c \end{vmatrix} = 0,$$

whence $$-\Delta - \lambda^2 a = 0, \text{ i.e. } \lambda = \pm\frac{(-\Delta)^{\frac{1}{2}}}{a^{\frac{1}{2}}}.$$

The conic is now $ax^2 + 2hxy + by^2 + 2\lambda y = 0$, and, by Newton's formula, the radius of curvature at the origin is

$$\text{Lt}\left(\frac{x^2}{2y}\right) = -\frac{\lambda}{a} = \mp\frac{(-\Delta)^{\frac{1}{2}}}{a^{\frac{3}{2}}}.$$

10. Let AN be the perpendicular from A on BC. Then since A can only move perpendicular to AN, the triangle must begin to turn about some point in AN. Let this point be I. Then the limiting frictions at A, B, C are perpendicular to IA, IB, IC. Let the pressures at A, B, C be R_1, R_2, R_3 and the weights ka, kb, kc.

Let $B\hat{I}N = \alpha$, $C\hat{I}N = \beta$. Then resolving parallel to BC for the forces in the plane of the triangle, we have

$$\mu R_1 = \mu R_2 \cos \alpha + \mu R_3 \cos \beta,$$

$$\text{i.e. } a = b \cos \alpha + c \cos \beta \quad \text{..................(i)},$$

since $$R_1 : R_2 : R_3 = a : b : c.$$

Also $$\frac{\tan \alpha}{\tan \beta} = \frac{BN}{NC} = \frac{\cot B}{\cot C} = \frac{\tan C}{\tan B} \quad \text{...............(ii)}.$$

Now both the equations (i) and (ii) are satisfied if $\alpha = C$, $\beta = B$, i.e. if I is the orthocentre. Hence, since the wire can only begin to turn about one point, that point must be the orthocentre.

Now taking moments about I, we have

$$\mu (AI \,.\, ka + BI \,.\, kb + CI \,.\, kc) \cos \alpha$$

$$= kb \sin \alpha \,.\, c \cos B - kc \sin \alpha \,.\, b \cos C;$$

$$\therefore \tan \alpha = \mu \,.\, \frac{2R^2 \,.\, \Sigma \sin 2A}{bc (\cos B - \cos C)},$$

and $$\Sigma \sin 2A = 4 \sin A \sin B \sin C;$$

$$\therefore \tan \alpha = \mu \,.\, \frac{2 \sin A}{\cos B - \cos C}.$$

11. Suppose the gun elevated at an angle α, and that the shot leaves the gun at an angle θ to the horizon. Let u be the velocity of the shot relative to the gun, in the direction of the bore of the gun, and let V be the backward velocity of the gun and shot. Then the actual horizontal velocity of the shot is $u \cos \alpha - V$. Now the horizontal components of the impulses of the explosion on the gun and shot are equal;

$$\therefore nV = u \cos \alpha - V;$$

$$\therefore V = \frac{u \cos \alpha}{n + 1}.$$

Hence the horizontal velocity of the shot

$$= u \cos \alpha - V = \frac{nu \cos \alpha}{n + 1}.$$

Also the vertical velocity of the shot is $u \sin \alpha$;

$$\therefore \tan \theta = \frac{u \sin \alpha}{nu \cos \alpha/(n + 1)} = \frac{n + 1}{n} \tan \alpha.$$

The equation to the subsequent trajectory is

$$y = x \tan\theta - \tfrac{1}{2}\frac{gx^2}{v^2\cos^2\theta} \quad (v \text{ being the initial velocity})$$

$$= x\tan\theta\left(1 - \frac{x}{a}\right),$$

since a is the range and therefore equal to $\dfrac{2v^2 \sin\theta\cos\theta}{g}$.

According to the data, this trajectory passes through the point (c, h), and we therefore have

$$h = c\tan\theta\left(1 - \frac{c}{a}\right),$$

$$\text{i.e. } \tan\theta = \frac{ah}{c(a-c)};$$

$$\therefore \tan\alpha = \frac{n}{n+1} \,.\, \tan\theta = \frac{n}{n+1} \cdot \frac{ah}{c(a-c)}.$$

12. Let O be the centre of the basin, A and B those of the balls M and m. Then at the instant of contact $OA = 4a$, $AB = 3a$, $OB = 5a$; therefore OAB is a right angle, and therefore the direction of motion of M at the impact is along AB.

If $AOB = \alpha$, the striking velocity is given by

$$u^2 = 2 \,.\, 4a\cos\alpha \,.\, g = \tfrac{32}{5} ga.$$

Let v, v' be the velocities of M, m after impact. Then, by Newton's law,

$$v - v'\cos\alpha = -u,$$

and the equation of momentum is

$$M(u - v)\cos\alpha = mv'.$$

From the data, these equations are

$$v - \tfrac{4}{5}v' = -u, \quad 5(u - v) = 4v',$$

whence $$v = 0, \quad v' = \tfrac{5}{4}u.$$

Hence m rises to a height $\dfrac{v'^2}{2g} = \dfrac{25}{16} \cdot \dfrac{u^2}{2g} = 5a$, i.e. to the level of the centre of the basin.

Also since $v' = \frac{5}{4}u, \quad \therefore \; v'^2 = \frac{25}{16}u^2,$

$$\text{i.e. } \tfrac{1}{2}mv'^2 = \tfrac{1}{2}Mu^2,$$

so that the spheres exchange kinetic energies, and this exchange continues indefinitely, since after the second impact M will evidently rise to its original position, and the whole motion will be repeated.

L.

1. Let $ABCD$ be the quadrilateral, EF the third diagonal. Then for the circle which is self-polar for the triangle EBC, BC is the polar of E, and therefore E and F are conjugate points. Hence the circle on EF as diameter cuts the self-polar circle orthogonally (see XXVI. 1). Similarly for all the other cases.

2. We can construct any number of other points on the curve by Pascal's Theorem, as follows: Let A, B, C, D, E be the five given points and AG any line through A. Let AG and CD meet in L, and AB and DE in M. Let LM cut BC in N. Then the intersection of NE and AG is a point on the conic. If now we take AG first parallel to CD and then to DE, we shall obtain two pairs of parallel chords, and hence the centre, which is the intersection of the lines joining their middle points.

We now have also the directions of two pairs of conjugate diameters and we can therefore construct the involution of conjugate diameters, and the pair of perpendicular lines belonging to this involution are the axes of the conic.

3. Suppose

$$\left[(1+x)y+\frac{1}{x}\right]^n = A_0 + A_1 y + \ldots + A_n y^n \;\ldots\ldots\ldots\ldots(\text{i}),$$

$$\left(1+x+\frac{1}{y}\right)^n = B_0 + B_1 \cdot \frac{1}{y} + \ldots + B_n \cdot \frac{1}{y^n} \;\ldots\ldots(\text{ii}).$$

The term independent of y in the product is $\sum_0^n A_r B_r$.

Now $A_r = {}^nC_r (1+x)^r \cdot \frac{1}{x^{n-r}}$ and $B_r = {}^nC_r (1+x)^{n-r}$;

$$\therefore\ A_r B_r = {}^nC_r^{\,2} (1+x)^n \cdot \frac{1}{x^{n-r}},$$

and the term independent of x in this is ${}^nC_r^{\,3}$.

Hence the term independent of both x and y in the product of (i) and (ii) is $\sum\limits_{r=0}^{r=n} {}^nC_r^{\,3}$, and this latter sum is therefore equal to the coefficient of $x^n y^n$ in

$$[x(1+x)y+1]^n [y(1+x)+1]^n = [1+y(1+x)^2(1+xy)]^n.$$

4. The number of ways in which $2n$ things can be divided into n pairs is $\frac{(2n)!}{2^n \cdot n!} = \phi(n)$.

Now, take one arrangement as fixed, and consider the number of possible re-arrangements. If we keep the first pair fixed, we can re-divide the other pairs in $\phi(n-1)$ ways. Hence the number of re-arrangements in which the first pair is not the same is $\phi(n) - \phi(n-1)$.

Among these there are $\phi(n-1) - \phi(n-2)$ in which the second pair is the same. Hence the number in which neither the first nor the second pair is the same is

$$\phi(n) - 2\phi(n-1) + \phi(n-2).$$

Among these there are

$$\phi(n-1) - 2\phi(n-2) + \phi(n-3),$$

in which the third pair is the same. Hence the number in which neither the first, second, nor third pairs are the same is

$$\phi(n) - 3\phi(n-1) + 3\phi(n-2) - \phi(n-3),$$

and so on, the coefficients evidently following the law of the Binomial Theorem. The given expression is thus the number of cases in which none of the n pairs is the same as before.

5. We have

$$s = 84\sqrt{2}, \quad s-a = 32\sqrt{2}, \quad s-b = 28\sqrt{2}, \quad s-c = 24\sqrt{2},$$

whence $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = 8 \times 12 \times 28,$

and
$$R=\frac{abc}{4\Delta}=\frac{65\sqrt{2}}{2},$$

$$\sin A=\frac{a}{2R}=\tfrac{4}{5};$$

$\therefore\ \cos A=\tfrac{3}{5},\ \sin B=\tfrac{56}{65},\ \cos B=\tfrac{33}{65}$, and $\sin C=\tfrac{12}{13},\ \cos C=\tfrac{5}{13}$.

Now if x is the perpendicular from N, the nine-point centre, on BC, we have

$$\begin{aligned}2x&=R\cos A+2R\cos B\cos C\\&=\frac{65\sqrt{2}}{2}\left(\tfrac{3}{5}+2\cdot\tfrac{33}{65}\cdot\tfrac{5}{13}\right),\end{aligned}$$

whence
$$x=\frac{837\sqrt{2}}{52};$$

$$\therefore\quad \triangle BNC=\tfrac{1}{2}xa=837,$$

and so for the others.

6. Let $\alpha=\dfrac{1}{2n-1},\ \beta=\dfrac{1}{2n+1}$, so that $\alpha-\beta=2\alpha\beta$.

Then
$$\begin{aligned}16\alpha^4\beta^4=(\alpha-\beta)^4&=\alpha^4+\beta^4-4\alpha\beta(\alpha^2+\beta^2)+6\alpha^2\beta^2\\&=\alpha^4+\beta^4-2(\alpha-\beta)(\alpha^2+\beta^2)+6\alpha^2\beta^2\\&=\alpha^4+\beta^4-2(\alpha^3-\beta^3)+10\alpha^2\beta^2\\&=\alpha^4+\beta^4-2(\alpha^3-\beta^3)+\tfrac{5}{2}(\alpha^2+\beta^2)-\tfrac{5}{2}(\alpha-\beta).\end{aligned}$$

Hence if S is the given sum, we have

$$16S=(2S_4-1)-2+\tfrac{5}{2}(2S_2-1)-\tfrac{5}{2},$$

where
$$S_r=\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^r}.$$

Now by comparing the infinite series and infinite product forms for $\cos x$, we easily find

$$S_2=\frac{\pi^2}{8},\quad S_4=\frac{\pi^4}{96};$$

$$\therefore\quad 16S=\frac{\pi^4}{48}+\frac{5\pi^2}{8}-8=\frac{\pi^4+30\pi^2-384}{48}.$$

7. Let P be (X, Y). Then the parameters of the feet of the normals through P are given by

$$am^3 + (2a - X)\,m - Y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and the radius of curvature at m is $\rho = 2a\,(m^2 + 1)^{\frac{3}{2}}$.

Now if in (i) we make the substitution $m^2 + 1 = t$, the equation becomes

$$(t-1)\,[a\,(t-1) + 2a - X]^2 = Y^2,$$

$$\text{i.e. } (t-1)\,(at - X + a)^2 = Y^2,$$

and the product of the roots is $\dfrac{(X-a)^2 + Y^2}{a^2} = \dfrac{SP^2}{a^2}$.

Hence the product of the values of ρ is

$$8a^3\left(\frac{SP^2}{a^2}\right)^{\frac{3}{2}} = 8SP^3.$$

8. If α, β, γ are the angular points, we have to shew that

$$3\,.\,\frac{\Sigma\cos\alpha}{3}\,.\,\frac{\Sigma\sin\alpha}{3} = \Sigma\cos\alpha\sin\alpha,$$

$$\text{i.e. } \Sigma\sin(\beta+\gamma) = 2\Sigma\sin\alpha\cos\alpha \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Now since the triangle is equilateral, the centroid coincides with the circumcentre. Hence

$$\frac{a}{3}\,.\,\Sigma\cos\alpha = \frac{a^2-b^2}{4a}\,\{\Sigma\cos\alpha + \cos(\alpha+\beta+\gamma)\},$$

$$\frac{b}{3}\,.\,\Sigma\sin\alpha = \frac{b^2-a^2}{4b}\,\{\Sigma\sin\alpha - \sin(\alpha+\beta+\gamma)\}.$$

From these
$$\frac{a^2}{a^2-b^2} = \tfrac{3}{4}\left\{1 + \frac{\cos(\alpha+\beta+\gamma)}{\Sigma\cos\alpha}\right\},$$

$$\frac{b^2}{a^2-b^2} = \tfrac{3}{4}\left\{-1 + \frac{\sin(\alpha+\beta+\gamma)}{\Sigma\sin\alpha}\right\},$$

whence, subtracting,

$$1 = \tfrac{3}{4}\left\{2 - \frac{\Sigma\sin(\beta+\gamma)}{\Sigma\cos\alpha\,.\,\Sigma\sin\alpha}\right\},$$

$$\text{i.e. } \Sigma\sin(\beta+\gamma) = \tfrac{2}{3}\,(\Sigma\cos\alpha)\,(\Sigma\sin\alpha),$$

which is equivalent to (i).

9. Taking the two parabolas in the form

$$y^2 = 4ax,$$

$$(x \sin \alpha + y \cos \alpha + k)^2 = 4a' (x \cos \alpha - y \sin \alpha + h),$$

any other conic through the four points is of the form

$$(x \sin \alpha + \ldots + \ldots)^2 - 4a' (x \cos \alpha - \ldots + \ldots) + \lambda (y^2 - 4ax) = 0,$$

$$\text{i.e. } x^2 \sin^2 \alpha + 2xy \sin \alpha \cos \alpha + (\cos^2 \alpha + \lambda) y^2 + \ldots = 0,$$

and the eccentricity is given by

$$\frac{e^4}{1-e^2} + 4 = \frac{(1+\lambda)^2}{\lambda \sin^2 \alpha},$$

i.e. for a given value of e there are in general two values of λ.

If the equation $(1 + \lambda)^2 = k\lambda$ has equal roots, then $k = 0$ or 4. Hence if there is only one conic,

$$\frac{e^4}{1-e^2} + 4 = 0 \text{ or } \frac{4}{\sin^2 \alpha}.$$

In the former case $(e^2 - 2)^2 = 0$, i.e. the conic is a rectangular hyperbola. In the latter case

$$e^4 \tan^2 \alpha + 4e^2 - 4 = 0.$$

10. Suppose the string makes an angle ϕ with the horizontal. Then taking moments about the fixed end of the rod,

$$T \,.\, c \sin \phi = W \,.\, l \cos \theta,$$

where $2l$ is the length of the rod.

Also $\qquad \dfrac{b}{2l} = \dfrac{\sin \theta}{\sin \phi}; \quad \therefore\ 2T \,.\, c \sin \theta = W \,.\, b \cos \theta.$

But $\qquad T = \lambda \dfrac{b-a}{a}; \quad \therefore\ \lambda \dfrac{b-a}{a} = \tfrac{1}{2} W \,.\, \dfrac{b}{c} \cot \theta,$

$$\text{i.e. } \lambda \left(1 - \frac{a}{b}\right) = \tfrac{1}{2} W \,.\, \frac{a}{c} \cot \theta.$$

11. We find (as in XXV. 11) that the accelerations of the particle along and perpendicular to the face of the wedge are

$$f_1 = g \sin \alpha, \quad f_2 = \frac{mg \cos \alpha \sin^2 \alpha}{M + m \sin^2 \alpha}.$$

Hence the resultant acceleration (i.e. the resultant of f_1 and f_2) is constant in magnitude and direction. Hence the particle describes a parabola with its axis in this direction.

If θ is the angle the resultant acceleration makes with the plane, then

$$\tan\theta = \frac{f_2}{f_1} = \frac{m\tan\alpha}{(M+m)\tan^2\alpha + M};$$

$$\therefore\ \tan(\theta+\alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{(M+m)(\tan\alpha + \tan^3\alpha)}{M(1+\tan^2\alpha)}$$

$$= \left(1 + \frac{m}{M}\right)\tan\alpha.$$

The vertical acceleration is

$$f_1\sin\alpha + f_2\cos\alpha = \frac{(M+m)\sin^2\alpha}{H}\cdot g,$$

and the initial vertical velocity is $V\sin\beta\sin\alpha$. Hence the vertical velocity will be zero at a height

$$h = \frac{V^2\sin^2\beta\,.\,H}{2(M+m)g}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Again the acceleration in the direction of the axis is

$$-(f_1\cos\theta + f_2\sin\theta),$$

and the initial velocity in this direction is $V\sin\beta\cos\theta$. Hence the velocity in the direction of the axis vanishes after a time

$$\frac{V\sin\beta}{f_1 + f_2\tan\theta} = \frac{V\sin\beta}{g\sin\alpha}\cdot\frac{H^2}{K^2} \text{ on reduction.}$$

In this time, the particle describes vertically a distance

$$\frac{V^2\sin^2\beta}{g}\cdot\frac{H^2}{K^2} - \tfrac{1}{2}\cdot\frac{(M+m)\sin^2\alpha}{H}\cdot g\cdot\frac{V^2\sin^2\beta}{g^2\sin^2\alpha}\cdot\frac{H^4}{K^4}$$

$$= \frac{V^2\sin^2\beta}{2g}\cdot\frac{H^2}{K^2}\left\{2 - \frac{(M+m)H}{K^2}\right\}\ \ldots\ldots\ldots\ldots\text{(ii)}.$$

We require the difference of the heights (i) and (ii), and this is

$$\frac{V^2\sin^2\beta}{2g}\cdot\frac{H}{(M+m)K^4}\{K^4 - 2HK^2(M+m) + H^2(M+m)^2\},$$

and the expression in the bracket is the square of

$$\begin{aligned}(M+m)H-K^2 \\ &= (M+m)(M+m\sin^2\alpha)-(M^2+2Mm\sin^2\alpha+m^2\sin^2\alpha) \\ &= Mm\cos^2\alpha,\end{aligned}$$

leading to the required expression.

12. In the corresponding simple harmonic motion let the amplitude be b. Then the constant of the motion is $\mu=\dfrac{g}{4a}$, and since the velocity at a distance $4a$ from the centre of acceleration is V, we have

$$V=\sqrt{\mu(b^2-16a^2)} \quad \text{.....................(i).}$$

The time to reach the point distant $4a$ from the centre is

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\left(\frac{4a}{b}\right),$$

and the time to reach the centre is $\dfrac{\pi}{2\sqrt{\mu}}$.

Hence the time required here is

$$\begin{aligned}&\frac{\pi}{2\sqrt{\mu}}-\frac{1}{\sqrt{\mu}}\cos^{-1}\left(\frac{4a}{b}\right) \\ &= \frac{\pi}{2\sqrt{\mu}}-\frac{1}{\sqrt{\mu}}\tan^{-1}\left(\frac{V/\sqrt{\mu}}{4a}\right) \text{ by (i)} \\ &= \sqrt{\frac{4a}{g}}\left(\frac{\pi}{2}-\tan^{-1}\frac{V}{\sqrt{4ag}}\right) \\ &= \sqrt{\frac{4a}{g}}\,.\,\tan^{-1}\left(\frac{\sqrt{4ag}}{V}\right).\end{aligned}$$

PART II

LI.

1. Let $ABCD$ be the square, O the centre, and let AB touch the circle in L. Then evidently

$$P\hat{O}L = \tfrac{1}{2}A\hat{P}S, \quad Q\hat{O}L = \tfrac{1}{2}P\hat{Q}B.$$

$\therefore\ P\hat{O}Q =$ half a right angle. Similarly for each of the angles POS, SOR. Hence OP, OR are the bisectors of the angle SOQ.

$$\therefore\ QP : PS = QR : RS.$$

(The figure is taken in which the line cuts AB, AD internally in P and S.)

2. Let TUV be the triangle formed by the tangents, TU touching the parabola at P. Draw the circle TUV and also a circle touching UV at U and passing through P. Let these circles intersect in S: then S is the focus, for $S\hat{U}V = S\hat{P}U$. To find the vertex draw perpendiculars from S on two of the tangents. The join of their feet is the tangent at the vertex.

3. (i) Putting $y - z = \alpha$, etc., the expression is

$$\Sigma(\beta-\gamma)^5 = 5(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\Sigma\alpha^2 - \Sigma\beta\gamma)$$
$$= 15(y+z-2x)(z+x-2y)(x+y-2z)(\Sigma x^2 - \Sigma yz).$$

(ii) The expression vanishes when

$$x = y, \quad x = -y, \quad x = 0, \quad y = 0,$$

and also when $a = c$ or $b = d$.

Hence we have as factors

$$xy(x^2 - y^2)(a-c)(b-d),$$

and the remaining factor will not contain x or y.

Putting $x = b$, $y = -a$, the expression becomes

$$-ab(a^2 - b^2)(a-c)(b-d)(bd - ac).$$

Hence the remaining factor must be $ac - bd$.

4. The sum of the cubes of *all* positive proper fractions with denominator 5 is

$$\frac{1}{5^3}[1^3 + 2^3 + 3^3 + \dots + (5n-1)^3] = \frac{1}{5^3}\left[\frac{5n(5n-1)}{2}\right]^2.$$

The sum of the cubes of those not in their lowest terms is

$$1^3 + 2^3 + 3^3 + \dots + (n-1)^3 = \left[\frac{n(n-1)}{2}\right]^2.$$

Hence the sum required is

$$\frac{1}{5} \cdot \frac{n^2(5n-1)^2}{4} - \frac{n^2(n-1)^2}{4} = \frac{1}{5} n^2 (5n^2 - 1).$$

5. Since $\Sigma a = 0$,

$$\therefore \ \beta^2 + \gamma^2 - a^2 = -2\beta\gamma = \frac{2b}{a}.$$

Hence the required equation is immediately obtained by the substitution $y = \frac{2b}{x}$ or $x = \frac{2b}{y}$.

6. Let p be the perpendicular from A on BC, r the in-radius. Then

$$\frac{a'}{a} = \frac{p-r}{p} = 1 - \frac{r}{p} = 1 - \frac{a}{2s}, \text{ since } r = \frac{\Delta}{s}, \quad p = \frac{2\Delta}{a}.$$

$$\therefore \ \Sigma \frac{a'}{a} = 3 - \frac{a+b+c}{2s} = 2.$$

7. The equation $\operatorname{cosec} 3\theta = 2$ is satisfied by

$$3\theta = 30°, \ 150° \text{ or } -210°,$$

and these give distinct values for $\operatorname{cosec} \theta$. Hence, putting $\operatorname{cosec} \theta = x$, the roots of the equation

$$\frac{3}{x} - \frac{4}{x^3} = \frac{1}{2} \text{ or } x^3 - 6x^2 + 8 = 0$$

are $\operatorname{cosec} 10°$, $\operatorname{cosec} 50°$ and $-\operatorname{cosec} 70°$, and the sum of these roots is 6.

8. The tangents at m, m' meet at amm', $a(m+m')$ and the normals at

$$a(m^2+mm'+m'^2)+2a, \quad -amm'(m+m').$$

Hence, putting $m+m'=\theta$, $mm'=\phi$, we may take

$$P \text{ as } a\phi, a\theta \text{ and } Q \text{ as } a(\theta^2-\phi+2), -a\theta\phi,$$

so that the equation to PQ is

$$x\theta(1+\phi)+y(\theta^2-2\phi+2)-a\theta(\phi^2+\theta^2-\phi+2)=0,$$

whence
$$k=\frac{a(\phi^2+\theta^2-\phi+2)}{1+\phi},$$

$$\text{i.e.}\quad a(\phi^2+\theta^2)-(a+k)\phi+2a-k=0,$$

shewing that P lies on the given circle.

9. If the rectangular hyperbola $xy=c^2$ cuts the ellipse at θ, we have

$$ab\cos\theta\sin\theta=c^2 \qquad\text{.......................(i).}$$

The tangents at this point are

$$bx\cos\theta+ay\sin\theta=ab, \quad bx\sin\theta+ay\cos\theta=2c^2.$$

If these cut at an angle α, we have

$$\tan\alpha=\frac{ab(\cos^2\theta-\sin^2\theta)}{(a^2+b^2)\sin\theta\cos\theta},$$

whence from (i)

$$\tan^2\alpha=\frac{a^2b^2(a^2b^2-4c^4)}{(a^2+b^2)^2c^4},$$

$$\text{i.e.}\quad c^4=\frac{a^4b^4}{(a^2+b^2)^2\tan^2\alpha+4a^2b^2}.$$

10. Let $ABCDEF$ be the hexagon, the force P acting along AB, $2P$ along BC, etc. Then the sum of the resolved parts perpendicular to AD is zero; therefore the resultant is parallel to DA, and its magnitude is

$$6P\cos 60^\circ+3P=6P.$$

Suppose its line of action is at distance x from AD, then taking moments about the centre

$$6P\,.\,x=21P\,.\,\frac{\sqrt{3}a}{2};\quad\therefore\ x=\frac{7}{4}\sqrt{3}\,.\,a.$$

11. If f be the acceleration of the sphere vertically downwards, f_1 that of the wedge horizontally, then

$$M'g - R\cos 60^\circ = M'f, \quad R\sin 60^\circ = Mf_1,$$

where R is the pressure between the sphere and the wedge.

Also the accelerations perpendicular to the common face are the same,

$$\therefore \; f\cos 60^\circ = f_1 \sin 60^\circ, \quad \text{i.e. } f_1 = \frac{f}{\sqrt{3}}.$$

Hence
$$M'g - M.\frac{f}{\sqrt{3}}.\frac{1}{\sqrt{3}} = M'f,$$

$$\text{i.e. } f = \frac{3M'}{M+3M'}g.$$

12. The only asymptotes are the axes and the curve is symmetrical in the first and third quadrants, in which it lies. The subtangent at any point is

$$y\Big/\frac{dy}{dx} = y\frac{dx}{dy} = y\left(-\frac{3a^4}{y^4}\right) = -3x.$$

LII.

1. Let O be the centre of the circle BGC, R_1 its radius, N the middle point of BC. Then

$$AO^2 + 2ON^2 = 3OG^2 + AG^2 + 2GN^2 = 3R_1^2 + \tfrac{2}{3}AN^2.$$

Also
$$ON^2 = R_1^2 - \tfrac{1}{4}a^2;$$

$$\therefore \; AO^2 - R_1^2 = \tfrac{1}{2}a^2 + \tfrac{2}{3}AN^2 = \tfrac{1}{2}a^2 + \tfrac{1}{3}(b^2 + c^2 - \tfrac{1}{2}a^2)$$
$$= \tfrac{1}{3}(a^2 + b^2 + c^2).$$

Hence, by symmetry, the theorem follows.

2. Let P and Q be the extremities of the arc, V the middle point of PQ. The direction of the axis can be found by drawing any two parallel chords, and joining their middle points. Draw VRT in this direction, meeting the arc in R, and such that $VR = RT$.

Then TP, TQ are tangents to the arc. We can now find the focus by making the angles QPS, PQS equal to PTV, QTV respectively, and the axis can then be drawn. Let TQ meet the axis in t, and draw QN perpendicular to the axis. Then the vertex is the middle point of tN.

3. Let s_r denote the sum $1^r + 2^r + \dots + n^r$, and a, b any two of the numbers 1, 2, 3, ... n. Then, if S is the required sum,

$$s_1^3 = s_3 + 3\Sigma a^2 b + 6S.$$

Also $$\Sigma a^2 b = s_1 s_2 - s_3; \quad \therefore \quad 6S = s_1^3 - 3s_1 s_2 + 2s_3,$$

i.e.

$$6S = \left\{\frac{n(n+1)}{2}\right\}^3 - 3 \cdot \frac{n(n+1)}{2} \cdot \frac{n(n+1)(2n+1)}{6} + 2\left\{\frac{n(n+1)}{2}\right\}^2$$

$$= \frac{n^2(n+1)^2}{8}\{n(n+1) - 2(2n+1) + 4\}$$

$$= \frac{n^2(n+1)^2}{8}(n-1)(n-2) = \frac{n^2(n^2-1)}{8}(n^2-n-2).$$

4. If u_n is the total number of plants in the nth year, then evidently

$$u_n = u_{n-1} + 6u_{n-2}.$$

Assuming $u_n = A\alpha^n$, this gives $\alpha^2 = \alpha + 6$, whence $\alpha = 3$ or -2. Hence u_n is of the form $A \cdot 3^n + B(-2)^n$. But in each of the first two years he has only one plant ;

$$\therefore \quad 3A - 2B = 1, \quad 9A + 4B = 1,$$

whence $$A = \tfrac{1}{5}, \quad B = -\tfrac{1}{5}, \quad \text{i.e. } u_n = \tfrac{1}{5}\{3^n - (-2)^n\}.$$

5. (i) $(3x-4)(x^2+2x+5) = 0$,

(ii) $(x^2+2x-1)(x^2+x-3) = 0$,

(iii) $(x-1)^3(x^2+3x+6) = 0$.

6. Let r be the radius of either of the two circles first drawn. Then

$$r(1 + \operatorname{cosec} 60^\circ) = a, \quad \text{whence } r = (2\sqrt{3} - 3)a.$$

If r' be the radius of the required circle, then

$$r' + \sqrt{(r + r')^2 - r^2}$$

$$= \text{perpendicular distance between } AD \text{ and } EF$$

$$= a \,.\, \frac{\sqrt{3}}{2}\,.$$

From this $\qquad 2rr' = \tfrac{3}{4}a^2 - a\sqrt{3}\,.\,r'\,;$

$$\therefore \quad r' = \frac{3}{4}\,.\,\frac{a^2}{2r + \sqrt{3}a} = \frac{3}{4}\,.\,\frac{a}{5\sqrt{3} - 6} = \frac{5\sqrt{3} + 6}{52}\,.\,a.$$

7. The roots of the equation $\tan 3\phi = \tan 3\theta$, considered as an equation in $\tan \phi$, are

$$\tan \theta, \quad \tan\left(\theta + \frac{\pi}{3}\right), \quad \tan\left(\theta + \frac{2\pi}{3}\right).$$

But the equation is

$$\tan^3 \phi - 3 \tan 3\theta\,.\,\tan^2 \phi - 3 \tan \phi + \tan 3\theta = 0,$$

and the sum of the squares of its roots is

$$(3 \tan 3\theta)^2 - 2\,(-3) = 9 \tan^2 3\theta + 6.$$

8. Let the eccentric angles of the angular points be θ, ϕ, ψ. Then, from the auxiliary circle,

$$\phi - \theta = 2\gamma, \quad \psi - \phi = 2\alpha, \quad 2\pi + \theta - \psi = 2\beta.$$

If (X, Y) is the C. of G., then

$$\frac{X}{a} = \frac{1}{3}\,.\,\Sigma \cos \theta, \quad \frac{Y}{b} = \frac{1}{3}\,.\,\Sigma \sin \theta\,;$$

$$\therefore \quad \frac{X^2}{a^2} + \frac{Y^2}{b^2} = \frac{1}{9}\,[3 + 2\Sigma \cos(\theta - \phi)]$$

$$= \frac{1}{9}\,(3 + 2\Sigma \cos 2\alpha),$$

and since $\Sigma\alpha = \pi$,

$$\therefore \quad 1 + \Sigma \cos 2\alpha = -4 \cos \alpha \cos \beta \cos \gamma\,;$$

$\therefore$ the locus of (X, Y) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{9}\,(1 - 8 \cos \alpha \cos \beta \cos \gamma).$$

9. To find where the line

$$\frac{x-x_0}{\cos\theta}=\frac{y-y_0}{\sin\theta}=r$$

meets the general conic, we substitute

$$r\cos\theta+x_0,\quad r\sin\theta+y_0$$

for x and y. The result is

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)$$
$$+2r\left[(ax_0+hy_0+g)\cos\theta+(hx_0+by_0+f)\sin\theta\right]+S_0=0.$$

This quadratic gives the lengths of the radii vectores drawn from $(x_0,\ y_0)$ in the direction θ. Hence the required locus will evidently be obtained by writing $\frac{r}{k}$ for r, and then replacing

$$r\cos\theta,\ r\sin\theta \text{ by } x-x_0 \text{ and } y-y_0.$$

10. Let O be the centre of the board, G that of the square removed, G_1 the C. of G. of the remainder, and C the point of suspension. Draw G_1N perpendicular to CO. Let the side of the board be $2a$.

Then since the weights of the two parts are in the ratio $15:1$, we have

$$OG_1=\tfrac{1}{15}OG=\tfrac{1}{15}\cdot\tfrac{3}{4}\cdot\sqrt{2}a.$$

$$\therefore\quad G_1N=ON=\frac{OG_1}{\sqrt{2}}=\frac{1}{20}a.$$

Hence

$$G_1\hat{C}N=\tan^{-1}\left(\frac{G_1N}{CN}\right)=\tan^{-1}\left(\frac{\frac{1}{20}a}{a-\frac{1}{20}a}\right)=\tan^{-1}\frac{1}{19}.$$

11. The velocity of M just before m begins to move is $\sqrt{2ga}$, and the common velocity just after is $\frac{M}{M+m}\cdot\sqrt{2ga}$, and the acceleration $\frac{M}{M+m}\cdot g$.

The velocity of m just before m' begins to move is given by

$$u^2=\frac{M^2}{(M+m)^2}\cdot 2ga+2\cdot\frac{M}{M+m}\cdot gc=\frac{2M}{M+m}\left(\frac{M}{M+m}\cdot a+c\right)g.$$

The common velocity after the jerk is $\dfrac{M+m}{M+m+m'} \cdot u$ and the acceleration is $\dfrac{M}{M+m+m'} \cdot g$.

With this acceleration the masses describe a distance $b-c$, and the velocity of m when it passes over the edge is given by

$$v^2 = \frac{(M+m)^2}{(M+m+m')^2} \cdot u^2 + 2 \cdot \frac{M}{M+m+m'} (b-c)\, g.$$

After m has passed over the edge, the acceleration of the system is $\dfrac{M+m}{M+m+m'} \cdot g$, and therefore when m' reaches the edge, its velocity is given by

$$V^2 = \frac{(M+m)^2}{(M+m+m')^2} \cdot u^2 + 2 \cdot \frac{M}{M+m+m'} (b-c)\, g + 2 \cdot \frac{M+m}{M+m+m'} \cdot gc\,;$$

therefore

$$\frac{V^2}{2g} = \frac{M\{Ma + (M+m)c\} + M(M+m+m')(b-c) + (M+m)(M+m+m')c}{(M+m+m')^2}.$$

12. We have

$$y_1 = \frac{e^x}{1+x^2} + e^x \tan^{-1} x = \frac{e^x}{1+x^2} + y,$$

$$\text{i.e. } (1+x^2)\, y_1 = e^x + (1+x^2)\, y \;\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Hence $$(1+x^2)\, y_2 + 2xy_1 = e^x + (1+x^2)\, y_1 + 2xy \;\ldots\ldots\ldots\text{(ii)}.$$

Subtracting (i) from (ii), the result easily follows.

LIII.

1. Let the perpendiculars AL, BM, CN on $B'C'$ etc. intersect in O, and the perpendiculars $B'M'$, $C'N'$ on AC, AB in O'. Then

$$AC'^2 - BC'^2 = AN'^2 - BN'^2 = O'A^2 - O'B^2,$$

and $$AB'^2 - CB'^2 = O'A^2 - O'C^2.$$

Subtracting

$$\begin{aligned}O'C^2 - O'B^2 &= (AC'^2 - AB'^2) - BC'^2 + CB'^2\\ &= (OC'^2 - OB'^2) - (OC'^2 - OA'^2 + A'B^2)\\ &\qquad\qquad + (OB'^2 - OA'^2 + A'C^2)\\ &= A'C^2 - A'B^2.\end{aligned}$$

Hence $O'A'$ is perpendicular to BC.

2. Let SR be the semi-latus rectum of one of the ellipses, RN the perpendicular on the minor axis. Then

$$SR = e\,.\,SX, \quad \text{i.e.} \quad CN\,.\,CA = CS\,.\,SX;$$

$$\therefore\ CA^2 - CN\,.\,CA = CS\,.\,CX - CS\,.\,SX = CS^2 = RN^2.$$

Hence, if we take a point V on the minor axis, such that $CV = CA$, we have $RN^2 = CA\,.\,VN$, shewing that R lies on a parabola, with vertex V, axis the minor axis, and latus rectum of length CA, and from the relation just written it is evident that A is on the parabola.

3. We have

$$1 \equiv \frac{1}{x}\{1 - (1 - x)\},$$

$$2(1 - x) \equiv \frac{1}{x}\{(1 - x) - (1 - x)(1 - 2x)\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$n(1 - x)(1 - 2x)\dots(1 - \overline{n-1}\,x) \equiv \frac{1}{x}\{(1 - x)(1 - 2x)\dots(1 - \overline{n-1}\,x)$$
$$- (1 - x)(1 - 2x)\dots(1 - nx)\}.$$

Adding these equations, the result follows.

4. The first determinant

$$= \begin{vmatrix} x+1, & 1, & 2 \\ x+3, & 2, & 3 \\ x+7, & 3, & 4 \end{vmatrix} = \begin{vmatrix} x+1, & 1, & 2 \\ 2, & 1, & 1 \\ 4, & 1, & 1 \end{vmatrix} = \begin{vmatrix} x+1, & 1, & 1 \\ 2, & 1, & 0 \\ 4, & 1, & 0 \end{vmatrix} = -2.$$

The second determinant

$$= \begin{vmatrix} a^2, & b^2, & c^2 \\ 2a+1, & 2b+1, & 2c+1 \\ 2, & 2, & 2 \end{vmatrix}$$

(the last row being obtained by adding the second row to the third, and subtracting twice the first)

$$= 2 \begin{vmatrix} a^2, & b^2, & c^2 \\ 2a+1, & 2b+1, & 2c+1 \\ 1, & 1, & 1 \end{vmatrix}$$

$$= 4 \begin{vmatrix} a^2, & b^2, & c^2 \\ a, & b, & c \\ 1, & 1, & 1 \end{vmatrix} = -4 \,.\, \Pi\,(b-c).$$

5. If $$f(x) \equiv x^3 + x^2 - ax - b,$$
then $$f(a) = a^3 - b, \quad f(0) = -b, \quad f(-1) = a - b.$$

This shews that there is a root between a and 0, and another between 0 and -1; therefore the remaining root must be real.

6. We have

$$\begin{aligned} 4 \cos 2x \sin (y-z) \sin (z-x) \sin (x-y) \\ &= 2 \cos 2x \left[\cos (y+z-2x) - \cos (y-z)\right] \sin (y-z) \\ &= \left[\cos (y+z-4x) + \cos (y+z)\right] \sin (y-z) \\ &\qquad\qquad - \cos 2x \sin 2(y-z). \end{aligned}$$

Hence

$$\begin{aligned} 4 (\Sigma \cos 2x) \sin (y-z) \sin (z-x) \sin (x-y) \\ &= \Sigma \cos (y+z-4x) \sin (y-z) + \Sigma \cos (y+z) \sin (y-z) \\ &\qquad\qquad - \Sigma \cos 2x \sin 2(y-z). \end{aligned}$$

The two latter sums vanish identically. Hence if the first also vanishes, and none of the differences $y-z$, etc. is zero, or a multiple of π, we must have

$$\Sigma \cos 2x = 0.$$

7. Let AN be the perpendicular on BC, O and O' the orthocentres. Then each of the angles OBN, $O'BN$ is $90° - C$, where ABC is the acute-angled triangle. Hence $ON = O'N$ and the distance required is $2ON$.

Now $$OB = BN \operatorname{cosec} C = c \cos B \operatorname{cosec} C$$
$$= b \cos B \operatorname{cosec} B = b \cot B,$$
and $$BN = c \cos B;$$
$$\therefore\ ON^2 = OB^2 - BN^2 = b^2 \cot^2 B - c^2 \cos^2 B.$$

8. Let (x', y') be the point on the parabola, and
$$\frac{x - x'}{\cos\theta} = \frac{y - y'}{\sin\theta} = r$$
the chord. This line meets the parabola where
$$(r \sin\theta + y')^2 = 4a(r\cos\theta + x'),$$
giving $r = 0$ or $\dfrac{4a\cos\theta - 2y'\sin\theta}{\sin^2\theta}$.

But $$y' = 2a \cot(\theta - \phi),$$
and therefore the length of the chord is
$$\frac{4a(\cos\theta - \cot\overline{\theta - \phi}\sin\theta)}{\sin^2\theta} = \frac{4a\sin\phi}{\sin(\theta - \phi)\sin^2\theta}.$$

9. The centre is the point $(1, \frac{1}{2})$, and the equation referred to the centre is
$$5x^2 + 4xy + 8y^2 = 9.$$
The axes are now
$$\frac{x^2 - y^2}{-3} = \frac{xy}{2}, \text{ i.e. } (2x - y)(x + 2y) = 0.$$

Hence, putting
$$X = \frac{2x - y}{\sqrt{5}}, \quad Y = \frac{x + 2y}{\sqrt{5}},$$
the equation becomes
$$4X^2 + 9Y^2 = 9,$$
so that the semi-axes are $\frac{3}{2}$ and 1, and
$$e^2 = \frac{\frac{9}{4} - 1}{\frac{9}{4}} = \frac{5}{9}, \text{ i.e. } e = \frac{\sqrt{5}}{3}.$$

Hence the foci are given by

$$X = \pm\frac{\sqrt{5}}{2}, \quad Y = 0,$$

$$\text{i.e. } 2x - y = \pm\frac{5}{2}, \quad x + 2y = 0.$$

These equations give the points $(1, -\frac{1}{2})$ and $(-1, \frac{1}{2})$. Hence referring back to the original origin, the foci are $(2, 0)$ and $(0, 1)$.

10. Let V be the vertex, G the centre of gravity, N the point of contact with the rail. Draw VO horizontal, and NO perpendicular to VN. Then GO must be vertical.

Now $$VO = VG\cos\theta = \tfrac{3}{4}h\cos\theta,$$

$$VN = c\sec(\theta - \alpha),$$

$$\therefore\; VO = VN\sec(\theta-\alpha) = c\sec^2(\theta-\alpha),$$

$$\therefore\; \tfrac{3}{4}h\cos\theta = c\sec^2(\theta-\alpha).$$

11. Suppose P moves downwards, and let T be the tension of the string. Then the acceleration of P is

$$f = \frac{P-2T}{P}\cdot g,$$

and that of Q upwards is

$$f' = \frac{T-Q}{Q}\cdot g.$$

But $f' = 2f$, whence

$$T = \frac{3PQ}{P+4Q}, \quad f = \frac{P-2Q}{P+4Q}\cdot g.$$

The acceleration of the C. of M. downwards is

$$\frac{P.f - Q.2f}{P+Q} = \frac{(P-2Q)^2}{(P+Q)(P+4Q)}\cdot g.$$

The same result is obtained when the motion is in the opposite direction.

12. $$\int \frac{dx}{x^3 - x} = \int \left(-\frac{1}{x^2} + \frac{1}{x-1} - \frac{1}{x} \right) dx$$

$$= \frac{1}{x} + \log \frac{x-1}{x},$$

$$\int \sec^4 x \, dx = \int (\tan^2 x + 1) \, d(\tan x) = \frac{1}{3} \tan^3 x + \tan x,$$

$$\int x^2 \log x \, dx = \frac{x^3}{3} \log x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^3}{3} \log x - \frac{x^3}{9},$$

$$\int x (a^2 + x^2)^{\frac{3}{2}} \, dx = \frac{1}{2} \int y^{\frac{3}{2}} \, dy \quad \text{where } y = a^2 + x^2$$

$$= \frac{1}{2} \cdot \frac{2}{5} y^{\frac{5}{2}} = \frac{1}{5} (a^2 + x^2)^{\frac{5}{2}}.$$

LIV.

1. Let $ABCD$ be the quadrilateral, and let the successive circles through AB etc. intersect in E, F, G, H.

Then

$$A\hat{H}E = 180^\circ - A\hat{B}E, \quad A\hat{H}G = 180^\circ - A\hat{D}G;$$

$$\therefore \; E\hat{H}G = A\hat{B}E + A\hat{D}G.$$

Similarly $$E\hat{F}G = C\hat{B}E + C\hat{D}G;$$

$$\therefore \; E\hat{H}G + E\hat{F}G = A\hat{B}C + A\hat{D}C = 2 \text{ right angles}.$$

2. Let V be the middle point of PQ, and let TV meet the curve in p, and the directrix in L. Draw TN, pN' perpendicular to the axis. Then

$$SP + SQ = 2VL = 2(TV + NX).$$

Now take a point M on the axis (away from S) such that

$$AM = \tfrac{1}{2}(SP + SQ) - SA,$$

so that M is fixed. We have

$$TN^2 = pN'^2 = 4AS \cdot AN' = 2AS(2AN + 2NN')$$
$$= 2AS(2AN + TV).$$

Also $$AM = TV + AN,$$

$$\therefore\ TN^2 = 2AS(AN + AM) = 2AS \cdot MN.$$

Hence the locus of T is a parabola, vertex M, and latus-rectum $2AS$.

3. (i) Multiplying by $r-2$, the $(p+1)$th term becomes

$$\frac{(p+1)!\,(r-2)}{(r+p)!} = \frac{1}{r!}\left\{\frac{(p+1)!}{(r+1)\ldots(r+p-1)} - \frac{(p+2)!}{(r+1)\ldots(r+p)}\right\}.$$

Hence $$(r-2)S_n = \frac{1}{(r-1)!} - \frac{(n+1)!}{(r+n-1)!}.$$

(ii) The nth term is

$$\frac{n}{1+n^2+n^4} = \frac{1}{2}\left(\frac{1}{1-n+n^2} - \frac{1}{1+n+n^2}\right)$$

$$= \frac{1}{2}\left\{\frac{1}{1+(n-1)+(n-1)^2} - \frac{1}{1+n+n^2}\right\};$$

$$\therefore\ S_n = \frac{1}{2}\left(1 - \frac{1}{1+n+n^2}\right) = \frac{1}{2}\cdot\frac{n(n+1)}{1+n+n^2}.$$

4. Let the quantities be $a_1, a_2 \ldots a_n$. Then

$$\frac{\Sigma a_1}{n} \nless \sqrt[n]{a_1 a_2 \ldots a_n},$$

$$\frac{\Sigma \frac{1}{a_1}}{n} \nless \sqrt[n]{\frac{1}{a_1}\cdot\frac{1}{a_2}\cdots\frac{1}{a_n}}.$$

Multiplying these inequalities together, the result follows.

5. If the equations have a common root, it will also be a root of

$$x(ax^3 + 3bx^2 + d) + (bx^3 + 3dx + e) = 0,$$

$$\text{i.e.} \quad ax^4 + 4bx^3 + 4dx + e = 0 \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Hence the equations (i) and

$$ax^3 + 3bx + d = 0$$

have a common root and therefore since the latter equation is the first derived of (i), the equation (i) has equal roots, the condition for which is

$$I^3 - 27J^2 = 0.$$

But for (i), $\qquad I = ae - 4bd, \quad J = -(ad^2 + b^2e).$

6. We have

$$a = \frac{AP \,.\, AQ \sin\alpha}{PQ}, \quad b = \frac{BP \,.\, BQ \sin\beta}{PQ},$$

$$\frac{AP}{BP} = \frac{\sin\beta_1}{\sin\alpha_1}, \quad \frac{AQ}{BQ} = \frac{\sin\beta_2}{\sin\alpha_2},$$

whence $$\frac{a}{b} = \frac{\sin\alpha}{\sin\beta} \cdot \frac{\sin\beta_1}{\sin\alpha_1} \cdot \frac{\sin\beta_2}{\sin\alpha_2}.$$

7. Putting each expression equal to k, the angles α, β, γ satisfy the equation

$$\tan 3\theta - \tan\theta = k.$$

If $\tan\theta = x$, this is

$$\frac{3x - x^3}{1 - 3x^2} - x = k,$$

or $$2x^3 + 3kx^2 + 2x - k = 0.$$

The roots of this are $\tan\alpha$, $\tan\beta$, $\tan\gamma$,

$$\therefore \tan\alpha\tan\beta\tan\gamma = \frac{k}{2}.$$

Also $$\tan(\alpha + \beta + \gamma) = \frac{-\frac{3k}{2} - \frac{k}{2}}{1 - 1} = \infty,$$

$$\therefore \alpha + \beta + \gamma \text{ is an odd multiple of } \frac{\pi}{2}.$$

8. Let the angular points of the triangle be (x_1, m_1x_1) etc. Then since the line joining the last two is perpendicular to $y=m_1x$, we have

$$\frac{m_2x_2 - m_3x_3}{x_2 - x_3} = -\frac{1}{m_1},$$

whence $$\frac{x_2}{m_3m_1+1} = \frac{x_3}{m_1m_2+1} = \frac{x_1}{m_2m_3+1} \text{ similarly.}$$

The co-ordinates of the centre of gravity are $\frac{1}{3}\Sigma x$, and $\frac{1}{3}\Sigma m_1 x$, and they are therefore proportional to

$$\Sigma m_2m_3 + 3 \text{ and } 3m_1m_2m_3 + \Sigma m_1.$$

9. The equation to the circumcircle must be of the form

$$ax^2 + 2hxy + by^2 + (lx + my + 1)(l'x + m'y) = 0 \ldots\ldots \text{(i)},$$

and the conditions for a circle are

$$a + ll' = b + mm', \quad 2h + lm' + l'm = 0,$$

whence $$l' = \frac{(b-a)l - 2hm}{l^2+m^2}, \quad m' = \frac{-(b-a)m - 2hl}{l^2+m^2},$$

$$\therefore\ a + ll' = \frac{am^2 - 2hlm + bl^2}{l^2+m^2},$$

and from (i) the square of the diameter is

$$\frac{l'^2 + m'^2}{(a+ll')^2},$$

which is equal to the given expression.

10. Let DN, EN be perpendicular to AB, AC respectively, and let G be the C. of G. of the lamina. Then NG must be vertical.

The distance of G from the vertical through A is

$$\frac{1}{3}\left\{b \sin\left(\frac{A}{2} + \phi\right) - c \sin\left(\frac{A}{2} - \phi\right)\right\}$$

$$= \frac{1}{3}\frac{a}{\sin A}\left\{\sin B \sin\left(\frac{A}{2} + \phi\right) - \sin C \sin\left(\frac{A}{2} - \phi\right)\right\}$$

$$= \frac{1}{3}\frac{a}{\sin A}\left\{\sin\frac{A}{2}\cos\phi\,(\sin B - \sin C) + \cos\frac{A}{2}\cos\phi\,(\sin B + \sin C)\right\}$$

$$= \frac{2}{3}\frac{a}{\sin A}\left(\sin^2\frac{A}{2}\sin\frac{B-C}{2}\cos\phi + \cos^2\frac{A}{2}\cos\frac{B-C}{2}\sin\phi\right),$$

and the distance of N from the same line

$$DN\cos\left(\frac{A}{2}-\phi\right)-AD\sin\left(\frac{A}{2}-\phi\right)$$

$$=DE\,.\frac{\sin\left(\frac{A}{2}+\phi\right)}{\sin A}\cos\left(\frac{A}{2}-\phi\right)-DE\,.\frac{\cos\left(\frac{A}{2}+\phi\right)}{\sin A}\sin\left(\frac{A}{2}-\phi\right)$$

$$=DE\,.\frac{\sin 2\phi}{\sin A}.$$

Equating these expressions, the result follows.

11. The accelerations of m, m' downwards are

$$g-\frac{T}{m} \text{ and } g-\frac{T}{m'},$$

while that of M towards the edge is $\frac{2P}{M}$.

From the geometry the sum of the first two must be twice the third,

$$\text{i.e. } 2g-T\left(\frac{1}{m}+\frac{1}{m'}\right)=\frac{4T}{M},$$

whence $$T=\frac{2mm'M}{M(m+m')+4mm'}\,.\,g,$$

giving the required value for $\frac{2T}{M}$.

12. The tangent at (x, y) is

$$X(x^2-ay)+Y(y^2-ax)-axy=0,$$

$$\therefore\quad \alpha=\frac{axy}{x^2-ay},\qquad \beta=\frac{axy}{y^2-ax},$$

$$\therefore\ \frac{1}{\alpha\beta}=\frac{(x^2-ay)(y^2-ax)}{a^2x^2y^2}=\frac{x^2y^2-a(x^3+y^3)+a^2xy}{a^2x^2y^2}$$

$$=\frac{x^2y^2-2a^2xy}{a^2x^2y^2}=\frac{xy-2a^2}{a^2xy},$$

$$\therefore\ xy=\frac{2a^2\alpha\beta}{\alpha\beta-a^2}.$$

LV.

1. Given the triangle, the construction for the inscribed square is as follows:—Draw CN perpendicular and equal to CB. Join AN cutting CB in L, and draw LH perpendicular to BC. Then LH is a side of the square.

Now let AD be the perpendicular from A on BC, and let HK, parallel to BC, cut AD in E. Then

$$CN : HL = CA : HA = DA : EA.$$

Hence the construction for this problem:—At D erect DE perpendicular to CB and equal to a side of the square. In CN take a point V such that NV is equal to a side of the square. Join VE cutting CB in L. Produce NL to meet DE in A. Then A is the vertex of the triangle.

2. Let the tangent at A meet an asymptote in L, and let AP, $A'P$ cut the same asymptote in Q and Q'.

Since AA' is perpendicular to PP', A must be the orthocentre of the triangle $A'PP'$;

$$\therefore\ P\hat{P}'A = A\hat{A}'P.$$

Now since AP, $A'P$ are supplemental chords, they are parallel to a pair of conjugate diameters, and are therefore equally inclined to the asymptote;

$$\therefore\ P\hat{Q}Q' = P\hat{Q}'Q = C\hat{Q}'A'.$$

Also CA, AL are equally inclined to the asymptote;

$$\therefore\ A\hat{L}C = A\hat{C}L.$$

Hence subtracting, $\quad P\hat{A}L = P\hat{A}'A = P\hat{P}'A,$

$\therefore$ AL touches the circle $PP'A$ at A.

3. We may exhaust the cases as follows:—

n unlike things,	no. of combinations	${}^{2n}C_n$
$(n-1)$,, ,,	,, ,,	${}^{2n}C_{n-1}$
....		
all like things,	,, ,,	${}^{2n}C_0$.

Hence the total number of combinations is

$$^{2n}C_n + {}^{2n}C_{n-1} + \ldots + {}^{2n}C_0$$
$$= \text{coefficient of } x^n \text{ in } (1+x)^{2n}(1+x+x^2+\ldots+x^n+\ldots)$$
$$= \quad ,, \qquad ,, \qquad (1+x)^{2n}(1-x)^{-1}.$$

4. (i) The series is the Binomial expansion of $\left(1-\frac{2}{10^2}\right)^{-\frac{1}{2}}$, and is therefore equal to $\frac{5}{7}\sqrt{2}$.

(ii) Putting $\frac{1}{10^2}=x$, the series is

$$1+2x+\frac{3}{2!}\,.\,x^2+\frac{4}{3!}\,.\,x^3+\ldots$$

and the general term is

$$\frac{n+1}{n!}\,.\,x^n=\frac{x^n}{(n-1)!}+\frac{x^n}{n!}.$$

Hence the sum of the series is $(1+x)\,e^x$, so that the sum of the series given is $\frac{101}{100}e^{\frac{1}{100}}$.

5. Multiplying the columns by 1, θ, θ^2, θ^3, where $\theta^4=1$, and adding, we see that

$$x+\theta a+\theta^2 b+\theta^3 c$$

is a factor of the determinant,

$$\text{i.e.} -(\theta a+\theta^2 b+\theta^3 c)$$

is a root of the equation. Hence the four roots are

$$-(a+b+c),\quad a-b+c,\quad b\pm(a-c)\,i.$$

6. (i) We have

$$\cos(\alpha+\beta-2\gamma)-\cos(\alpha-\beta)=2\sin(\beta-\gamma)\sin(\gamma-\alpha),$$
$$\sin(2\alpha+\gamma)-\sin(2\beta+\gamma)=2\cos(\alpha+\beta+\gamma)\sin(\alpha-\beta).$$

The product of the expressions on the left is

$$\tfrac{1}{2}[\sin(3\alpha+\beta-\gamma)+\sin(\alpha-\beta+3\gamma)-\sin(3\alpha-\beta+\gamma)-\sin(\alpha+\beta+\gamma)$$
$$-\sin(\alpha+3\beta-\gamma)-\sin(\beta-\alpha+3\gamma)+\sin(\alpha+\beta+\gamma)+\sin(3\beta+\gamma-\alpha)].$$

Also $\quad \sin(3\alpha+\beta-\gamma)-\sin(3\alpha-\beta+\gamma)=2\cos 3\alpha\sin(\beta-\gamma)$,

and so for the other pairs. Hence the expression above is

$$\Sigma \cos 3\alpha \sin(\beta-\gamma).$$

(ii) $\sin 9\theta = \sin 3\theta\,(3-4\sin^2 3\theta)$

$$\begin{aligned} &= \sin 3\theta\,(4\cos^2 3\theta - 1) \\ &= \sin\theta\,(4\cos^2\theta-1)\left[4(4\cos^3\theta-3\cos\theta)^2-1\right] \\ &= \sin\theta\,(4\cos^2\theta-1)(64\cos^6\theta-96\cos^4\theta+36\cos^2\theta-1). \end{aligned}$$

7. Dividing each expression by

$$\cos^2\alpha\cos^2\beta\cos^2\gamma,$$

and putting $\tan\alpha = a$, etc. we have

$$\frac{x}{a(b+c)^2}=\frac{y}{b(c+a)^2}=\frac{z}{c(a+b)^2}.$$

But $a+b+c=0$,

$$\therefore\ \frac{x}{a^3}=\frac{y}{b^3}=\frac{z}{c^3},$$

$$\therefore\ x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0,$$

and the rational form of this is

$$(x+y+z)^3=27xyz.$$

8. Let the circles be

$$S \equiv x^2+y^2-2kx+\delta^2=0,$$

$$S' \equiv x^2+y^2-2k'x+\delta^2=0.$$

Then the locus is

$$\sqrt{S}\pm\sqrt{S'}=k-k' \quad \text{.....................(i).}$$

Also $\quad S-S' \equiv -2(k-k')x$,

therefore for the locus

$$\sqrt{S}\mp\sqrt{S'}=-2x \quad \text{.....................(ii).}$$

From (i) and (ii),

$$2\sqrt{S}=-2x+k-k',$$

and rationalising and reducing this becomes

$$4y^2=4(k+k')x+(k-k')^2-4\delta^2,$$

a parabola, of latus-rectum $k+k'$.

9. Let the chords of intersection be

$$x \cos \alpha + y \sin \alpha - p = 0 \quad \text{and} \quad x \cos \alpha' + y \sin \alpha' - p' = 0,$$

so that the equation to the circle must be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda (x \cos \alpha + y \sin \alpha - p)(x \cos \alpha' + y \sin \alpha' - p') = 0.$$

The conditions for a circle give

$$\frac{1}{a^2} + \lambda \cos \alpha \cos \alpha' = \frac{1}{b^2} + \lambda \sin \alpha \sin \alpha', \quad \alpha + \alpha' = n\pi \ldots \text{(i)}.$$

Also, since the circle goes through the origin,

$$-1 + \lambda pp' = 0.$$

From (i) $$\pm \lambda = \frac{1}{b^2} - \frac{1}{a^2},$$

$$\therefore \; pp' = \frac{1}{\lambda} = \frac{a^2b^2}{a^2 - b^2} \text{ numerically.}$$

10. Let P, Q, R be the forces. Then taking moments about the incentre and the circumcentre, we have

$$P + Q + R = 0,$$

$$P \cos A + Q \cos B + R \cos C = 0,$$

$$\therefore \; P : Q : R = \cos B - \cos C : \cos C - \cos A : \cos A - \cos B.$$

Now $$\cos B - \cos C = \frac{c^2 + a^2 - b^2}{2ca} - \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(b - c)\{a^2 - (b + c)^2\}}{2abc}$$

$$= -(b - c)(b + c - a)\frac{a + b + c}{2abc},$$

and similarly for the others.

11. Let u be the original velocity of m_1, and v, v' the velocities of m_1, m_2 after the first impact. Then

$$m_1 v + m_2 v' = m_1 u, \quad v - v' = -eu,$$

$$\therefore \; v = \frac{(m_1 - em_2)\, u}{m_1 + m_2}, \quad v' = \frac{m_1 (1 + e)\, u}{m_1 + m_2}.$$

Now let V, V' be the velocities of m_2, m_3 after m_2 strikes m_3. Then

$$m_2 V + m_3 V' = m_2 v',$$
$$V - V' = - ev',$$

whence $$V = \frac{(m_2 - em_3)\, v'}{m_2 + m_3} = \frac{m_1 (m_2 - em_3)(1 + e)}{(m_1 + m_2)(m_2 + m_3)} u,$$

so that if $V = v$, we have

$$m_1 (m_2 - em_3)(1 + e) = (m_1 - em_2)(m_2 + m_3),$$

which is equivalent to the given result.

12. Here λ, $-\mu$ are the roots of

$$\frac{x^2}{c + \theta} + \frac{y^2}{\theta} = 1,$$

$$\therefore \ \lambda - \mu = x^2 + y^2 - c, \quad \lambda\mu = cy^2.$$

Hence if $\dfrac{\partial\lambda}{\partial x} = \lambda_x$, etc. we have

$$\left.\begin{matrix} \lambda_x - \mu_x = 2x, & \lambda_x \mu + \mu_x \lambda = 0 \\ \lambda_y - \mu_y = 2y, & \lambda_y \mu + \mu_y \lambda = 2cy \end{matrix}\right\}.$$

$$\therefore \ \frac{\lambda_x}{\lambda} = \frac{\mu_x}{-\mu} = \frac{2x}{\lambda + \mu}$$
$$= \frac{\lambda_x \mu_y - \mu_x \lambda_y}{2cy},$$

$$\therefore \ \lambda_x \mu_y - \mu_x \lambda_y = \frac{4cxy}{\lambda + \mu}.$$

LVI.

1. Let OD, OF be perpendicular to BC, AB and let LA meet the pedal line in K. We have $O\hat{L}A = O\hat{B}A$, and also since $OBDF$ is cyclic, $O\hat{B}A = O\hat{D}F$, i.e. $O\hat{D}F = O\hat{L}A$, i.e. the points O, D, L, K are cyclic,

$$\therefore \ D\hat{K}L = D\hat{O}L = \text{a right angle}.$$

2. Let the latus-rectum meet the circle in U and U', and let PP' cut the axis in N. Then

$$SU^2 = UN^2 - SN^2 = QN^2 - SN^2.$$

Now $\qquad QN : NX = SR : SX, \quad \therefore \; QN = e \, . \, NX = SP,$

$$\therefore \; SU^2 = SP^2 - SN^2 = PN^2, \text{ i.e. } SU = PN.$$

3. The nth term is the reciprocal of

$$\frac{1}{a} + (n-1)\left(\frac{1}{a+b} - \frac{1}{a}\right),$$

and is therefore equal to

$$\frac{a(a+b)}{a-(n-2)b} = (a+b)\left[1-(n-2)\frac{b}{a}\right]^{-1}$$

$$= (a+b)\left[1+(n-2)\frac{b}{a} + (n-2)^2\frac{b^2}{a^2}\right] \text{ approximately}$$

$$= a + (n-1)b + (n-2)(n-1)\frac{b^2}{a}.$$

4. Suppose the given fraction equal to $1 + \Sigma \dfrac{A_r}{x-r}$. Then

$$A_r = \frac{(r+1)(r+2)\dots(r+n)}{(r-1)\dots 1(-1)\dots(r-n)}$$

$$= (-1)^{n-r}\frac{(n+r)!}{(r-1)!\,(n-r)!\,r!} = (-1)^{n-r}\frac{(n+r)!}{n!\,(r-1)!}\,.\,{}^nC_r.$$

Now $\qquad \dfrac{A_r}{x-r} = -\dfrac{A_r}{r}\left(1-\dfrac{x}{r}\right)^{-1}.$

Hence the required coefficient is

$$-\Sigma\frac{A_r}{r}\,.\,\frac{1}{r^p} = -\Sigma\frac{A_r}{r^{p+1}}.$$

5. Let s_n denote the sum of the nth powers. Then

$$s_1 = s_2 = 0, \qquad s_3 = -3a,$$

$$s_4 + as_1 + 4b = 0, \quad \therefore \; s_4 = -4b,$$

$$s_5 + as_2 + bs_1 = 0, \quad \therefore \; s_5 = 0.$$

Also $\qquad x^{20} = -(ax+b)^5,$

$$\therefore \; s_{20} = -(a^5s_5 + 5a^4bs_4 + 10a^3b^2s_3 + 10a^2b^3s_2 + 5ab^4s_1 + 4b^5)$$

$$= -5a^4b(-4b) + (-10a^3b^2)(-3a) - 4b^5$$

$$= 50a^4b^2 - 4b^5.$$

6. The identity to be proved is, clearing of fractions,

$$\Sigma \sin A \cos A \tan (B-C)(1+\tan B \tan C)=0,$$

i.e. $\Sigma \sin A \cos A (\tan B - \tan C)=0,$

i.e. $\Sigma \sin A \cos^2 A \sin (B-C)=0,$

i.e. $\Sigma \cos^2 A (\cos 2B - \cos 2C)=0,$

i.e. $\Sigma (1+\cos 2A)(\cos 2B - \cos 2C)=0,$

which is evidently true.

7. We have

$$\begin{aligned} 2^{11} \sin^7 \theta \cos^5 \theta &= 2^6 \sin^2 \theta \sin^5 2\theta \\ &= 2^5 (1-\cos 2\theta) \sin^5 2\theta \\ &= 2^2 (2 \sin 2\theta - \sin 4\theta)(1-\cos 4\theta)^2 \\ &= 2\,(2 \sin 2\theta - \sin 4\theta)(3 - 4\cos 4\theta + \cos 8\theta) \\ &= 12 \sin 2\theta - 6 \sin 4\theta - 8 (\sin 6\theta - \sin 2\theta) \\ &\qquad + 4 \sin 8\theta + 2 (\sin 10\theta - \sin 6\theta) \\ &\qquad\qquad - (\sin 12\theta - \sin 4\theta) \\ &= 20 \sin 2\theta - 5 \sin 4\theta - 10 \sin 6\theta + 4 \sin 8\theta \\ &\qquad + 2 \sin 10\theta - \sin 12\theta. \end{aligned}$$

8. If $S+\lambda S'$ is a point circle, we have

$$(g+\lambda g')^2 + (f+\lambda f')^2 + 2(fg+\lambda f'g')(1+\lambda)=0,$$

i.e. $\lambda^2 (g'+f')^2 + 2\lambda (g+f')(g'+f) + (g+f)^2 = 0.$

Replacing λ by $-\dfrac{S}{S'}$, we obtain the equation given.

9. Since the axis of the parabola is parallel to the axis of x, its equation must be of the form

$$y^2 = px + qy.$$

This curve meets $y=x$ where $x=0$ or $p+q$. Hence in this case we must have $q=-p$, so that the equation is $y^2=p(x-y)$, and since the curve passes through $(4, -3)$ we obtain $p=\frac{9}{7}$.

Also the directrix must be of the form $x - k = 0$, so that if (α, β) is the focus, the equation to the parabola must be

$$(x-\alpha)^2 + (y-\beta)^2 = (x-k)^2.$$

Comparing this with the above form, we find

$$\alpha - k = \tfrac{9}{14}, \quad \beta = -\tfrac{9}{14}, \quad k^2 - \alpha^2 - \beta^2 = 0,$$

whence easily $\alpha = 0$, so that the focus is $(0, -\frac{9}{14})$.

10. If r is the radius of the quadrant, that of the base of the cone is $\dfrac{r}{4}$. Hence if A be the point of suspension, V the vertex, G the C. of G., and N the centre of the base, we have

$$VA = r, \quad AN = \frac{r}{4},$$

$$\therefore\ VN = \frac{\sqrt{15}}{4} r, \quad \therefore\ GN = \frac{1}{3} \cdot \frac{\sqrt{15}}{4} r = \frac{\sqrt{5}}{4\sqrt{3}} r,$$

and the angle required is

$$\tan^{-1}\left(\frac{AN}{GN}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}\right).$$

11. Let R, R' be the pressures between the ends of the rod and the planes, f and f' the accelerations of the rod horizontally and vertically (these will be the same for every point, since the rod remains horizontal). Then the accelerations of the planes must be

$$\frac{f \sin\alpha + f' \cos\alpha}{\sin\alpha} \quad \text{and} \quad \frac{f' \cos\alpha' - f \sin\alpha'}{\sin\alpha'}$$

horizontally, since they are always in contact with the ends of the rod.

Hence their equations of motion are

$$\left.\begin{aligned} \frac{R \sin^2\alpha}{M} &= f \sin\alpha + f' \cos\alpha \\ \frac{R' \sin^2\alpha'}{M'} &= f' \cos\alpha' - f \sin\alpha' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Also for the rod, resolving horizontally

$$R' \sin \alpha' - R \sin \alpha = mf,$$

and since there is no rotation,

$$R \cos \alpha = R' \cos \alpha',$$

$$\therefore \quad \frac{R}{\cos \alpha'} = \frac{R'}{\cos \alpha} = \frac{mf}{\sin (\alpha' - \alpha)}.$$

Hence from (i)

$$\frac{R \sin^2 \alpha \cos \alpha'}{M} - \frac{R' \sin^2 \alpha' \cos \alpha}{M'} = f \sin (\alpha + \alpha'),$$

$$\therefore \quad m \left(\frac{\sin^2 \alpha \cos^2 \alpha'}{M} - \frac{\sin^2 \alpha' \cos^2 \alpha}{M'} \right) = \sin (\alpha' - \alpha) \sin (\alpha + \alpha')$$

$$= \sin^2 \alpha' - \sin^2 \alpha,$$

$$\therefore \quad m \left(\frac{\tan^2 \alpha}{M} - \frac{\tan^2 \alpha'}{M'} \right) = \tan^2 \alpha' \sec^2 \alpha - \tan^2 \alpha \sec^2 \alpha'$$

$$= \tan^2 \alpha' - \tan^2 \alpha,$$

$$\text{i.e.} \quad \left(\frac{m}{M} + 1 \right) \tan^2 \alpha = \left(\frac{m}{M'} + 1 \right) \tan^2 \alpha'.$$

12. Since the curve is of the form $u_3 + u_1 = 0$, (u_n denoting a homogeneous function of x, y of degree n), the asymptotes are $u_3 = 0$ and therefore all pass through the origin.

Again, from the equation,

$$(3ax^2 - 2klxy + h) \frac{dx}{dy} - k (lx^2 + 3my^2 - 1) = 0.$$

Hence, if $\dfrac{dx}{dy} = 0$, we have $lx^2 + 3my^2 - 1 = 0$, shewing that the points at which the tangents are parallel to the axis of y are the intersections of the cubic with this conic, and are therefore six in number.

LVII.

1. Let AB be a side of the pentagon, O the centre, L the middle point of the arc AB, and N the point of intersection of OL and AB. Then the triangle OAL is similar to that described in Euclid IV. 10, and

$$L\hat{A}N = \tfrac{1}{2}L\hat{O}B = \tfrac{1}{2}L\hat{O}A.$$

Hence if on NO we take $NK = NL$, we have

$$L\hat{A}K = L\hat{O}A, \quad \therefore \ OK = KA = AL.$$

Now
$$\begin{aligned} OK^2 + OL^2 &= 2ON^2 + 2NK^2 \\ &= 2(OA^2 + AK^2 - 2AN^2), \end{aligned}$$

i.e. $$AL^2 + OA^2 = 2OA^2 + 2AL^2 - AB^2,$$

$$\therefore \ OA^2 + AL^2 = AB^2.$$

2. Let S be the given focus, B the given extremity of the minor axis, SY the perpendicular on the given tangent. Now if C is the centre, SB = semi-major axis = CY and therefore C lies on a circle, centre Y and radius SB. Also C lies on the circle on SB as diameter and is therefore determined. Now join SC, which is the direction of the major axis, and take CA, CA' each equal to SB. Then A, A' are the vertices.

3. This result appears from multiplying the determinants

$$\begin{vmatrix} a^2, & ac, & c^2 \\ a'^2, & a'c', & c'^2 \\ a''^2, & a''c'', & c''^2 \end{vmatrix} \text{ and } \begin{vmatrix} c^2, & -2ac, & a^2 \\ c'^2, & -2a'c', & a'^2 \\ c''^2, & -2a''c'', & a''^2 \end{vmatrix},$$

the second of which is twice the first.

4. The nth term is

$$\begin{aligned} \frac{(n+1)^3}{n(n+2)} \cdot x^{n-1} &= \left[n + 1 + \frac{n+1}{n(n+2)}\right] x^{n-1} \\ &= (n+1)\,x^{n-1} + \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+2}\right) x^{n-1}. \end{aligned}$$

Hence the sum of the infinite series is

$$\frac{(1-x)^{-2}-1}{x}+\frac{1}{2}\left[-\frac{\log(1-x)}{x}-\frac{\log(1-x)+x+\frac{1}{2}x^2}{x^3}\right]$$

$$=\frac{2-x}{(1-x)^2}-\frac{1}{2x^2}-\frac{1}{4x}-\frac{1}{2x^3}(1+x^2)\log(1-x).$$

5. By hypothesis we have

$$(a_1-\lambda)(a_2-\lambda)\ldots(a_n-\lambda)+k\equiv\epsilon(x_1-\lambda)(x_2-\lambda)\ldots(x_n-\lambda),$$

where ϵ is a quantity independent of λ, and equating coefficients of λ^n, we see that $\epsilon=1$.

Hence

$$(x_1-\lambda)(x_2-\lambda)\ldots(x_n-\lambda)-k\equiv(a_1-\lambda)(a_2-\lambda)\ldots(a_n-\lambda),$$

from which the required result evidently follows.

6. We have

$$AI=4R\sin\frac{B}{2}\sin\frac{C}{2},\quad AI_1=4R\cos\frac{B}{2}\cos\frac{C}{2},$$

$$\therefore\ II_1=4R\cos\frac{B+C}{2}=4R\sin\frac{A}{2}.$$

Also $$r_1-r=4R\sin^2\frac{A}{2},\quad\therefore\ II_1^2=4R(r_1-r).$$

Thus, if $IOI_1=\theta$, we have

$$\cos\theta=\frac{(R^2-2Rr)+(R^2+2Rr_1)-4R(r_1-r)}{2\sqrt{R^2-2Rr}\sqrt{R^2+2Rr_1}}$$

$$=\frac{R-r_1+r}{\sqrt{R-2r}\sqrt{R+2r_1}};$$

$$\therefore\ \tan^2\theta=\frac{(R-2r)(R+2r_1)}{(R-r_1+r)^2}-1=\frac{4R(r_1-r)-(r_1+r)^2}{(R-r_1+r)^2}.$$

Now $$R-r_1+r=R-4R\sin^2\frac{A}{2}=R(2\cos A-1),$$

$$r_1+r=4R\sin\frac{A}{2}\cos\frac{B-C}{2},$$

$$\therefore\ 4R(r_1 - r) - (r_1 + r)^2 = 16R^2 \sin^2 \frac{A}{2} - 16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2}$$

$$= 16R^2 \sin^2 \frac{A}{2} \sin^2 \frac{B-C}{2},$$

$$\therefore\ \tan\theta = \frac{4 \sin \frac{A}{2} \sin \frac{B-C}{2}}{2\cos A - 1} = \frac{2(\sin B - \sin C)}{2\cos A - 1}.$$

7. The area of a regular polygon of n sides inscribed in a circle of radius R is

$$\frac{1}{2} nR^2 \sin \frac{2\pi}{n}.$$

Also if θ is acute, $\sin\theta$ lies between θ and $\theta - \frac{1}{6}\theta^3$. Hence $50 \sin \frac{\pi}{50}$ lies between π and

$$50\left\{\frac{\pi}{50} - \frac{1}{6}\left(\frac{\pi}{50}\right)^3\right\} = \pi - \frac{1}{6} \cdot \frac{\pi^3}{50^2}.$$

Hence the ratio of the areas lies between 1 and $1 - \frac{1}{6} \cdot \frac{\pi^2}{50^2}$.

Now $\pi^2 < 10$; $\quad \therefore\ \frac{1}{6} \cdot \frac{\pi^2}{50^2} < \frac{10}{6 \times 2500} < \frac{1}{1500}$.

8. If the normals at $P(m)$ and $Q(m')$ meet at a point μ on the curve, then m, m' are the roots of

$$m^2 + m\mu + 2 = 0,$$

and the co-ordinates of T are $(2a, -a\mu)$, while the equation to PQ is

$$2x + \mu y + 4a = 0.$$

The equation to the circle TPQ must be of the form

$$y^2 - 4ax + \lambda(2x + \mu y + 4a)(2x - \mu y + k) = 0 \ldots\ldots \text{(i)},$$

and the condition for a circle gives

$$1 - \lambda\mu^2 = 4\lambda, \quad \text{i.e. } \lambda = \frac{1}{4 + \mu^2}.$$

Further the circle passes through T and substituting the co-ordinates of T and the value of λ in (i), we find $k=0$, so that the equation to the circle becomes

$$(\mu^2+4)(y^2-4ax)+4x^2-\mu^2y^2+4a(2x-\mu y)=0,$$

$$\text{or}\quad x^2+y^2-a(\mu^2+2)x-a\mu y=0.$$

The centre is therefore

$$x=\tfrac{1}{2}a(\mu^2+2),\quad y=\tfrac{1}{2}a\mu,$$

so that its locus is

$$2y^2=a(x-a).$$

9. The tangents at the points $\beta\pm\alpha$ on $\frac{l}{r}=1+e\cos\theta$ intersect where $\theta=\beta$, and the equation to the tangent at $(\beta-\alpha)$ being

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\overline{\beta-\alpha}),$$

we have
$$\frac{l}{ST}=e\cos\beta+\cos\alpha.$$

But $\frac{l}{SP}=1+e\cos(\beta-\alpha)$, $\frac{l}{SQ}=1+e\cos(\beta+\alpha)$,

whence easily
$$l\left(\frac{1}{SP}+\frac{1}{SQ}-\frac{2\cos\alpha}{ST}\right)=2\sin^2\alpha.$$

10. Let x be the distance of the C. of G. of the machine from the fulcrum O, A the position of the moveable weight for a given weight W. Then

$$W.a+Q.x=P.OA.$$

When the fulcrum is moved to O', let the true weight of the body appearing to weigh W be $W-W'$. Then

$$(W-W')(a+\alpha)+Q(x+\alpha)=P.O'A.$$

Subtracting

$$W\alpha-W'(a+\alpha)+Q\alpha=P(O'A-OA)=-P\alpha,$$

$$\therefore\ W'=\frac{(W+P+Q)\alpha}{a+\alpha}.$$

11. Let V be the velocity of the nail and hammer-head just after impact, V' the final velocity when all are in motion. Then

$$mv = (m + p) V = (M + m + p) V'.$$

Also the change in Kinetic Energy is equal to the work done by penetration against the resistance. Hence if x be the distance penetrated, we have

$$\frac{1}{2}(m + p) V^2 - \frac{1}{2}(M + m + p) V'^2 = R \cdot x,$$

$$\text{i.e.}\quad R \cdot x = \frac{1}{2}\frac{m^2v^2}{m + p} - \frac{1}{2}\frac{m^2v^2}{M + m + p}$$

$$= \frac{1}{2} \cdot \frac{Mm^2v^2}{(m + p)(M + m + p)}.$$

12. Calling the given integral u_3, we have, integrating by parts,

$$u_3 = \Big[x^3e^x\Big]_0^1 - 3u_2 = e - 3u_2,$$

and similarly $\qquad u_2 = e - 2u_1, \qquad u_1 = e - u_0.$

But $u_0 = e - 1$, and hence we find

$$u_3 = 6 - 2e = 6 - 2(2{\cdot}7183\ldots) = {\cdot}56\ldots.$$

LVIII.

1. Denote the given straight lines by 1, 2, 3. Through any point O on 1, draw OH parallel to 2. Then through 2 we can draw a plane parallel to the plane containing 1 and OH. Let 3 meet this pair of parallel planes in A and B. Through A draw AE parallel to 2 meeting 1 in E, and complete the parallelogram $EABF$. Draw FG, BK parallel to 1 meeting 2 in G and K. Draw GM parallel to 3 meeting 1 in M, and complete the parallelogram $KGML$.

2. Let AA' be the major axis of one section, X its intersection with the directrix, XLL' perpendicular to the axis of the cone, C the centre of the section, AN, $A'N'$, CM perpendiculars on the axis of the cone. Let O be the vertex of the cone, and suppose $OA < OA'$. Then $CM = \frac{1}{2}(A'N' - AN)$. But AN, $A'N'$ are in a fixed ratio to OA, OA', therefore CM is proportional to

$$\tfrac{1}{2}(OA' - OA) = \tfrac{1}{2}(L'A' - LA) = \tfrac{1}{2}(SA' - SA) = \tfrac{1}{2}SS' = CS.$$

But $CS \,.\, XS = b^2$, and $XS^2 = XL \,.\, XL'$, so that XS is the same for both sections. Hence CM is proportional to b^2.

3. (i) Take n quantities each equal to $\dfrac{1}{n}$, and $(n+1)$ each equal to $\dfrac{1}{n+1}$. Then since their A.M. is > the G.M. we have

$$\frac{2}{2n+1} > \left\{\frac{1}{n^n} \cdot \frac{1}{(n+1)^{n+1}}\right\}^{\frac{1}{2n+1}},$$

$$\text{i.e. } n^n (n+1)^{n+1} > (n+\tfrac{1}{2})^{2n+1}.$$

(ii) Take $(n-1)$ quantities equal to $(n-1)$, and $(n+1)$ equal to $(n+1)$. Then

$$\frac{(n-1)^2 + (n+1)^2}{2n} > \{(n-1)^{n-1}(n+1)^{n+1}\}^{\frac{1}{2n}},$$

$$\text{i.e. } \left(n + \frac{1}{n}\right)^{2n} > (n-1)^{n-1}(n+1)^{n+1}.$$

4. Since $n+1$ and $2n+1$ are prime, n must be even. If n were of the form $3m+1$, then $2n+1$ would be $M(3)$. If n were of the form $3m-1$, then $n+1$ would be $M(3)$. Hence n must be a multiple of 3 and therefore of 6.

Again, if n were of the form $5m-1$, then $(n+1)$ would be $M(5)$, and if n were of the form $5m+2$, then $2n+1$ would be $M(5)$. Hence n must be of one of the forms

$$5m, \quad 5m+1, \quad 5m-2,$$

i.e. either n, $n-1$ or $n-3$ is a multiple of 5.

5. Multiply the equation in succession by

$$1, \quad x, \quad x^2 \ldots x^n.$$

Then assuming that $x^n = 1$, we get

$$a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0,$$

$$a_2x^{n-1} + a_3x^{n-2} + \dots + a_1 = 0,$$

$$a_3x^{n-1} + a_4x^{n-2} + \dots + a_2 = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_nx^{n-1} + a_1x^{n-2} + \dots + a_{n-1} = 0,$$

whence eliminating x^{n-1}, x^{n-2}, ..., x, 1, the determinant in question is zero.

Conversely, multiplying the columns of the determinant by x^{n-1}, x^{n-2}, ... x, 1, and adding, it appears that the determinant contains the factor

$$a_1x^{n-1} + a_2x^{n-2} + \dots + a_n,$$

and hence, if the determinant is zero, the equation in question must be satisfied.

6. Let D be the middle point of AB. Then since the angle B is common to the triangles ABC, DBC, the ratio of the radii of the circles is clearly $CD : AC$, and therefore the ratio of their areas is $CD^2 : AC^2$.

But $$CD^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{4}c^2,$$

so that the ratio required is

$$2a^2 + 2b^2 - c^2 : 4b^2,$$

or $$2 \sin^2 A + 2 \sin^2 B - \sin^2 C : 4 \sin^2 B,$$

and putting $$2 \sin^2 A = 1 - \cos 2A, \text{ etc.},$$

this ratio takes the form given.

7. The roots of $\sin 3\theta = \frac{\sqrt{3}}{2}$, considered as an equation in $\sin\theta$, are $\sin 20°$, $\sin 40°$ and $-\sin 80°$. But the equation is

$$8 \sin^3\theta - 6 \sin\theta + \sqrt{3} = 0,$$

and the product of the roots is $-\frac{\sqrt{3}}{8}$;

$$\therefore \quad \sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{\sqrt{3}}{8} \text{ and } \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

The second result is obtained from $\cos 3\theta = \frac{1}{2}$ in the same way.

8. The equation to the normal at (x', y') may be written

$$\frac{x - x'}{px'/a^2} = \frac{y - y'}{py'/b^2} = r,$$

where $p^2\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right) = 1$. To find where this meets the ellipse we have

$$\frac{x'^2}{a^2}\left(1 + \frac{pr}{a^2}\right)^2 + \frac{y'^2}{b^2}\left(1 + \frac{pr}{b^2}\right)^2 = 1,$$

whence $$r = 0, \text{ or } -2/p^3\left(\frac{x'^2}{a^6} + \frac{y'^2}{b^6}\right),$$

$$\therefore \quad \frac{2}{n} = p^3\left(\frac{x'^2}{a^6} + \frac{y'^2}{b^6}\right).$$

Also the equation to the perpendicular semi-diameter is

$$\frac{x}{py'/b^2} = \frac{y}{-px'/a^2} = r,$$

and where this meets the ellipse, we have

$$\frac{1}{a^2}\cdot\frac{p^2y'^2}{b^4} + \frac{1}{b^2}\cdot\frac{p^2x'^2}{a^4} = \frac{1}{r^2}, \text{ i.e. } \frac{p^2}{a^2b^2} = \frac{1}{r^2};$$

$$\therefore \quad pd = ab.$$

Further $$(a^2 + b^2)\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right) = 1 + a^2b^2\left(\frac{x'^2}{a^6} + \frac{y'^2}{b^6}\right);$$

$$\text{i.e. } (a^2 + b^2)\cdot\frac{d^2}{a^2b^2} = 1 + a^2b^2\cdot\frac{2}{n}\cdot\frac{d^3}{a^3b^3};$$

$$\text{i.e. } (a^2 + b^2)\,d^2 - a^2b^2 = \frac{2}{n}\cdot d^3ab.$$

9. The line $y = mx + c$ meets the first circle where

$$(1 + m^2)\,x^2 + 2\,(mc - k)\,x + c^2 + \delta^2 = 0.$$

Hence if (x_1, y_1), (x_2, y_2) are the points of intersection,

$$x_1 + x_2 = \frac{2\,(k - mc)}{1 + m^2}, \quad x_1x_2 = \frac{c^2 + \delta^2}{1 + m^2};$$

$$\therefore\ (x_1-x_2)^2 = 4\,\frac{(k-mc)^2-(c^2+\delta^2)(1+m^2)}{(1+m^2)^2}$$

and the square of the intercept is

$$(x_1-x_2)^2+(y_1-y_2)^2 = (1+m^2)(x_1-x_2)^2$$
$$= 4\,\frac{k^2-c^2-2kmc-\delta^2(1+m^2)}{1+m^2}.$$

Hence, if the intercepts are equal,

$$k^2-2kmc = k'^2-2k'mc,$$

$$\text{i.e. } c = \frac{k+k'}{2m}.$$

Hence the line is

$$y = mx + \frac{k+k'}{2m},$$

and its envelope is $\quad y^2 = 2(k+k')\,x.$

10. Let A be the point of contact with the ground, N with the rail, G the centre of gravity of the man and the ladder. Draw AO making an angle λ with the vertical towards N, and NO perpendicular to the ladder. Then, in limiting equilibrium, O must be vertically above G.

If the man has ascended a distance x, then

$$a+nx = (n+1)\,AG.$$

Also $\dfrac{AG}{GO} = \dfrac{\sin\lambda}{\sin(\alpha-\lambda)}$, $\dfrac{GN}{GO} = \cos\alpha$; $\therefore\ \dfrac{AG}{GN} = \dfrac{\sin\lambda}{\cos\alpha\sin(\alpha-\lambda)}$;

$$\therefore\ \frac{AG}{AN} = \frac{\sin\lambda}{\sin\alpha\cos(\alpha-\lambda)};$$

$$\text{i.e. } \frac{a+nx}{n+1}\sin\alpha\cos(\alpha-\lambda) = \sin\lambda\,.\,\frac{h}{\cos\alpha},$$

$$\text{i.e. } a+nx = 2(n+1)\,h\sin\lambda\sec(\alpha-\lambda)\operatorname{cosec}2\alpha.$$

11. Let a be the amplitude, A the extreme position of the particle. Draw a circle, centre O and radius OA, and draw PQ perpendicular to OA to meet the circle in Q. Then if P moves with simple harmonic motion, Q describes the circle with uniform angular velocity, and the periodic times are the same, so that, if

ω is the angular velocity of Q, $T=\dfrac{2\pi}{\omega}$. Also the velocity of P is equal to the resolved velocity of Q in the direction OA, i.e. $v=a\omega\sin\theta$, where $A\hat{O}Q=\theta$. Hence, since $OP=a\cos\theta$, we have $\theta=\tan^{-1}\left(\dfrac{v}{\omega\,.\,OP}\right)$, and the time required is $\dfrac{2\theta}{\omega}$. Putting $\omega=\dfrac{2\pi}{T}$, the result follows.

12. Since $\log(\sec x+\tan x)+\log(\sec x-\tan x)$

$$=\log(\sec^2 x-\tan^2 x)=\log 1=0,$$

it follows that the function changes sign with x, i.e. it is an odd function of x. Suppose then

$$\log(\sec x+\tan x)=a_1x+\frac{a_3x^3}{3\,!}+\frac{a_5x^5}{5\,!}+\ldots.$$

Then, differentiating,

$$\sec x=a_1+\frac{a_3x^2}{2\,!}+\frac{a_5x^4}{4\,!}+\ldots.$$

But $$\cos x=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\ldots.$$

Multiplying these together, and equating all the coefficients after the first to zero, we get

$$a_1=1,\quad a_3-a_1=0,$$

$$\frac{a_1+a_5}{4\,!}-\frac{a_3}{(2\,!)^2}=0,\quad \frac{a_7-a_1}{6\,!}-\frac{a_5-a_3}{2\,!\,4\,!}=0,$$

whence $$a_3=1,\quad a_5=5,\quad a_7=61.$$

LIX.

1. If r is the radius, and O the centre,

$$AQ\,.\,AQ'=r^2-OA^2=OA\,.\,OB-OA^2=OA\,.\,AB;$$

$\therefore$ the triangles OAQ', BAQ are similar, and $A\hat{B}Q=A\hat{Q}'O$. Similarly $A\hat{B}P=A\hat{P}'O$; $\therefore$ $P\hat{B}Q=A\hat{P}'O-A\hat{Q}'O=A\hat{P}O-A\hat{Q}O$, which is similarly equal to $P'\hat{B}Q'$.

2. Let ABC be any triangle, I, I_1, I_2, I_3 the centres of the inscribed and escribed circles, ID, I_1L, I_2M, I_3N perpendiculars on BC, O the middle point of BC. Then

$$ON = CN - OC = s - \tfrac{1}{2}a = \tfrac{1}{2}(b+c) = OM \text{ similarly.}$$

Again, from similar triangles,

$$\frac{ID}{BD} = \frac{BN}{I_3N}, \quad \frac{ID}{DC} = \frac{I_3N}{CN};$$

$$\therefore \quad \frac{ID^2}{BD \cdot DC} = \frac{BN}{CN} = \frac{s-a}{s}.$$

Similarly $$\frac{I_1L^2}{BL \cdot LC} = \frac{s}{s-a}.$$

Applying these results to the triangle SPS', the side SS' corresponding to BC, it appears that the loci of I_2, I_3 are the tangents at the vertices of the ellipse, and the loci of I, I_1 ellipses on SS' as major and minor axes respectively.

3. If x, y, z be the roots of the equation

$$\lambda^3 - p\lambda^2 + q\lambda - r = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i}),$$

then by the question, if the given equations are consistent, they must also be roots of

$$a \cdot \frac{r}{\lambda} + b(p^2 - 2q - \lambda^2) + \left(c + \frac{r}{\lambda}\right)(p - \lambda) = 0,$$

$$\text{i.e.} \;\; b\lambda^3 + c\lambda^2 - \{b(p^2 - 2q) + pc - r\}\lambda - r(a+p) = 0 \;\ldots(\text{ii}).$$

The equations (i) and (ii) must therefore be identical, so that

$$p = -\frac{c}{b}, \text{ and } r = \frac{r(a+p)}{b}, \text{ i.e. } a + p = b.$$

Hence $$a - \frac{c}{b} = b, \text{ i.e. } ab = b^2 + c.$$

If this condition is satisfied, the equations have an infinity of solutions. Otherwise they are inconsistent, unless two of the quantities are equal, in which case the three equations reduce to two, and yield a finite number of solutions.

4. The series is the absolute term in the expansion of

$$\left[2\left(1+\frac{1}{x}\right)\right]^n + n\cdot\frac{x}{2}\left[2\left(1+\frac{1}{x}\right)\right]^{n-1}$$

$$+\frac{n(n-1)}{2\,!}\left(\frac{x}{2}\right)^2\left[2\left(1+\frac{1}{x}\right)\right]^{n-2}+\ldots$$

$$=\left\{2\left(1+\frac{1}{x}\right)+\frac{x}{2}\right\}^n=\frac{(x+2)^{2n}}{(2x)^n}.$$

It is therefore the coefficient of x^n in $\dfrac{1}{2^n}(x+2)^{2n}$, which is $\dfrac{(2n)!}{(n\,!)^2}$.

5. Here $\alpha=2(\beta-\gamma)$, $\alpha+\beta+\gamma=0$, whence

$$\frac{\alpha}{-4}=\frac{\beta}{1}=\frac{\gamma}{3}=k, \text{ suppose.}$$

Then $\quad \Sigma\beta\gamma=-13k^2=a \ldots\text{(i)}, \qquad \alpha\beta\gamma=-12k^3=-b\ldots\text{(ii)};$

$$\therefore\ k=-\frac{13b}{12a}, \text{ and the roots are } \frac{13b}{3a},\ -\frac{13b}{12a},\ -\frac{13b}{4a}.$$

Further, eliminating k from (i) and (ii), we have

$$\left(-\frac{a}{13}\right)^3=\left(\frac{b}{12}\right)^2, \text{ i.e. } 144a^3+2197b^2=0.$$

6. Let $ABC\ldots$ be the polygon of m sides, and P any vertex of the other polygon, and let $P\hat{O}A=\theta$, O being the common centre. Then

$$PA^2=a^2+b^2-2ab\cos\theta,\quad PB^2=a^2+b^2-2ab\cos\left(\theta+\frac{2\pi}{m}\right), \text{ etc.}$$

Hence

$$\Sigma PA^2=m(a^2+b^2)-2ab\sum_{r=0}^{r=m-1}\cos\left(\theta+\frac{2r\pi}{m}\right).$$

The value of the latter sum is zero, since

$$\sin(m\times\tfrac{1}{2}\text{ difference})=\sin\left(m\cdot\frac{\pi}{m}\right)=0.$$

Hence $\Sigma PA^2=m(a^2+b^2)$, and this will be true for every vertex of the second polygon, so that the sum required is $mn(a^2+b^2)$.

7. We have

$$\left(2\cos\frac{x}{2}+1\right)\left(2\cos\frac{x}{2}-1\right)=4\cos^2\frac{x}{2}-1=2\cos x+1,$$

$$\left(2\cos\frac{x}{2^2}+1\right)\left(2\cos\frac{x}{2^2}-1\right)=2\cos\frac{x}{2}+1,$$

....

$$\left(2\cos\frac{x}{2^n}+1\right)\left(2\cos\frac{x}{2^n}-1\right)=2\cos\frac{x}{2^{n-1}}+1.$$

Multiplying these together, we get

$$\left(2\cos\frac{x}{2}-1\right)\left(2\cos\frac{x}{2^2}-1\right)\ldots\left(2\cos\frac{x}{2^n}-1\right)=\frac{2\cos x+1}{2\cos\frac{x}{2^n}+1}.$$

Now, if n be increased without limit, $\text{Lt}\cos\frac{x}{2^n}=1$, and therefore the limiting value of the product is $\frac{1}{3}(2\cos x+1)$.

8. If the angular points are m, m', m'' we have

$$\Sigma\frac{1}{m}=3k, \quad \Sigma m=0.$$

Also the side (m', m'') is

$$x+\frac{y}{m'm''}-\frac{a(m'+m'')}{m'm''}=0,$$

$$\text{i.e.}\quad x-\frac{3km-1}{m^2}\cdot y-a\left(3k-\frac{1}{m}\right)=0,$$

or
$$m^2(x-3ak)-m(3ky-a)+y=0,$$

and its envelope is

$$(3ky-a)^2=4y(x-3ak), \quad \text{i.e.}\quad (3ky+a)^2=4xy.$$

9. The centre is $(\frac{1}{6}, -\frac{1}{6})$, and the equation referred to the centre is

$$7x^2+2xy+7y^2=\frac{4}{3},$$

and writing $X=\frac{x-y}{\sqrt{2}}$, $Y=\frac{x+y}{\sqrt{2}}$, this becomes

$$3X^2+4Y^2=\frac{2}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

whence the squares of the semi-axes are $\frac{2}{9}$ and $\frac{1}{6}$ and

$$e^2 = \frac{\frac{2}{9} - \frac{1}{6}}{\frac{2}{9}} = \frac{1}{4}, \quad \therefore \ e = \frac{1}{2}.$$

The directrices are now

$$X = \pm \frac{\frac{\sqrt{2}}{3}}{\frac{1}{2}} = \pm \frac{2\sqrt{2}}{3}; \text{ i.e. } x - y = \pm \tfrac{4}{3},$$

or transferring back to the original origin

$$x - \tfrac{1}{6} - (y + \tfrac{1}{6}) = \pm \tfrac{4}{3},$$

$$\text{i.e. } x - y = -1 \text{ and } x - y = \tfrac{5}{3}.$$

The figure can be traced from the equation (i).

10. Draw DN perpendicular to the plane ABC. Then the rods being all equal, N is the circumcentre of ABC. Considering the forces on BD, two of them, viz., the weight and the pressure at B, are vertical, and intersect BN. Hence the horizontal force must act along that line for equilibrium to be possible. Let R be the pressure at either B or C and let AN produced meet BC in M. Let T be the tension of either string.

Then for the equilibrium of BD, taking moments about D,

$$-W \,.\, \tfrac{1}{2} \sin \alpha + R \,.\, \sin \alpha = T \,.\, \cos \alpha \ \ldots\ldots\ldots\ldots \text{(i)},$$

and for the whole system, taking moments about A,

$$2R \,.\, AM = 2W(AN + \tfrac{1}{2} MN) + W \,.\, \tfrac{1}{2} AN \ \ldots\ldots\ldots \text{(ii)}.$$

Now, if $AN = r$, $MN = r \cos 2\theta = r(2\cos^2\theta - 1)$;

$$\therefore \ AM = 2r \cos^2 \theta.$$

Thus the equation (ii) becomes

$$4R \cos^2 \theta = (\tfrac{3}{2} + 2 \cos^2 \theta) W,$$

and therefore from (i)

$$T = (R - \tfrac{1}{2} W) \tan \alpha = \tfrac{3}{8} W \,.\, \frac{\tan \alpha}{\cos^2 \theta}.$$

11. If the velocity be due to a height h, then h is least when the focus of the path is in PQ, and we then have

PQ = sum of the perpendiculars from P and Q on the directrix

$= h + h - PQ \sin \alpha$.

Hence, if $PQ = c$, we have

$$h = \frac{c\,(1 + \sin \alpha)}{2}.$$

Similarly for the path from Q to P, we get

$$h' = \frac{c\,(1 - \sin \alpha)}{2};$$

$$\therefore \quad \frac{h}{h'} = \frac{1 + \sin \alpha}{1 - \sin \alpha} = \tan^2 \left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

12. Clearly the curve is contained within a rectangle, sides $2a$, $2b$, with centre at the origin. It passes through the origin. Also, if $\pi - \theta$ be written for θ, y remains unchanged, and x changes sign: while if $\pi + \theta$ be written for θ, x remains unchanged, and y changes sign. It thus appears that the shape of the curve is that of a figure of 8, the two loops being equal.

The whole area is the value of $\int x \frac{dy}{d\theta} d\theta$, taken round the curve, i.e.

$$\int_0^{2\pi} a \sin 2\theta \,.\, b \cos \theta d\theta = 4 \int_0^{\frac{\pi}{2}} ab \sin 2\theta \cos \theta d\theta$$

$$= 2ab \int_0^{\frac{\pi}{2}} (\sin 3\theta + \sin \theta)\, d\theta$$

$$= 2ab \Big[-\tfrac{1}{3} \cos 3\theta - \cos \theta \Big]_0^{\frac{\pi}{2}}$$

$$= \tfrac{8}{3} ab.$$

LX.

1. Let AN be the perpendicular from A on BC, and R the radius of the circumcircle. Then $AB \,.\, AC = 2R \,.\, AN$, therefore R is given. Let O be the circumcentre, I the incentre, and let AO, AI meet the circumcircle in L, D respectively. Produce LD to meet AN produced in X. Then since

$$O\hat{A}I = I\hat{A}N \left(= \frac{B - C}{2}\right)$$

and ADL is a right angle, $AX = AL = 2R$, therefore X is a fixed point. Produce AD to E, making $DE = DI$. Then since $DI = DC$, ICE is a right angle, therefore $A\hat{E}C = \frac{B}{2}$, therefore triangles AEC, AIB are similar;

$$\therefore\ AI \,.\, AE = AB \,.\, AC.$$

But $$AI \,.\, AE = AD^2 - ID^2 = AX^2 - IX^2,$$

i.e. $IX^2 = 4R^2 - AB \,.\, AC$, therefore the locus of I is a circle with centre X.

2. Let AA', $\alpha\alpha'$ be the major axes, AN, $A'N'$, αn, $\alpha' n'$ perpendiculars on the axis of the cone. Then

$$b^2 = AN \,.\, A'N', \quad b'^2 = \alpha n \,.\, \alpha' n';$$

$$\therefore\ \frac{b^2}{b'^2} = \frac{AN}{\alpha n} \cdot \frac{A'N'}{\alpha' n'} = \frac{OA}{O\alpha} \cdot \frac{OA'}{O\alpha'} = \frac{OA^2}{O\alpha'^2},$$

where O is the vertex of the cone, since $OA \,.\, O\alpha = OA' \,.\, O\alpha'$.

But the triangles OAA', $O\alpha\alpha'$ are similar,

$$\therefore\ \frac{OA}{O\alpha'} = \frac{AA'}{\alpha\alpha'} = \frac{a}{a'},$$

$$\therefore\ \frac{b}{b'} = \frac{a}{a'}, \text{ i.e. the sections are similar.}$$

3. The series is

$$ab\,(1-1)^{r+1} + (r+1)\,(a+b)\,(1-1)^r$$
$$- \left[(r+1) - \frac{(r+1)\,r}{2\,!} \,.\, 2^2 + \frac{(r+1)\,r\,(r-1)}{3\,!} \,.\, 3^2 - \ldots\right].$$

The last series is $2\,!$ times the coefficient of x^2 in $(1-e^x)^{r+1}$, and is therefore zero, since the expansion begins with x^{r+1}. Hence the value of the original series is zero.

4. With the given conditions, the series

$$S = u_0 + u_1 x + u_2 x^2 + \ldots \quad (u_0 = b)$$

will be a recurring series, with scale of relation

$$2u_n - u_{n-1} - u_{n-2} = 0,$$

and the generating function is

$$S = \frac{2u_0 + (2u_1 - u_0)x}{2 - x - x^2} = \frac{2b + ax}{2 - x - x^2} = \frac{b + \frac{1}{2}ax}{(1-x)(1+\frac{1}{2}x)}$$

$$= \tfrac{2}{3}(b + \tfrac{1}{2}a) \cdot \frac{1}{1-x} + \tfrac{1}{3}(b-a) \cdot \frac{1}{1+\frac{1}{2}x};$$

$$\therefore\ u_n = \tfrac{2}{3}(b + \tfrac{1}{2}a) + \tfrac{1}{3}(b-a)(-\tfrac{1}{2})^n$$

$$= \tfrac{1}{3}\left[\{1 - (-\tfrac{1}{2})^n\}a + \{2 + (-\tfrac{1}{2})^n\}b\right].$$

5. We have

$$[x - (\alpha^2 + \alpha^4)][x - (\alpha + \alpha^3)] = x^2 + x + (\alpha^3 + \alpha^2 + 2) \ldots\ldots\text{(i)},$$

since $\quad \alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1, \quad$ and $\quad \alpha^5 = 1.$

So also

$$[x - (\alpha + \alpha^2)][x - (\alpha^3 + \alpha^4)] = x^2 + x + (\alpha^4 + \alpha + 2) \ldots\text{(ii)}.$$

Multiplying (i) and (ii), the product is

$$(x^2 + x)^2 + 3(x^2 + x) + 1 = x^4 + 2x^3 + 4x^2 + 3x + 1.$$

Hence the roots of the given equation are

$$\alpha^2 + \alpha^4, \quad \alpha + \alpha^3, \quad \alpha + \alpha^2, \quad \alpha^3 + \alpha^4.$$

6. Putting $\tan\theta = t$, $\tan\alpha = t_1$, etc., then by hypothesis t_3, t_4 are the roots of

$$t(1 - t_1t_2)^2 + t_1(1 - tt_2)^2 + t_2(1 - tt_1)^2 = 0,$$

$$\text{i.e. } t_1t_2(t_1 + t_2)t^2 + (1 - 6t_1t_2 + t_1^2t_2^2)t + t_1 + t_2 = 0;$$

$$\therefore\ t_3t_4 = \frac{1}{t_1t_2}, \quad t_3 + t_4 = -\frac{1 - 6t_1t_2 + t_1^2t_2^2}{t_1t_2(t_1 + t_2)},$$

$$\text{i.e. } t_1t_2t_3t_4 = 1, \quad (t_1 + t_2)(t_3 + t_4) + t_1t_2 + t_3t_4 - 6 = 0.$$

The symmetry of these results shews that t_3, t_4 are interchangeable with t_1, t_2.

7. If $\tan 2n\theta = \infty$, then $\theta = \dfrac{(2r+1)\pi}{4n} \quad (r = 0, 1, \ldots 2n-1).$

Hence putting $\cot^2\theta = x$ in the equation

$$1 - \frac{2n(2n-1)}{2!}\tan^2\theta + \frac{2n(2n-1)(2n-2)(2n-3)}{4!}\tan^4\theta - \ldots$$
$$+ \tan^{2n}\theta = 0,$$

the roots of the equation

$$x^n - \frac{2n(2n-1)}{2!}x^{n-1} + \ldots + 1 = 0$$

are the n quantities $\cot^2\dfrac{\pi}{4n}$, $\cot^2\dfrac{3\pi}{4n}$, etc., and their sum is

$$\frac{2n(2n-1)}{2!} = 2n^2 - n;$$

$$\therefore\ \Sigma \operatorname{cosec}^2\frac{(2r+1)\pi}{4n} = n + \Sigma\cot^2\frac{(2r+1)\pi}{4n} = 2n^2.$$

8. Let (X, Y) be the point from which the tangents are drawn, $lx + my = 1$ the chord required. Then, for some value of k, the conic

$$(ax^2 - by^2 - 1)(aX^2 - bY^2 - 1) - (axX - byY - 1)^2$$
$$+ k(lx + my - 1)(axX + byY - 1) = 0$$

must coincide with $ax^2 + by^2 = 1$.

Equating therefore the coefficients of xy, x and y to zero, we get

$$2abXY + k(aXm + bYl) = 0 \quad \text{.................(i)},$$
$$2aX + k(-l - aX) = 0,\ \text{ i.e. } kl = (2-k)aX \quad \text{......(ii)},$$
$$-2bY + k(-m - bY) = 0,\ \text{ i.e. } km = -(2+k)bY \quad \text{...(iii)}.$$

Substituting from (ii) and (iii) in (i), we find

$$k = 1, \quad l = aX, \quad m = -3bY,$$

so that the chord is $\quad axX - 3bYy = 1,$

which touches $ax^2 + 9by^2 = 1$ at the point $(X, -\frac{1}{3}Y)$.

9. The equation to the tangent at the point $(1, -1)$ is

$$3x - x + y - 3y - 4(x+1) - 4(y-1) - 4 = 0$$

or $$x + 3y + 2 = 0.$$

Hence since the chord of curvature also passes through the point $(1, -1)$, the equation to the circle of curvature must be of the form

$$3x^2 + 2xy + 3y^2 - 8x - 8y - 4 + \lambda(x + 3y + 2)(x + ky - 1 + k) = 0.$$

The conditions for a circle are

$$3 + \lambda = 3 + 3\lambda k,$$
$$2 + \lambda (3 + k) = 0,$$

whence $k = \frac{1}{3}$, $\lambda = -\frac{3}{5}$, and the equation is

$$5 (3x^2 + 2xy + 3y^2 - 8x - 8y - 4) - (x + 3y + 2)(3x + y - 2) = 0,$$

reducing to the form given.

10. Let A be the end of the rod on the ground, G its centre, D its point of contact with the hemisphere (centre O). At A draw AC making an angle λ with the vertical on the side of the hemisphere, and let OD, AC meet at C. Then C must be vertically above G. Hence we have

$$\frac{AD}{AC} = \sin (\lambda + \theta), \quad \frac{AC}{AG} = \frac{\cos\theta}{\sin\lambda},$$

where θ is the required angle;

$$\therefore \frac{AD}{AG} = \frac{\sin (\lambda + \theta)\cos\theta}{\sin\lambda}.$$

But $$\frac{AD}{AG} = \frac{r\cot\theta}{\frac{1}{2}nr} = \frac{2\cos\theta}{n\sin\theta},$$

$$\therefore n \sin\theta \sin (\lambda + \theta) = 2 \sin\lambda,$$

$$\text{i.e. } n (\cot\theta + \cot\lambda) = 2\operatorname{cosec}^2\theta = 2 + 2\cot^2\theta;$$

$$\therefore 2\cot^2\theta - n\cot\theta + 2 - n\cot\lambda = 0;$$

$$\therefore \cot\theta = \tfrac{1}{4}\left(n \pm \sqrt{n^2 + 8n\cot\lambda - 16}\right).$$

But we must have $2AG > AD > AG$, i.e. $n > \cot\theta > \dfrac{n}{2}$.

Hence the upper sign must be taken, and there is the further condition

$$\sqrt{n^2 + 8n\cot\lambda - 16} < 3n.$$

11. If a particle be projected from height x, then just before the first impact with the ground its horizontal velocity is

$$u = \sqrt{2g(h - x)},$$

and its vertical velocity $v = \sqrt{2gx}$.

Hence the distance described before it ceases to rebound is

$$\frac{uv}{g}+\frac{2u\,.\,ev}{g}+\frac{2u\,.\,e^2v}{g}+\dots \text{ ad inf.}$$

$$=\frac{uv}{g}+\frac{2u\,.\,ev}{g}\,.\,\frac{1}{1-e}=\frac{uv}{g}\,.\,\frac{1+e}{1-e}=2\sqrt{(h-x)\,x}\,.\,\frac{1+e}{1-e}.$$

But $(h-x)\,x=\frac{1}{4}h^2-(\frac{1}{2}h-x)^2$, and therefore the greatest value of $\sqrt{(h-x)\,x}$ is $\frac{1}{2}h$. Hence the greatest value of the above distance is

$$h\,.\,\frac{1+e}{1-e}.$$

12. Let $u=\dfrac{1}{x}$, $v=\log x$, so that

$$u_n=\frac{(-1)^n\,.\,n\,!}{x^{n+1}},\quad v_n=\frac{(-1)^{n-1}\,.\,(n-1)\,!}{x^n},$$

the suffixes denoting differentiations with regard to x.

By Leibnitz' Theorem

$$(uv)_n=u_nv+{}^nC_1u_{n-1}v_1+{}^nC_2u_{n-2}v_2+\dots,$$

the general term being

$${}^nC_r\,.\,u_{n-r}v_r=\frac{n\,!}{r\,!\,(n-r)\,!}\,.\,\frac{(-1)^{n-r}\,.\,(n-r)\,!}{x^{n-r+1}}\,.\,\frac{(-1)^{r-1}\,.\,(r-1)\,!}{x^r}$$

$$=\frac{(-1)^{n-1}\,.\,n\,!}{r}\,.\,\frac{1}{x^{n+1}};$$

$$\therefore\ (uv)_n=\frac{(-1)^n\,.\,n\,!}{x^{n+1}}\left(\log x-\sum_1^n\frac{1}{r}\right).$$

LXI.

1. Let A, B be the centres of the given circles, a and b their radii, SVU a direct common tangent, S being the centre of similitude. Suppose the circle to be described touches the given circles in P and Q and the line of centres in T. Then PQ passes through S: let it meet the circle B again in Q'. Then

$$\frac{SP}{SQ'}=\frac{a}{b};\quad\therefore\ \frac{SP\,.\,SQ}{SQ\,.\,SQ'}=\frac{a}{b},\ \text{i.e.}\ \frac{ST^2}{SV^2}=\frac{a}{b}=\frac{ab}{b^2}.$$

Hence we get the following construction :—Let the first circle cut the line of centres in D, D'. On AD' take $AR = b$, and describe a semi-circle on DR as diameter. Draw AL perpendicular to the line of centres to meet this semi-circle in L. Draw LM parallel to the line of centres, meeting the common tangent UV in M, and draw MT perpendicular to AB. Then T is the point of contact. To find the other points of contact, draw radii AE, BF perpendicular to AB, and join TE, TF. These lines will cut the circles in the required points.

2. Draw the tangent pq parallel to PQ and draw pK, qK perpendicular to TP, TQ respectively. Then, evidently, since p, q are the middle points of TP, TQ, therefore K is the middle point of TU. Let O be the middle point of TK. Then T, p, K, S, q all lie on a circle, centre O,

$$\therefore\ T\hat{K}Q = T\hat{O}q = 2T\hat{p}q = 2T\hat{P}Q,$$

and similarly $T\hat{K}P = T\hat{Q}P$. Hence, if QT be produced to Q',

$$P\hat{K}Q = 2P\hat{T}Q' = P\hat{S}Q\ ;$$

therefore K lies on the circle SPQ.

3. In the identity

$$1 + a_1 - a_1(1 + a_2) + a_1a_2(1 + a_3) - \dots$$
$$- a_1a_2 \dots a_{2n-1}(1 + a_{2n}) \equiv 1 - a_1a_2 \dots a_n,$$

put $\quad a_1 = \frac{2}{2n}, \quad a_2 = \frac{3}{2n-1}, \quad a_3 = \frac{4}{2n-2} \dots a_{2n} = \frac{2n+1}{1}.$

We then have

$$\frac{2n+2}{2n} - \frac{2}{2n} \cdot \frac{2n+2}{2n-1} + \frac{2.3}{2n(2n-1)} \cdot \frac{2n+2}{2n-2} - \dots$$
$$- \frac{2.3 \dots 2n}{2n(2n-1) \dots 2} \cdot \frac{2n+2}{1} = 1 - (2n+1),$$

and the series to be summed is

$$S = 1 - \frac{1}{2n} + \frac{1.2}{2n(2n-1)} - \frac{1.2.3}{2n(2n-1)(2n-2)} + \dots + 1,$$
$$\therefore\ (2n+2)(1 - S) = -2n,$$
$$\text{i.e. } S - 1 = \frac{n}{n+1} = 1 - \frac{1}{n+1}.$$

4. Let

$$n^2(n+1)^2 \equiv A + Bn + Cn(n-1)$$
$$+ Dn(n-1)(n-2) + n(n-1)(n-2)(n-3).$$

Putting $n = 0, 1, 2, 3$ in succession, we find

$$A = 0, \quad B = 4, \quad C = 14, \quad D = 8;$$

$$\therefore \frac{n^2(n+1)^2}{n!} = \frac{4}{(n-1)!} + \frac{14}{(n-2)!} + \frac{8}{(n-3)!} + \frac{1}{(n-4)!},$$

and therefore the sum of the series is

$$4e + 14e + 8e + e = 27e.$$

5. Multiplying the equation by $x - 1$, it becomes

$$x^{n+1} - 2x^n + 1 = 0.$$

Let S_r denote the sum of the rth powers of the roots of this equation. Then since the terms in $x, x^2, \ldots x^{n-1}$ are all wanting,

$$\therefore S_{-1} = S_{-2} = S_{-3} = \ldots = S_{-(n-1)} = 0.$$

Also dividing by x^{n-r}, substituting the roots and adding we get

$$S_{r+1} - 2S_r + S_{-(n-r)} = 0;$$
$$\therefore S_{r+1} = 2S_r \quad (r \not> n-1).$$

But $S_1 = 2$, therefore in general $S_m = 2^m \quad (m \not> n)$.

Hence for the original equation, the sum of the mth powers is $2^m - 1$.

6. Applying the formula $abc = 4R\Delta$ to the triangles BGC, etc., we have

$$a \,.\, BG \,.\, GC = \tfrac{4}{3} l_1 \,.\, \Delta, \text{ etc.};$$

$$\therefore \tfrac{4}{3}\Delta \,.\, \frac{l_2 l_3}{l_1} = \frac{bc}{a} \,.\, GA^2;$$

$$\therefore a^2 \,.\, \frac{l_2 l_3}{l_1} = 3R \,.\, GA^2.$$

Now $\quad GA^2 = \tfrac{4}{9} AD^2$, where AD is the median

$$= \tfrac{4}{9}\left(\tfrac{1}{2}b^2 + \tfrac{1}{2}c^2 - \tfrac{1}{4}a^2\right) = \tfrac{2}{9}b^2 + \tfrac{2}{9}c^2 - \tfrac{1}{9}a^2;$$

$$\therefore \Sigma GA^2 = \tfrac{1}{3} \,.\, \Sigma a^2,$$

$$\therefore \Sigma\left(a^2 \,.\, \frac{l_2 l_3}{l_1}\right) = R \,.\, \Sigma a^2.$$

7. We have in general

$$\tan 7\theta = \frac{{}^7C_1 \,.\, t - {}^7C_3 \,.\, t^3 + {}^7C_5 \,.\, t^5 - t^7}{1 - {}^7C_2 \,.\, t^2 + {}^7C_4 \,.\, t^4 - {}^7C_6 \,.\, t^6}, \text{ where } t = \tan\theta.$$

Hence in this case

$$\tan 7\theta = \frac{7 \,.\, \frac{1}{2} - 35 \,.\, \frac{1}{8} + 21 \,.\, \frac{1}{32} - \frac{1}{128}}{1 - 21 \,.\, \frac{1}{4} + 35 \,.\, \frac{1}{16} - 7 \,.\, \frac{1}{64}} = \tfrac{29}{278} \text{ on reduction.}$$

8. Let $lx + my = 1$ be any such straight line. The equations to the pairs of lines joining the origin to the points of intersection of this line with the conics are

$$ax^2 + by^2 = (lx + my)^2 \quad \text{and} \quad a'x^2 + b'y^2 = (lx + my)^2,$$

$$\text{i.e. } (a - l^2)\,x^2 - 2lmxy + (b - m^2) = 0,$$

and
$$(a' - l^2)\,x^2 - 2lmxy + (b' - m^2) = 0.$$

The condition that these form a harmonic pencil is

$$(a - l^2)\,(b' - m^2) + (a' - l^2)\,(b - m^2) - 2l^2m^2 = 0,$$

$$\text{i.e. } l^2\,(b + b') + m^2\,(a + a') = ab' + a'b,$$

which is also the condition that the line should touch the conic in question.

9. Taking the given point as origin, let the conic be given by the general equation, and let the two lines be $y = mx$, $y = m'x$. Then any conic through P, P', Q, Q' is of the form

$$S + \lambda\,(y - mx)\,(y - m'x) = 0,$$

and this is a rectangular hyperbola if

$$a + \lambda mm' + b + \lambda = 0, \quad \text{i.e. } \lambda = -\frac{a + b}{1 + mm'},$$

and the equation then takes the form

$$(1 + mm')\,S - (a + b)\,(y - mx)\,(y - m'x) = 0 \,\ldots\ldots\ldots\text{(i)}.$$

But since the lines are conjugate for the conic S,

$$\therefore \quad a + (m + m')\,h + bmm' = 0,$$

so that (i) takes the form

$$h\,(1 + mm')\,S - (a + b)\left[hy^2 + (a + bmm')\,xy + hmm'x^2\right] = 0,$$

shewing that it passes through the intersections of the conics

$$hS - (a + b)\,(hy + ax)\,y = 0,$$

and
$$hS - (a + b)\,(hx + by)\,x = 0.$$

10. If the pulley is at D, we have

$$BD^2 = AB^2 + AD^2 - 2AB \,.\, AD \cos BAD$$

$$= a^2 + a^2 \cos^2 \alpha - 2a^2 \cos \alpha \sin (\alpha - \phi), \text{ where } AB = a.$$

Taking moments about A, we have

$$W \,.\, \tfrac{2}{3} a \cos \alpha \cos \phi = P \,.\, a \sin ABD.$$

Also, from the triangle ABD, $\dfrac{\sin ABD}{\cos (\alpha - \phi)} = \dfrac{a \cos \alpha}{BD}$,

$$\therefore \; W \,.\, \tfrac{2}{3} \cos \alpha \cos \phi = P \,.\, \frac{a \cos \alpha \cos (\alpha - \phi)}{BD},$$

$$\text{i.e. } 2W \cos \phi = 3P \,.\, a \cos (\alpha - \phi)/BD.$$

11. Let T be the tension of the string after the blow on the mass m, f the acceleration of m'. Then

$$m\left(f - \frac{v^2}{l}\right) = mg - T, \text{ where } v^2 = 2gnl.$$

$$m'f = T - m'g.$$

Eliminating f, we find

$$T = \frac{2mm'}{m + m'} (n + 1) g,$$

i.e. the tension is increased in the ratio $n + 1 : 1$.

12. (i) Putting $x = \sec \theta$, the integral becomes

$$\int \frac{\sec \theta \tan \theta \, d\theta}{\sec \theta \tan \theta} = \theta = \sec^{-1} x.$$

(ii) The integral is

$$\int \frac{\sec^2 x \, dx}{(a + b \tan x)^2} = \int \frac{dt}{(a + bt)^2}, \text{ where } t = \tan \theta,$$

$$= -\frac{1}{b(a + bt)}.$$

(iii) Putting $\cos x = y$, the integral is

$$-\int (1 - y^2) \, y^4 \, dy = \tfrac{1}{7} y^7 - \tfrac{1}{5} y^5.$$

LXII.

1. Let AB, CD be the two given finite straight lines. Draw planes through A and B perpendicular to AB, and planes through C and D perpendicular to CD. These four planes will form a parallelopiped of unlimited length. Take a point P in the edge common to the faces containing A and C. Let PA, PC cut the next edges in Q and R, and let QD, RB cut the fourth edge in S and S'.

Then as P moves along the edge, P and Q cut two parallel straight lines proportionally, since PQ always passes through the fixed point A, and similarly for Q and R. Hence P and S cut two parallel straight lines proportionally, and therefore PS always passes through a fixed point, say K. Similarly PS' passes through another fixed point K'. Hence if we join KK' this will be the required position of PS.

2. Let TQ, TR be two of the perpendicular tangents, TQ', TR' the other tangents from T, which are also perpendicular. Let the normal at Q meet RR' in P'. Then (Q being on the outer conic) the pole of TQ for the inner conic lies on QP'. But it also lies on RR', therefore the pole is P'.

Hence if TQ meets RR' in P, it follows that PP' divides RR' harmonically, and since PQP' is a right angle, QR, QR' are equally inclined to QP', i.e. TQ bisects the angle RQR' externally.

Similarly TR bisects the angle QRQ' internally, and since QTR is a right angle it follows that QR' and $Q'R$ must be parallel, and both of them must therefore be parallel to CT, if C is the centre. Hence CT must bisect QR, say in V.

Hence $VT = VR$, and since CV is half the next side of the parallelogram, it follows that the perimeter is $4CT$. But T lies on a fixed circle, centre C, and therefore CT is constant for all positions of the rectangle.

3. Since
$$\Pi\,(y+z) = \Sigma x^2 y + 2xyz$$
$$= \Sigma a^3 + 2d^3,$$
and $x^2y^2z^2 \,.\, \Pi\,(y+z) = a^3b^3c^3$, we have at once
$$d^6\,(\Sigma a^3 + 2d^3) = a^3b^3c^3.$$

4. We have
$$\frac{1-ax}{(1-bx)(1-cx)} = \frac{1}{b-c}\left[(b-a)(1-bx)^{-1} - (c-a)(1-cx)^{-1}\right],$$
and the coefficient of x^n in the expansion is
$$\frac{1}{b-c}\left[b^n\,(b-a) - c^n\,(c-a)\right].$$
Hence, if the coefficients of x^n and x^{n-1} are both zero, we must have
$$b^n\,(b-a) = c^n\,(c-a)$$
and
$$b^{n-1}\,(b-a) = c^{n-1}\,(c-a),$$
leading, on division, to $b=c$ (since neither $b-a$ nor $c-a$ is zero), which is contrary to the hypothesis that b and c are unequal.

5. By the ordinary multiplication rule
$$\phi\,(\lambda)\,\phi\,(-\lambda) = \begin{vmatrix} A-\lambda^2, & H, & G \\ H, & B-\lambda^2, & F \\ G, & F, & C-\lambda^2 \end{vmatrix},$$
where $\quad A = a^2 + h^2 + g^2, \quad F = gh + f\,(b+c)$, etc.

Expanding we have
$$\lambda^6 - L\lambda^4 + M\lambda^2 - N = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$
where $\quad L = \Sigma a^2 + 2\Sigma f^2, \quad M = \Sigma\,(bc - f^2)^2 + 2\Sigma\,(gh - af)^2,$
$$N = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}^2$$
so that L, M, N are essentially positive, so that by Descartes' rule all the real roots of (i) in λ^2 are positive.

Also there is obviously no root of the form βi, and if there were a root of the form $\alpha+\beta i$, by writing $A-\alpha$, $B-\alpha$, $C-\alpha$ for A, B, C, we should get an equation of similar form with a root βi. Hence the roots of (i) in λ^2 are all real and positive and therefore the roots of $\phi(\lambda)=0$ are all real.

6. The radii of the circles are R, $\frac{1}{2}R$, and the square of the distance between their centres is $\frac{1}{4}R^2(1-8\cos A\cos B\cos C)$. The required condition is then

$$R^2+\tfrac{1}{4}R^2=\tfrac{1}{4}R^2(1-8\cos A\cos B\cos C),$$

whence $$2\cos A\cos B\cos C=-1.$$

But $$1-\Sigma\cos^2 A-2\cos A\cos B\cos C\equiv 0,$$

$$\therefore\ \Sigma\cos^2 A=2,\ \text{i.e. } \Sigma\sin^2 A=1.$$

7. (i) Here

$$4S=3\cos\theta+\cos 3\theta-\tfrac{1}{3}(3\cos 3\theta+\cos 3^2\theta)$$
$$+\frac{1}{3^2}(3\cos 3^2\theta+\cos 3^3\theta)-\dots\text{ ad inf.}$$

$=3\cos\theta$, since the series is convergent.

(ii) Here

$$2S=(\cos\theta-\cos 3\theta)+a(\cos\theta-\cos 5\theta)+a^2(\cos\theta-\cos 7\theta)+\dots$$
$$=\cos\theta(1+a+a^2+\dots)-(\cos 3\theta+a\cos 5\theta+a^2\cos 7\theta+\dots)$$
$$=\frac{\cos\theta}{1-a}-\frac{\cos 3\theta-a\cos\theta}{1-2a\cos 2\theta+a^2}=\frac{2\sin\theta\sin 2\theta}{(1-a)(1-2a\cos 2\theta+a^2)}.$$

8. Let P be the point θ, and let QR be $lx+my=1$. Then, for some value of λ,

$$x^2+y^2-b^2+2\lambda x(lx+my-1)=0\ \dots\dots\dots\dots\text{(i)}$$

must coincide with

$$[bx(\sin\theta-1)-ay\cos\theta+ab\cos\theta]$$
$$[bx(\sin\theta+1)-ay\cos\theta-ab\cos\theta]=0\ \dots\text{(ii)},$$

these latter being the lines joining P to the extremities of the minor axis.

The equation (ii) is

$$b^2x^2\cos^2\theta - a^2y^2\cos^2\theta + 2abxy\sin\theta\cos\theta$$
$$- 2ab^2x\cos\theta + a^2b^2\cos^2\theta = 0.$$

Comparing with (i) we get

$$\frac{1+2\lambda l}{b^2\cos\theta} = -\frac{1}{a^2\cos\theta} = \frac{\lambda m}{ab\sin\theta} = \frac{\lambda}{ab^2},$$

whence $\lambda = -\dfrac{b^2}{a\cos\theta}, \quad m = \dfrac{\sin\theta}{b}, \quad l = \dfrac{a^2+b^2}{2ab^2}\cos\theta,$

so that the equation to QR is

$$\frac{a^2+b^2}{2ab^2}\cos\theta \,.\, x + \frac{\sin\theta}{b} \,.\, y = 1,$$

the envelope of which is the given ellipse.

9. If the lines $Ax^2 + 2Hxy + By^2 = 0$ are common conjugate diameters of the conics $ax^2 + 2hxy + by^2 = 1$, $a'x^2 + 2h'xy + b'y^2 = 1$, we have the conditions

$$bA + aB - 2hH = 0, \quad b'A + a'B - 2h'H = 0,$$
$$\therefore\ (b-b')A + (a-a')B - 2(h-h')H = 0.$$

But this is the condition that the same lines should form a harmonic pencil with

$$(a-a')x^2 + 2(h-h')xy + (b-b')y^2 = 0,$$

and these are the common central chords of the two conics.

10. The surfaces of the bowl and lid are in the ratio $2:1$ and the centre of gravity of the bowl is at the middle point of the central radius OA. Hence if G be the centre of gravity of the whole, $OG = \frac{1}{3}a$, where a is the radius.

Taking the floor and wall as axes let L, M be the points of contact, LK, MK the directions of limiting friction. Then the equation to LK is

$$y = -\frac{1}{\mu}(x-a),$$

and to MK, $$y - a = \mu x.$$

At their intersection K, we have $x = \dfrac{1-\mu}{1+\mu^2} \cdot a$, and K is vertically above G, so that we must have $a - x < OG$,

$$\text{i.e. } a - \frac{1-\mu}{1+\mu^2} \cdot a < \tfrac{1}{3} a,$$

$$\text{i.e. } 2\mu^2 + 3\mu - 1 < 0,$$

or
$$(\mu + \tfrac{3}{4})^2 < \tfrac{17}{16}, \quad \text{i.e. } \mu < \frac{\sqrt{17} - 3}{4}.$$

11. Let ω be the angular velocity of each mass m, f the acceleration of nm after release. This mass will begin to move along the bisector of the angle α. Hence if T' be the tension of either part of the thread, we have for either mass m,

$$T' = m\left(a\omega^2 - f \cos\frac{\alpha}{2}\right),$$

and for nm,
$$2T' \cos\frac{\alpha}{2} = nmf;$$

$$\therefore \; T'\left(1 + \frac{1}{n} \cdot 2\cos^2\frac{\alpha}{2}\right) = ma\omega^2 = T, \text{ the original tension,}$$

$$\therefore \; \frac{T'}{T} = \frac{n}{n + 2\cos^2\dfrac{\alpha}{2}} = \frac{n}{n + 1 + \cos\alpha}.$$

12. The coefficients of the highest powers of x and y are $y^3 - a^3$ and x. This gives the real asymptotes

$$y = a \text{ and } x = 0,$$

and shews that the other two are imaginary.

At the point $(-a, 0)$, $\dfrac{dx}{dy}$ vanishes, and the tangent is parallel to the axis of y. Also since $y^3 = \dfrac{a^3(a+x)}{x}$, therefore, as x passes through the value $-a$, the value of y changes from negative to positive. Hence at the point $(-a, 0)$ the curve must cross the tangent, i.e. $(-a, 0)$ is a point of inflexion. There is also another point of inflexion at

$$\left(-\tfrac{2}{3}a, \; -\frac{1}{\sqrt[3]{2}}a\right).$$

LXIII.

1. Let AD, BC meet in P, and AC, BD in Q. Then the triangle OPQ is self-polar for the circle; therefore PQ is the polar of O and is therefore fixed. Further the pencil $P(CA, OQ)$ is harmonic, therefore since $AO = OB$, PQ is parallel to AB.

2. Let S be the given focus, P the point of contact, S' the second focus of the ellipse, and let PS' meet the ellipse in Q. Then since SP, $S'P$ are equally inclined to the tangent, therefore PS' is parallel to the axis of the parabola. Let it meet the directrix in L. Then $LS' = SP + PS' = 2a$. Now draw a line YZ parallel to the directrix and at a distance $4a$ from it, and let LS' meet this line in L'. Then $S'L' = 2a$;

$$\therefore\ SQ = 2a - S'Q = QL'.$$

Further the tangent to the ellipse at Q is equally inclined to SQ, QL'. Hence if a parabola be drawn with S as focus, and YZ as directrix, it will touch the ellipse at Q.

3. The nth term of s_1 is $na + \dfrac{n(n-1)}{2!} \cdot b$;

$\therefore$ the nth term of s_2 is $\dfrac{n(n+1)}{2!} \cdot a + \dfrac{(n+1)n(n-1)}{3!} \cdot b$,

and so on. Generally the nth term of s_p is

$$\frac{n(n+1)\dots(n+p-1)}{p!} \cdot a + \frac{(n-1)n(n+1)\dots(n+p-1)}{(p+1)!} \cdot b$$

$$= \frac{(n+p-1)!}{p!\,(n-1)!} \cdot a + \frac{(n+p-1)!}{(p+1)!\,(n-2)!} \cdot b$$

$$= \frac{(n+p-1)!}{(p+1)!\,(n-1)!} \{(p+1)a + (n-1)b\}.$$

4. The total number of possible arrangements is 8 ! If no two odd integers are together, the arrangement must be either

(i) odd and even integers alternately,

or (ii) either one even integer or two between each successive pairs of odd integers, with odd integers occupying the first and last places.

In case (i) the number of arrangements is $2 \,.\, (4\,!)^2$,

.... (ii) .... $4\,! \times 3\,! \times (4 \times 3)$.

Hence the required chance is

$$\frac{2\,.\,(4\,!)^2 + (4\,! \times 3\,! \times 12)}{8\,!} = \frac{48 + 72}{5\,.\,6\,.\,7\,.\,8} = \tfrac{1}{14}.$$

5. If the equation has three roots equal to a, it must be equivalent to

$$(x^3 - 3ax^2 + 3a^2x - a^3)(x^2 + 3ax + a^2) = 0,$$

since the terms in x^4 and x are missing. This reduces to

$$x^5 - 5a^2x^3 + 5a^3x^2 - a^5 = 0\,;$$

$$\therefore\ 5a^2 = -\frac{b}{a}, \quad 5a^3 = \frac{c}{a}, \quad a^5 = -\frac{d}{a},$$

whence
$$-\frac{c}{b} = \frac{b^2}{5ac} = \frac{5bd}{c^2} = a.$$

The given equation satisfies these conditions, a being $\frac{3}{2}$, and is equivalent to

$$(2x - 3)^3 (4x^2 + 18x + 9) = 0.$$

6. Suppose the result true for any n angles, so that we have

$$\Sigma \cos(\pm a_1 \pm a_2 \pm \dots \pm a_{n-1} \pm \theta) = 2^n \cos a_1 \dots \cos a_{n-1} \cos\theta.$$

In this result put $\theta = a_n + a_{n+1}$, and then $\theta = a_n - a_{n+1}$, and add the results. The left-hand side is clearly

$$\Sigma \cos(\pm a_1 \pm a_2 \pm \dots \pm a_n \pm a_{n+1}),$$

and the right-hand side is

$$2^n \cos a_1 \dots \cos a_{n-1} [\cos(a_n + a_{n+1}) + \cos(a_n - a_{n+1})]$$
$$= 2^n \cos a_1 \dots \cos a_{n-1} (2 \cos a_n \cos a_{n+1})$$
$$= 2^{n+1} \cos a_1 \dots \cos a_{n+1}.$$

Hence, if the result is true for n angles, it is true for $n+1$ angles. But

$$\Sigma \cos(\pm a_1 \pm a_2) = 4 \cos a_1 \cos a_2.$$

Hence the result is true for two angles, and therefore by induction is true generally.

7. If $(1+x)\tan\alpha = (1-x)\tan\beta$, we have

$$\frac{e^{i\alpha}-e^{-i\alpha}}{e^{i\alpha}+e^{-i\alpha}} = \frac{(1-x)(e^{i\beta}-e^{-i\beta})}{(1+x)(e^{i\beta}+e^{-i\beta})};$$

$$\therefore\ e^{2i\alpha} = \frac{e^{i\beta}+xe^{-i\beta}}{xe^{i\beta}+e^{-i\beta}} = e^{2i\beta}\cdot\frac{1+xe^{-2i\beta}}{1+xe^{2i\beta}}.$$

Hence $2i\alpha = 2n\pi i + 2i\beta + \log(1+xe^{-2i\beta}) - \log(1+xe^{2i\beta})$;

$$\therefore\ 2n\pi i + 2i\beta - 2i\alpha = \log(1+xe^{2i\beta}) - \log(1+xe^{-2i\beta})$$

$$= \sum_1^\infty (-1)^{n-1}\cdot\frac{x^n}{n}\cdot(e^{2ni\beta}-e^{-2ni\beta})$$

$$= \sum_1^\infty (-1)^{n-1}\cdot\frac{x^n}{n}\cdot(2i\sin 2n\beta).$$

8. If θ be the angle between the axes, the conics may be taken in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{(x\cos\theta - y\sin\theta)^2}{a'^2}+\frac{(x\sin\theta+y\cos\theta)^2}{b'^2}=1,$$

and the central common chords are

$$\left(\frac{1}{a^2}-\frac{\cos^2\theta}{a'^2}-\frac{\sin^2\theta}{b'^2}\right)x^2+\left(\frac{1}{b^2}-\frac{\sin^2\theta}{a'^2}-\frac{\cos^2\theta}{b'^2}\right)y^2$$

$$+2xy\sin\theta\cos\theta\left(\frac{1}{a'^2}-\frac{1}{b'^2}\right)=0,$$

and these coincide if

$$\left(\frac{1}{a^2}-\frac{\cos^2\theta}{a'^2}-\frac{\sin^2\theta}{b'^2}\right)\left(\frac{1}{b^2}-\frac{\sin^2\theta}{a'^2}-\frac{\cos^2\theta}{b'^2}\right)$$

$$=\sin^2\theta\cos^2\theta\left(\frac{1}{a'^2}-\frac{1}{b'^2}\right)^2,$$

i.e. $$\frac{1}{a^2b^2}-\frac{1}{a^2}\left(\frac{\sin^2\theta}{a'^2}+\frac{\cos^2\theta}{b'^2}\right)-\frac{1}{b^2}\left(\frac{\cos^2\theta}{a'^2}+\frac{\sin^2\theta}{b'^2}\right)=-\frac{1}{a'^2b'^2},$$

or $$\left(\frac{1}{a^2b^2}+\frac{1}{a'^2b'^2}\right)(\cos^2\theta+\sin^2\theta)=\left(\frac{1}{a^2b'^2}+\frac{1}{a'^2b^2}\right)\cos^2\theta$$

$$+\left(\frac{1}{a^2a'^2}+\frac{1}{b^2b'^2}\right)\sin^2\theta,$$

whence $$\tan^2\theta = -\frac{\left(\frac{1}{a^2}-\frac{1}{a'^2}\right)\left(\frac{1}{b^2}-\frac{1}{b'^2}\right)}{\left(\frac{1}{a^2}-\frac{1}{b'^2}\right)\left(\frac{1}{b^2}-\frac{1}{a'^2}\right)} = \frac{(a'^2-a^2)(b'^2-b^2)}{(a'^2-b^2)(a^2-b'^2)}.$$

9. The normal at α is

$$\frac{2a}{r}\cdot\frac{\sin\alpha}{1+\cos\alpha} = \sin\theta + \sin(\theta-\alpha) = 2\sin\left(\theta-\frac{\alpha}{2}\right)\cos\frac{\alpha}{2}.$$

Hence the vectorial angles of the points, the normals at which pass through (r, θ), are given by the equation

$$\frac{a}{r}\tan\frac{\phi}{2} = \sin\left(\theta-\frac{\phi}{2}\right)\cos\frac{\phi}{2}.$$

Putting $\tan\frac{\phi}{2}=t$, this is

$$\frac{a}{r}\,.\,t(1+t^2) = \sin\theta - t\cos\theta,$$

an equation in which there is no term in t^2. Hence the sum of its roots is zero.

10. Let A and B be the points of contact. Draw AN making an angle λ with the normal to the surface away from the vertex, and BN making an angle $90°-\lambda$ with AB towards A. Then N is vertically below G the centre of the rod. Then from the figure

$$\frac{AB}{c} = \frac{\cos\alpha}{\cos(\alpha-\beta)},\quad \frac{AN}{AB} = \frac{\cos\lambda}{\cos(\beta-\alpha-2\lambda)},\quad \frac{AG}{AN} = \frac{\cos(\lambda+\alpha)}{\cos\beta}.$$

Multiplying these ratios the result follows.

11. Let u, u' be the velocities before impact, v and v' afterwards. Then

$$v+v' = u+u' \;\ldots\ldots\text{(i)},\qquad v-v' = -e(u-u') \;\ldots\ldots\text{(ii)},$$

and by the question $\quad v^2-u^2 = e(u'^2-v'^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$

From (i) and (iii), we get

$$v+u = e(u'+v') \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

Solving (i) and (ii) for v, v' and substituting in (iv), we get

$$(1-e)\,u + (1+e)\,u' + 2u = 2eu' + e\,[(1+e)\,u + (1-e)\,u'],$$

$$\text{i.e. } u\,(3-2e-e^2) = u'\,(-1+2e-e^2),$$

$$\text{i.e. } u\,(3+e) = -u'\,(1-e).$$

12. The mean value for equidistant ordinates is

$$\frac{1}{2a}\int_{-a}^{a} y\,dx = \frac{b}{2a}\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,dx = \frac{b}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$

$$= \frac{b}{4}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{\pi b}{4}.$$

LXIV.

1. Since the ratio compounded of $\dfrac{BP}{PC}, \dfrac{CQ}{QA}, \dfrac{AR}{RB}$ is unity, therefore that compounded of

$$\frac{BP}{PC},\ \frac{CQ}{QA},\ \frac{AR'}{R'B}$$

is negative unity; therefore the points P, Q, R' are collinear. Hence the middle points in question are those of the diagonals of the complete quadrilateral $ABPQ$ and they are therefore collinear.

2. Let the tangents at P and Q meet in Z, those at P' and Q in Z'. Then since the angles at Z, Z' are right angles, and the parabolas, being confocal, intersect at right angles at Q; therefore QZ, $P'Z'$ are parallel, as also are QZ', PZ. Hence

$$SP : SQ = SZ : SZ',$$

$$SP' : SQ = SZ' : SZ, \qquad \therefore\ SP \,.\, SP' = SQ^2.$$

3. Since $a_r = a_0 + rd$, the sum required is

$$\Sigma c_r\,(a_0 + rd)^2 = a_0{}^2 \,.\, \Sigma c_r + 2a_0 d \,.\, \Sigma r c_r + d^2 \,.\, \Sigma r^2 c_r.$$

Now $\Sigma c_r = 2^n$, $\Sigma r c_r = n \,.\, 2^{n-1}$, $\Sigma r\,(r-1)\,c_r = n\,(n-1) \,.\, 2^{n-2}$.

Hence the sum is

$$a_0^2 \cdot 2^n + 2a_0 d \cdot n \cdot 2^{n-1} + d^2\left[n(n-1)\,2^{n-2} + n \cdot 2^{n-1}\right]$$
$$= 2^{n-2}\left[4a_0^2 + 4a_0 d \cdot n + (n^2+n)\,d^2\right]$$
$$= 2^{n-2}\left[(2a_0 + nd)^2 + nd^2\right]$$
$$= 2^{n-2}\left[(a_0 + a_n)^2 + nd^2\right].$$

4. Suppose $x^2 + ax + b = (x-\alpha)(x-\beta)$. Then

$$\frac{1}{x^2+ax+b} = \frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right)$$
$$= \frac{1}{\alpha-\beta}\left\{-\frac{1}{\alpha}\left(1-\frac{x}{\alpha}\right)^{-1} + \frac{1}{\beta}\left(1-\frac{x}{\beta}\right)^{-1}\right\}.$$

Hence if the expansion be in ascending powers of x,

$$p_{n-1} = \frac{1}{\alpha-\beta}\left(-\frac{1}{\alpha^n} + \frac{1}{\beta^n}\right).$$

In the other case putting $x = \frac{1}{y}$, we have

$$\frac{y^2}{by^2+ay+1} = \frac{y^2}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha y} - \frac{\beta}{1-\beta y}\right);$$
$$\therefore\ q_{n+1} = \frac{\alpha^n - \beta^n}{\alpha-\beta} = \alpha^n\beta^n p_{n-1} = b^n p^{n-1}.$$

5. Calling the given expressions p and q, we have

$$p + q = \Sigma\alpha^3(\beta^2+\gamma^2) = \Sigma\alpha^3(-2a-\alpha^2) = -2as_3 - s_5,$$

where $s_n = \Sigma\alpha^n$. But from the equation we easily get

$$s_3 + 3b = 0, \qquad s_5 + as_3 + bs_2 = 0,$$

whence $$s_5 = 5ab;\quad \therefore\ p + q = ab.$$

Further $$p - q = -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\,.\,\Sigma\beta\gamma;$$

$$\therefore\ (p-q)^2 = -(27b^2 + 4a^3)\,.\,a^2;$$

$$\therefore\ 4pq = (p+q)^2 - (p-q)^2$$
$$= a^2b^2 + a^2(27b^2+4a^3) = 4(a^5 + 7a^2b^2).$$

6. The quantities l, m, n are proportional to the areas of the triangles OIA, etc.

Now $$\triangle OIA = \tfrac{1}{2} OA \,.\, AI \sin OAI$$
$$= \tfrac{1}{2} R \,.\, r \operatorname{cosec} \frac{A}{2} \sin \frac{B-C}{2}$$
$$= \tfrac{1}{2} R \,.\, r \,.\, \frac{\tan \dfrac{B}{2} - \tan \dfrac{C}{2}}{\tan \dfrac{B}{2} + \tan \dfrac{C}{2}} \,.\, \cot \frac{A}{2}.$$

Hence since $r_1 = s \tan \dfrac{A}{2}$ etc., the result follows.

7. We have
$$[1-(\alpha+\beta)x^2][1-(\alpha-\beta)x^2]^{-1}$$
$$= [1-(\alpha+\beta)x^2][1+(\alpha-\beta)x^2+(\alpha-\beta)^2x^4+\dots]$$
$$= 1 - 2\beta \,.\, x^2 - 2\beta(\alpha-\beta)x^4 - 2\beta(\alpha-\beta)^2x^6 - \dots \qquad \dots\text{(i)}.$$

Also $$\cos^{\frac{1}{n}} x = (1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \dots)^{\frac{1}{n}}$$
$$= 1 - \frac{1}{n}(\tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 + \dots) - \frac{n-1}{2n^2}(\tfrac{1}{2}x^2 - \dots)^2$$
$$- \frac{(n-1)(2n-1)}{6n^3}(\tfrac{1}{2}x^2 - \dots)^3 - \dots \qquad \dots\dots\text{(ii)}.$$

We want to make these expansions coincide as far as x^4. We must therefore have
$$-2\beta = -\frac{1}{2n}, \quad -2\beta(\alpha-\beta) = \frac{1}{24n} - \frac{n-1}{8n^2},$$
leading to $$\beta = \frac{1}{4n}, \quad \alpha = \frac{1}{6}.$$

The coefficient of x^6 in (i) is $-\dfrac{(2n-3)^2}{288n^3}$, and in (ii)
$$-\frac{1}{720n} + \frac{n-1}{48n^2} - \frac{(n-1)(2n-1)}{48n^3} = \frac{-16n^2+30n-15}{720n^3}.$$
Hence the error is
$$\frac{-5(2n-3)^2 + 32n^2 - 60n + 30}{1440n^3} = \frac{4n^2-5}{480n^3}.$$

8. Let the common tangent be $lx + my = 1$. Then

$$a^2l^2 + b^2m^2 = 1, \quad a'^2l^2 + b'^2m^2 = 1,$$

whence
$$l^2 = \frac{b'^2 - b^2}{a^2b'^2 - a'^2b^2}, \quad m^2 = \frac{a^2 - a'^2}{a^2b'^2 - a'^2b^2}.$$

The points of contact are $(a^2l,\ b^2m)$ and $(a'^2l,\ b'^2m)$ and the square of the distance between them is

$$(a^2 - a'^2)^2\, l^2 + (b^2 - b'^2)^2\, m^2 = \frac{(a^2 - a'^2)(b'^2 - b^2)(a^2 - a'^2 + b'^2 - b^2)}{(a^2b'^2 - a'^2b^2)},$$

while the square of the intercept on the axes is

$$\frac{1}{l^2} + \frac{1}{m^2} = \frac{(a^2b'^2 - a'^2b^2)(a^2 - a'^2 + b'^2 - b^2)}{(a^2 - a'^2)(b'^2 - b^2)},$$

and these are in the required ratio.

9. The normal at $(x'y')$ is $xx' - yy' = x'^2 - y'^2$. Hence the feet of normals through $(X,\ Y)$ lie on the curve

$$x^2 - y^2 - xX + yY = 0 \text{ .... (i).}$$

If, then, the normals at the ends of $px + qy = 1$, $p'x + q'y = 1$ intersect at $(X,\ Y)$, the conic (i) must, for some value of λ, coincide with

$$xy - c^2 + \lambda(px + qy - 1)(p'x + q'y - 1) = 0.$$

Identifying the equations, we get

$$pp' = -qq' = \frac{p + p'}{X} = -\frac{q + q'}{Y},$$

$$1 + \lambda(pq' + p'q) = 0, \quad \lambda = c^2,$$

whence
$$\frac{p'}{q} = \frac{q'}{-p} = -\frac{1}{c^2} \cdot \frac{1}{q^2 - p^2};$$

$$\therefore\ X = \frac{1}{p} + \frac{1}{p'} = \frac{1}{p} - c^2 \cdot \frac{q^2 - p^2}{q},$$

$$Y = \frac{1}{q} + \frac{1}{q'} = \frac{1}{q} + c^2 \cdot \frac{q^2 - p^2}{p}.$$

10. The friction at B acts at right angles to AB. Hence taking moments about A for AB, we have

$$\mu R \,.\, a = W \sin \alpha \,.\, a \sin \theta + T \,.\, a \sin 2\theta,$$

where T is the tension in BC, and θ the angle either rod makes with the line of greatest slope.

Also for C, resolving along CA,

$$2T \cos \theta + \mu R = W \sin \alpha,$$

and resolving perpendicular to the plane $R = W \cos \alpha$.

From these

$$\mu \cos \alpha = \sin \alpha \sin \theta + \sin \theta (\sin \alpha - \mu \cos \alpha),$$

$$\text{i.e. } \sin \theta = \frac{\mu \cos \alpha}{2 \sin \alpha - \mu \cos \alpha} = \frac{\mu}{2 \tan \alpha - \mu}.$$

11. Let u be the velocity at the vertex, v that at any subsequent point P. Let PN be the ordinate at P, and let PG, the normal at P, make an angle θ with the axis. Then

$$NG = 2a, \quad PN = 2a \tan \theta, \quad AN = a \tan^2 \theta,$$

$$\rho = \frac{2}{\sqrt{a}} \,.\, SP^{\frac{3}{2}} = \frac{2}{\sqrt{a}} (a + AN)^{\frac{3}{2}} = 2a \sec^3 \theta.$$

Now $$\frac{v^2}{\rho} = g \cos \theta - \frac{R}{m}, \text{ and } v^2 - u^2 = 2g \,.\, AN;$$

$$\therefore \; R = mg \cos \theta - m \,.\, \frac{u^2 + 2ga \tan^2 \theta}{\rho}$$

$$= mg \left(\cos \theta - \frac{\tan^2 \theta}{\sec^3 \theta} \right) - \frac{mu^2}{\rho}$$

$$= mg \cos^3 \theta - \frac{mu^2}{\rho} = \frac{m (2ga - u^2)}{\rho},$$

i.e. R is proportional to $\frac{1}{\rho}$.

12. Let PQ make an angle 2θ with either CP or CQ. Then $r = a \cos 2\theta \,.\, \tan \theta$, and this is a maximum or minimum if

$$-2 \sin 2\theta \,.\, \tan \theta + \cos 2\theta \,.\, \sec^2 \theta = 0,$$

$$\text{i.e. } \cos 2\theta = \sin^2 2\theta = 1 - \cos^2 2\theta,$$

whence $\cos 2\theta = \dfrac{-1 \pm \sqrt{5}}{2}$, and the negative value is impossible.

Further $\qquad \tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \dfrac{3-\sqrt{5}}{1+\sqrt{5}} = \sqrt{5} - 2$;

$$\therefore\ r = a \,.\, \frac{\sqrt{5}-1}{2} \,.\, \sqrt{\sqrt{5}-2} = \frac{a}{2} \,.\, \sqrt{10\sqrt{5}-22}.$$

LXV.

1. Let O be the circumcentre, and draw OH perpendicular to OA. With centre A, and radius equal to a side of the given square, draw a circle cutting OH in H. Bisect AH in L and draw LN perpendicular to AH to meet AO produced in N. Draw ND perpendicular to AN cutting BC in D. Then D is the point required. For $BD \,.\, DC = R^2 - OD^2$;

$$\therefore\ BD \,.\, DC + AD^2 = OA^2 - OD^2 + AD^2 = OA^2 + AN^2 - ON^2$$
$$= OA^2 + HN^2 - ON^2 = OA^2 + OH^2 = AH^2.$$

2. Let the tangents intersect in T, and let P be one of the points of contact. Then, if S' be the second focus, TS' is a fixed straight line. Draw SY perpendicular to PT and let $S'P$, TS' meet SY in H and L respectively. Let PG be the normal at P, meeting TL in M, and let TG meet SY in K. Then we have

$$YK : PG = YT : PT = YL : PM,$$

$$HS : PG = HS' : PS' = HL : PM; \quad \therefore\ YK : HS = YL : HL.$$

But Y, H, L are fixed points, since $SY = YH$; therefore K is a fixed point, and the locus of G is a fixed line through T.

3. Considering the left side, the total number of possibilities is 3^n, for each x may be 0, 1 or -1. The number of cases in which no x is zero is 2^n; therefore the number of cases in which the left side vanishes is $3^n - 2^n$. Similarly for the right side. Hence the number of solutions in which each side of the equation is zero is

$$(3^n - 2^n)^2 = 3^{2n} - 3^n \,.\, 2^{n+1} + 2^{2n}.$$

The number of cases in which the left side is unity is $\frac{1}{2} \cdot 2^n = 2^{n-1}$. Hence the number of solutions in which each side is unity is 2^{2n-2}, and this is also the number in which each side is -1. Hence the total number of solutions is

$$3^{2n} - 3^n \cdot 2^{n+1} + 2^{2n} + 2 \cdot 2^{2n-2} = 3\,(3^{2n-1} - 3^{n-1} \cdot 2^{n+1} + 2^{2n-1}).$$

4. The numerators form a recurring series with scale $1 - 3x^2 - 2x^3$ and generating function

$$\frac{1+2x}{1-3x^2-2x^3} = \frac{1+2x}{(1+x)^2(1-2x)}$$

$$= \frac{8}{9} \cdot \frac{1}{1-2x} - \frac{1}{3} \cdot \frac{1}{(1+x)^2} + \frac{4}{9} \cdot \frac{1}{1+x}.$$

Hence the $(n+1)$th numerator is

$$\frac{8}{9} \cdot 2^n - \frac{1}{3}(-1)^n (n+1) + \frac{4}{9}(-1)^n;$$

therefore the $(n+1)$th term of the series is

$$\frac{8}{9} \cdot \frac{2^n}{n!} - (-1)^n \cdot \frac{1}{3} \cdot \frac{1}{(n-1)!} + (-1)^n \frac{1}{9} \cdot \frac{1}{n!},$$

and therefore the sum is

$$\frac{8}{9}e^2 + \frac{1}{3}e^{-1} + \frac{1}{9}e^{-1} = \frac{4}{9}(2e^2 + e^{-1}).$$

5. We have

$$(m-n)(x^4 + 6ax^2 + 4bx + c) \equiv m(x-n)^4 - n(x-m)^4,$$

whence equating coefficients, we have

$$a = -mn, \quad b = mn(m+n), \quad c = -mn(m^2 + mn + n^2);$$

$$\therefore\ m + n = -\frac{b}{a}, \quad c = a\left(\frac{b^2}{a^2} + a\right),$$

$$\text{i.e. } ac = a^3 + b^2,$$

and m, n are the roots of

$$t^2 + \frac{b}{a} \cdot t - a = 0, \quad \text{i.e. } at^2 + bt - a^2 = 0.$$

In the example given the condition is satisfied, and m, n are the roots of $t^2+t-2=0$, i.e. $m=1$, $n=-2$, so that the equation is

$$(x+2)^4+2(x-1)^4=0,$$

$$\text{i.e. } x+2=\alpha \,.\, 2^{\frac{1}{4}}(x-1),$$

where α is one of the fourth roots of -1.

6. Let U be the nine-point centre, P the orthocentre, and I the incentre. Then the triangle is obtuse-angled if

$$IU < UP,$$

$$\text{i.e. } \tfrac{1}{2}R - r < \tfrac{1}{2}R\sqrt{1-8\cos A\cos B\cos C},$$

$$\text{i.e. } 1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} < \sqrt{1-8\cos A\cos B\cos C} \ \ldots\text{(i)}.$$

Now
$$\begin{aligned} 2\cos A\cos B &= \cos(A-B)+\cos(A+B) \\ &< 1-\cos C \\ &< 2\sin^2\frac{C}{2}, \end{aligned}$$

and two similar inequalities. Multiplying these, we get

$$\cos A\cos B\cos C < \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2};$$

$$\therefore\ 1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} < 1-8\cos A\cos B\cos C \ \ldots\text{(ii)}.$$

But since $\sin\frac{A}{2}$, etc. are all positive,

$$\therefore\ 1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} < 1;$$

$$\therefore\ \left(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2 < 1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \ \ldots\text{(iii)},$$

and the inequality (i) follows from (ii) and (iii).

7. We have

$$\sin\theta\cos^2\theta = \tfrac{1}{4}\,.\,2\sin 2\theta\cos\theta = \tfrac{1}{4}(\sin 3\theta+\sin\theta).$$

Hence the general term in the expansion in powers of θ is

$$\tfrac{1}{4}(-1)^n \cdot \frac{3^{2n+1}+1}{(2n+1)!} \cdot \theta^{2n+1},$$

and for $n=0$, 1, 2 this gives the first three terms as stated.

8. If BC makes an angle θ with the axis of x, the coordinates of C are

$$x = 2a \sin \alpha \sin (\alpha + \theta), \quad y = a \sin \theta,$$

$$\text{i.e. } \sin \alpha \cos \theta = \frac{x}{2a \sin \alpha} - \frac{y}{a} \cos \alpha,$$

$$\sin \alpha \sin \theta = \frac{y}{a} \sin \alpha \,;$$

therefore the locus is

$$\frac{x^2}{4a^2 \sin^2 \alpha} - \frac{xy}{a^2} \cot \alpha + \frac{y^2}{a^2} = \sin^2 \alpha,$$

$$\text{i.e. } x^2 - 2xy \sin 2\alpha + 4y^2 \sin^2 \alpha = 4a^2 \sin^4 \alpha,$$

and the area of the locus is

$$\pi \cdot \frac{4a^2 \sin^4 \alpha}{\sqrt{4 \sin^2 \alpha - \sin^2 2\alpha}} = \pi \cdot 2a^2 \sin^2 \alpha.$$

9. The conic must be of the form

$$(x - my + am^2)^2 + \lambda (y^2 - 4ax) = 0,$$

and, calculating the tangential coefficients, the equation to the director circle is

$$x^2 + y^2 - 2 (2a\lambda + am^2) x - 4amy + a^2m^4 + 4a^2m^2 - 4a^2\lambda = 0.$$

The centre of this circle is the centre of the conic, its coordinates being

$$x = 2a\lambda + am^2, \quad y = 2am,$$

and, by the data,

$$x^2 + y^2 - c^2 = a^2m^4 + 4a^2m^2 - 4a^2\lambda,$$

leading to $$4a^2 (\lambda^2 + \lambda m^2 + \lambda) = c^2 \quad \text{..................(i).}$$

Now $$y^2 + 4ax = 8a^2m^2 + 8a^2\lambda, \quad y^2 - 4ax = - 8a^2\lambda \quad \text{......(ii).}$$

From (i) and (ii) we get the locus as given.

10. Let X and Y be the horizontal and vertical components of the reaction at C, and $2a$ the length of either rod. For AC,

the components are those towards B. Hence taking moments about A for AC, we have

$$X \,.\, 2a\sin\alpha + Y \,.\, 2a\cos\alpha = W \,.\, a\cos\alpha,$$

$$\text{i.e. } X + Y\cot\alpha = \tfrac{1}{2}W\cot\alpha \quad \text{...............(i).}$$

Similarly, taking moments about B for BC, and using the equal and opposite components, we get

$$X \,.\, 2a\sin\alpha = Y \,.\, 2a\cos\alpha + W' \,.\, a\cos\alpha,$$

$$\text{i.e. } X - Y\cot\alpha = \tfrac{1}{2}W'\cot\alpha \quad \text{...............(ii).}$$

The equations (i) and (ii) give the required values for X and Y.

11. Let the velocities of the shell parallel to any rectangular axes be u, v: those of the pieces parallel to the same axes u_1, v_1; u_2, v_2; u_3, v_3. Then, since there is no change of momentum,

$$\Sigma m_1u_1 = (\Sigma m_1)\,u, \quad \Sigma m_1v_1 = (\Sigma m_1)\,v,$$

and twice the change in kinetic energy is

$$\Sigma m_1u_1^2 + \Sigma m_1v_1^2 - (\Sigma m_1)(u^2+v^2)$$

$$= \Sigma m_1u_1^2 + \Sigma m_1v_1^2 - \frac{(\Sigma m_1u_1)^2 + (\Sigma m_1v_1)^2}{\Sigma m_1}.$$

Simplifying, the square terms disappear, and the coefficient of m_2m_3 is

$$\frac{1}{\Sigma m_1}(u_2^2 + u_3^2 + v_2^2 + v_3^2 - 2u_2u_3 - 2v_2v_3)$$

$$= \frac{1}{\Sigma m_1}\{(u_2-u_3)^2 + (v_2-v_3)^2\} = \frac{1}{\Sigma m_1} \,.\, v_{23}^2,$$

and so for the others.

12. The point of contact, being the point of intersection with the consecutive tangent, is given by

$$x \,.\, \psi(a) + y \,.\, \phi(a) = 1, \quad x \,.\, \psi'(a) + y \,.\, \phi'(a) = 0,$$

whence $$\frac{x}{\phi'(a)} = \frac{y}{-\psi'(a)} = \frac{1}{\psi(a)\,\phi'(a) - \psi'(a)\,\phi(a)},$$

and the normal is the line through this point perpendicular to the tangent, so that its equation is

$$x \,.\, \phi(a) - y \,.\, \psi(a) = \frac{\phi(a)\,\phi'(a) + \psi(a)\,\psi'(a)}{\psi(a)\,\phi'(a) - \psi'(a)\,\phi(a)}.$$

LXVI.

1. We have

$$AD^2 - BD^2 = AP^2 - BP^2 = AC^2 - BC^2,$$

since P is the orthocentre of ABC;

$$\therefore\ AD^2 + BC^2 = AC^2 + BD^2 = AB^2 + CD^2 \text{ similarly } \ldots\text{(i)}.$$

Now let AP meet BC in L and join DL. Then from (i)

$$BD^2 - CD^2 = AB^2 - AC^2 = BL^2 - CL^2;$$

therefore DL is perpendicular to BC. Now draw AM perpendicular to DL. Then since BC is perpendicular both to AL and DL, it is perpendicular to the plane ALD and therefore to AM, i.e. AM is perpendicular to both BC and DL and therefore to the plane BCD.

Hence $BM^2 - DM^2 = AB^2 - AD^2 = BC^2 - CD^2$ by (i); therefore CM is perpendicular to BD, i.e. M is the orthocentre of BCD, and similarly for the other faces. Hence each of the four perpendiculars must intersect every other one, and therefore all four must meet in a point.

2. Let AB, CD intersect in E, and BC, AD in F, and PR, QS in O. Then O is on the polar of E; therefore E is on the polar of O, and similarly so also is F; therefore EF is the polar of O. But the triangle formed by the three diagonals is self-polar for the conic, therefore the pole of EF is the intersection of AC, BD, i.e. O is the intersection of AC and BD.

3. If $$\frac{1}{(1-x)(1-2x)\ldots(1-nx)} = \sum_{p=1}^{p=n} \frac{A_p}{1-px} \quad \ldots\ldots\ldots\text{(i)},$$

then

$$A_p = \frac{1}{\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\ldots\left(1-\frac{n}{p}\right)} = \frac{p^{n-1}}{(p-1)(p-2)\ldots(p-n)}$$

$$= \frac{(-1)^{n-p}\cdot p^{n-1}}{(p-1)!\,(n-p)!}.$$

But the sum required is the coefficient of x^r in the expansion of (i), and this is

$$\sum_{p=1}^{p=n} A_p \cdot p^r = \sum_{p=1}^{p=n} \frac{(-1)^{n-p} \cdot p^{n+r-1}}{(p-1)!\,(n-p)!}.$$

4. Since

$$1 \equiv p-(p-1), \qquad 3 \equiv p-(p-3), \text{ etc.},$$

$$\therefore \quad 1 \,.\, 3 \,.\, 5 \ldots (p-2) = M(p) + (-1)^{\frac{p-1}{2}} \{2 \,.\, 4 \,.\, 6 \ldots (p-1)\};$$

$$\therefore \quad 1^2 \,.\, 3^2 \,.\, 5^2 \ldots (p-2)^2 = M(p) + (-1)^{\frac{p-1}{2}} (p-1)!$$

$$= M(p) + (-1)^{\frac{p-1}{2}} \{M(p) - 1\}$$

by Wilson's Theorem;

$$\therefore \quad 1^2 \,.\, 3^2 \,.\, 5^2 \ldots (p-2)^2 = M(p) + (-1)^{\frac{p+1}{2}}.$$

5. The sum of the roots is zero. Again

$$\alpha\beta(\beta-\gamma)(\gamma-\alpha) = \alpha\beta(\beta\gamma-\gamma^2-\alpha\beta+\gamma\alpha)$$

$$= \alpha\beta(-2\gamma^2-\alpha\beta), \text{ since } \Sigma\alpha = 0.$$

$$\therefore \quad \Sigma\alpha\beta(\beta-\gamma)(\gamma-\alpha) = -\Sigma\alpha^2\beta^2 = -(\Sigma\alpha\beta)^2 = -9p^2.$$

Further,

$$(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 = -27(q^2+4p^3);$$

therefore the product of the roots is $(-q)\{\pm\sqrt{-27(q^2+4p^3)}\}$. Hence the equation is as given, the sign of the square root being undetermined.

6. Let x, y be the lengths of the tangents from A and C, r the radius of the inscribed circle. Then

$$x = r\cot\frac{A}{2}, \quad y = r\cot\frac{C}{2}.$$

But $$\frac{A+C}{2} = 90^\circ; \quad \therefore \; xy = r^2.$$

Similarly

$$(a-x)(c-y) = r^2; \quad \therefore \; cx + ay = ac.$$

Also $a-x=b-y$, and from these

$$(a+c)\,x = ac + a^2 - ab = a\,(a+c-b) = ad\,;$$

$$\therefore\quad x = \frac{ad}{s},\quad y = \frac{bc}{s},\quad a - x = \frac{ab}{s},\quad c - y = \frac{cd}{s}.$$

7. By the usual formula,

$$\Sigma \cos(2n+1)\,\alpha = \frac{\cos 8\alpha \sin 8\alpha}{\sin\alpha}$$

$$= \frac{\sin 16\alpha}{2\sin\alpha} = -\tfrac{1}{2},$$

since $$16\alpha = 2\pi - \alpha.$$

Also $$2\Sigma\cos^2(2n+1)\,\alpha = \Sigma\left[1 + \cos(4n+2)\,\alpha\right]$$

$$= 8 + \Sigma\cos(4n+2)\,\alpha$$

$$= 8 + \frac{\cos 16\alpha \sin 16\alpha}{\sin 2\alpha}$$

$$= 8 + \frac{\sin 32\alpha}{2\sin 2\alpha} = 8 - \tfrac{1}{2},$$

since $$32\alpha = 4\pi - 2\alpha.$$

8. The tangent at β to the first conic is

$$\frac{l}{r} = e\cos(\theta - \alpha) + \cos(\theta - \beta),$$

or $$(e\cos\alpha + \cos\beta)\,x + (e\sin\alpha + \sin\beta)\,y = l,$$

and the tangent at $\frac{\pi}{2} + \beta$ to the second is

$$(e'\cos\alpha' - \sin\beta)\,x + (e'\sin\alpha' + \cos\beta)\,y = l'.$$

If these coincide, we have

$$\frac{e\cos\alpha + \cos\beta}{e'\cos\alpha' - \sin\beta} = \frac{e\sin\alpha + \sin\beta}{e'\sin\alpha' + \cos\beta} = \frac{l}{l'},$$

whence

$$l'\cos\beta + l\sin\beta = le'\cos\alpha' - l'e\cos\alpha,$$

$$l'\sin\beta - l\cos\beta = le'\sin\alpha' - l'e\sin\alpha.$$

Squaring and adding, we get

$$l'^2 + l^2 = l^2e'^2 + l'^2e^2 - 2ll'ee'\cos(\alpha - \alpha').$$

9. The line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \cos\beta$ will touch the ellipse in question. Let this be the equation to PQ. Then if the eccentric angles of P and Q are α, α', we must have

$$\theta = \frac{\alpha + \alpha'}{2}, \quad \beta = \frac{\alpha - \alpha'}{2}; \quad \therefore \ \theta + \beta = \alpha.$$

If T is the pole of PQ, its co-ordinates are

$$\frac{a\cos\theta}{\cos\beta}, \quad \frac{b\sin\theta}{\cos\beta},$$

and θ_1 is the angle between SP and ST.

The equations to these lines are

$$bx\sin\alpha - ay(\cos\alpha - e) - aeb\sin\alpha = 0,$$

$$bx\sin\theta - ay(\cos\theta - e\cos\beta) - aeb\sin\theta = 0;$$

therefore

$$\cot\theta_1 = \frac{b^2\sin\alpha\sin\theta + a^2(\cos\alpha - e)(\cos\theta - e\cos\beta)}{ab\left[\sin\alpha(\cos\theta - e\cos\beta) - \sin\theta(\cos\alpha - e)\right]}$$

$$= \frac{a}{b}\cdot\frac{(1-e^2)(\cos\beta - \cos\alpha\cos\theta) + (\cos\alpha - e)(\cos\theta - e\cos\beta)}{\sin\beta - e\cos\alpha\sin\beta}$$

$$= \frac{a}{b}\cdot\frac{(\cos\beta - e\cos\theta)(1 - e\cos\alpha)}{\sin\beta(1 - e\cos\alpha)}$$

$$= \frac{a}{b}\cdot\frac{\cos\beta - e\cos\theta}{\sin\beta}.$$

Similarly

$$\cot\theta_2 = \frac{a}{b}\cdot\frac{\cos\beta + e\cos\theta}{\sin\beta}.$$

$$\therefore \quad \cot\theta_1 + \cot\theta_2 = 2\cdot\frac{a}{b}\cot\beta.$$

10. Suppose each rod makes an angle α with the horizontal, so that $\cos\alpha = \frac{c}{a}$. Let R be the pressure between either rod and the cylinder, T the tension of the string. Then resolving vertically for either rod, we have

$$R\cos\alpha = W.$$

Also, taking moments for either rod about the joint,

$$W \cdot \frac{a}{2} \cos \alpha + T \cdot a \sin \alpha = R \cdot b \tan \alpha;$$

$$\therefore \quad T \cdot a \sin \alpha = W \cdot b \frac{\sin \alpha}{\cos^2 \alpha} - W \cdot \frac{a}{2} \cos \alpha;$$

$$\therefore \quad T = W \cdot \frac{b}{a} \cdot \frac{1}{\cos^2 \alpha} - \tfrac{1}{2} W \cot \alpha,$$

$$\text{i.e.} \quad T = W \cdot \frac{ab}{c^2} - \tfrac{1}{2} W \cdot \frac{c}{\sqrt{a^2 - c^2}}.$$

11. The impacts of A take place after times

$$\frac{2V}{g}, \quad \frac{2V}{g} + \frac{2eV}{g}, \quad \frac{2V}{g} + \frac{2eV}{g} + \frac{2e^2V}{g}, \text{ etc.,}$$

i.e. after times

$$\frac{2V}{g}, \quad \frac{2V}{g} \cdot \frac{1-e^2}{1-e}, \quad \frac{2V}{g} \cdot \frac{1-e^3}{1-e}, \text{ etc.,}$$

and the corresponding impacts of B at times $\dfrac{V}{g}$ later.

Hence the nth impact of B takes place before the $(n+1)$th impact of A provided

$$\frac{V}{g} + \frac{2V}{g} \cdot \frac{1-e^n}{1-e} < \frac{2V}{g} \cdot \frac{1-e^{n+1}}{1-e},$$

$$\text{i.e.} \quad 1 < 2e^n, \qquad \text{i.e.} \quad n < \frac{\log 2}{\log (e^{-1})}.$$

The rth impact of B takes place at time $\dfrac{V}{g}$ after the rth impact of A, when the velocity of A is $e^r V$ upwards.

Hence the velocity of A at the rth impact of B is

$$e^r V - \frac{V}{g} \cdot g = (e^r - 1) V \text{ upwards.}$$

Also the $(r+1)$th impact of A takes place at time

$$\frac{2V}{g} \cdot \frac{1-e^{r+1}}{1-e} - \left(\frac{V}{g} + \frac{2V}{g} \cdot \frac{1-e^r}{1-e}\right)$$

after the rth impact of B, and this time is $(2e^r - 1)\dfrac{V}{g}$.

Hence the velocity of B upwards is then

$$e^r V - (2e^r - 1)\frac{V}{g} \cdot g = (1 - e^r)\, V.$$

12. If $NQ = y'$, the required area on either side of the major axis is

$$\int_{-a}^{a} (y' - y)\, dx,$$

(x, y) being the co-ordinates of P.

Now $$y' - y = \frac{c^2}{y} - y = \frac{c^2 - b^2 \sin^2\theta}{b \sin\theta},$$

where θ is the eccentric angle of P. The above integral is therefore

$$\int_0^\pi \frac{c^2 - b^2 \sin^2\theta}{b \sin\theta} \cdot a \sin\theta\, d\theta$$

$$= \frac{a}{b}\int_0^\pi (c^2 - b^2 \sin^2\theta)\, d\theta$$

$$= \frac{a}{b}\left(c^2 \cdot \pi - b^2 \cdot \frac{\pi}{2}\right) = \frac{a\,(2c^2 - b^2)}{2b} \cdot \pi,$$

and the whole area is twice this expression.

LXVII.

1. If DEF is the given triangle inscribed in ABC, then since EF and the angle A are both given, the circle AEF is given in magnitude, and so also are the circles BFD, CDE. Now let $D'E'F'$ be a triangle having its sides equal to those of DEF. Describe a circle through $E'F'$ equal to the circle AEF and so on. Let the centres of the circles through $F'D'$ and $D'E'$ be O and O'. On OO' as diameter describe a circle, and place in it a chord $OP = \frac{1}{2} BC$. Draw a line through D' parallel to OP cutting the circles in B', C'. Join $B'F'$ cutting the remaining circle in A'. Join $A'E'$, $E'C'$. Then since $A' + B' + C' = A + B + C = 180°$, $A'E'C'$ is a straight line, and

the triangle $A'B'C'$ is obviously equal in all respects to ABC. Now make $BD = B'D'$, etc., and we construct the triangle DEF with the required sides.

2. If two tangents be drawn from P to a parabola, focus S, then one makes the same angle with SP as the other does with the axis of the parabola. Now let PQR be the triangle formed by the common tangents, S and S' the foci. Then

$$\begin{aligned} S'\hat{Q}P &= \text{angle between } QR \text{ and the axis of one parabola,} \\ SQP &= \text{angle between } QR \text{ and the axis of other parabola ;} \\ \therefore\ SQS' &= \text{angle between the axes of the parabolas} \\ &= \text{a right angle.} \end{aligned}$$

Therefore Q lies on the circle on SS' as diameter. Similarly for P and R.

3. We have

$$C_m = {}^{2n}C_m \times {}^{4n-2m}C_{2n},$$

so that the given series is the coefficient of x^{2n} in

$$\sum_{m=0}^{m=n} (-1)^m\, {}^{2n}C_m\, (x^2)^m\, [(1+x)^2]^{2n-m},$$

and in this sum we may replace the upper limit by $2n$, since the remaining terms only contain powers of x higher than x^{2n}. But the sum then becomes the expansion of

$$[(1+x)^2 - x^2]^{2n},$$

i.e. $(1+2x)^{2n}$ and the coefficient of x^{2n} is 2^{2n}.

4. Suppose a, b, c, d positive integers. Take a quantities each equal to $\frac{1}{a^2}$, b each equal to $\frac{1}{b^2}$, etc. Then since their A.M. is $\nless$ their G.M., we have

$$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}{a+b+c+d} \nless \left(\frac{1}{a^{2a}}\cdot\frac{1}{b^{2b}}\cdots\right)^{\frac{1}{a+b+c+d}},$$

$$\text{i.e. } \left(\frac{\Sigma\, bcd}{\Sigma a}\right)^{\Sigma a} \nless (abcd)^{\Sigma a}\,(a^{-2a}b^{-2b}c^{-2c}d^{-2d})$$
$$\nless a^{b+c+d-a}\, b^{c+d+a-b} \ldots .$$

If the quantities are not integers, we may choose k so that ka, kb, kc, kd are integers, and k then disappears from the foregoing inequality.

5. The roots being connected by the equations

$$x_1 + x_2 + x_3 = 0,$$
$$x_1 + 2x_2 + 3x_3 = k,$$

if we take $x_2 = -2\lambda$, we find

$$x_1 = \lambda - \tfrac{1}{2}k, \quad x_3 = \lambda + \tfrac{1}{2}k.$$

Hence the product of the roots is

$$-2\lambda(\lambda^2 - \tfrac{1}{4}k^2);$$
$$\therefore \quad 2\lambda^3 - \tfrac{1}{2}k^2\lambda - b = 0 \qquad \text{(i)}.$$

Also, since -2λ satisfies the given equation,

$$\therefore \quad -8\lambda^3 - 2a\lambda + b = 0 \qquad \text{(ii)}.$$

From (i) and (ii)

$$-2(a + k^2)\lambda - 3b = 0,$$
$$\text{i.e. } \lambda = -\frac{3b}{2(a + k^2)}.$$

6. If we eliminate z between

$$A\cos z + B\sin z = 1, \quad A'\cos z + B'\sin z = 1,$$

the result is

$$(A - A')^2 + (B - B')^2 = (AB' - A'B)^2.$$

Hence in this case

$$\frac{(\cos y - \cos x)^2}{\cos^4 a} + \frac{(\sin y - \sin x)^2}{\sin^4 a} = \frac{\sin^2(x - y)}{\cos^4 a \sin^4 a}.$$

If $\frac{x - y}{2} \neq n\pi$, we may divide by $2\sin^2\frac{x - y}{2}$, and we then get

$$\frac{1 - \cos(x + y)}{\cos^4 a} + \frac{1 + \cos(x + y)}{\sin^4 a} = \frac{1 + \cos(x - y)}{\cos^4 a \sin^4 a},$$

whence

$$1 - 2\sin^2 a\cos^2 a + (\cos^2 a - \sin^2 a)\cos(x + y) = 1 + \cos(x - y),$$

i.e. $\quad 2\sin^2 a \,.\, \cos x\cos y + 2\cos^2 a \,.\, \sin x\sin y + 2\sin^2 a\cos^2 a = 0,$

$$\text{i.e. } \frac{\cos x\cos y}{\cos^2 a} + \frac{\sin x\sin y}{\sin^2 a} + 1 = 0.$$

7. Suppose that OB_1 passes through the centre, and that A_1, the corresponding angular point of the polygon, is between O and B_1. Then

$$OB_r = \frac{c^2 - a^2}{OA_r} = \frac{c^2 - a^2}{\left(c^2 + a^2 - 2ac\cos\frac{2r\pi}{n}\right)^{\frac{1}{2}}}.$$

Now

$$c^{2n} - 2a^n c^n \cos n\theta + a^{2n} = \prod_0^{n-1} \left\{c^2 + a^2 - 2ac\cos\left(\theta + \frac{2r\pi}{n}\right)\right\} \quad \ldots\text{(i)}.$$

Putting $\theta = 0$, we get

$$(c^n - a^n)^2 = \prod_0^{n-1} \left(c^2 + a^2 - 2ac\cos\frac{2r\pi}{n}\right).$$

Hence $$OB_1 . OB_2 \ldots OB_n = (c^2 - a^2)^n/(c^n - a^n).$$

If B_1 is between O and A_1, then

$$OA_r = \left(c^2 + a^2 + 2ac\cos\frac{2r\pi}{n}\right), \text{ etc.,}$$

so that in the identity (i) we must put $\theta = \pi$, and, since n is odd, the left-hand side is then $(c^n + a^n)^2$.

8. Let the parabola be $y^2 = 4ax$ and the sides of the self-conjugate triangle $l_1x + m_1y - 1 = 0$, etc.

Then for some values of λ, μ, ν, the conic

$$\lambda(l_1x + m_1y - 1)^2 + \mu(l_2x + m_2y - 1)^2 + \nu(l_3x + m_3y - 1)^2 = 0$$

must coincide with $y^2 = 4ax$.

Hence

$$\Sigma\lambda l_1^2 = 0, \quad \Sigma\lambda l_1 m_1 = 0, \quad \Sigma\lambda m_1 = 0;$$

$$\therefore \begin{vmatrix} l_1^2, & l_1m_1, & m_1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = 0 \quad \ldots\ldots\ldots\text{(i)}.$$

But the line perpendicular to $l_1x + m_1y - 1 = 0$ where it meets the axis of x is

$$m_1x - l_1y = \frac{m_1}{l_1} \quad \text{or} \quad l_1m_1x - l_1^2y = m_1,$$

and (i) is also the condition that this and the two corresponding lines should be concurrent.

9. Any conic through the four points is

$$ax^2 + by^2 + 2gx + 2fy + c + \lambda(\alpha x^2 + \beta y^2 - 1) = 0.$$

If this is a circle

$$a + \lambda\alpha = b + \lambda\beta, \quad \text{i.e. } \lambda = -\frac{a-b}{\alpha-\beta},$$

and the square of the radius is

$$\frac{g^2 + f^2}{(a+\lambda\alpha)^2} - \frac{c-\lambda}{a+\lambda\alpha}.$$

If the conic is a rectangular hyperbola, then

$$a + \lambda\alpha + b + \lambda\beta = 0, \quad \text{i.e. } \lambda = -\frac{a+b}{\alpha+\beta},$$

and the conic is now

$$(a+\lambda\alpha)(x^2 - y^2) + 2gx + 2fy + c - \lambda = 0,$$

which may be written

$$\left(x + \frac{g}{a+\lambda\alpha}\right)^2 - \left(y - \frac{f}{a+\lambda\alpha}\right)^2 = \frac{g^2 - f^2}{(a+\lambda\alpha)^2} - \frac{c-\lambda}{a+\lambda\alpha},$$

and the expression on the right is the square of the real semi-axis.

10. Let O be the centre of the central vertical section, A and B be the points of contact of this section with the planes, F and F' the limiting frictions at A and B, G the moment of the couple. Then supposing the cylinder about to turn in the direction AB, we have, resolving horizontally,

$$F \sin(\alpha - \epsilon) = F' \sin(\beta + \epsilon),$$

and vertically,

$$F \cos(\alpha - \epsilon) + F' \cos(\beta + \epsilon) = W,$$

whence
$$\frac{F}{\sin(\beta+\epsilon)} = \frac{F'}{\sin(\alpha-\epsilon)} = \frac{W}{\sin(\alpha+\beta)},$$

and taking moments about O,

$$G = (F + F')\,a \sin\epsilon,$$

giving the required value of the couple.

If the motion were in the direction BA, we should obtain the same expression with the sign of ϵ changed in the bracket, and this is greater than the given expression, since

$$\sin(\beta-\epsilon)+\sin(\alpha+\epsilon)-\sin(\beta+\epsilon)-\sin(\alpha-\epsilon)$$
$$=2(\cos\alpha-\cos\beta)\sin\epsilon,$$

which is positive under the given conditions.

If $\epsilon > \alpha < \beta$, the motion must be in the direction BA, otherwise the horizontal forces cannot balance. Hence in this case we must change the sign of ϵ in the result.

11. The velocity after the jerk is

$$V'=\frac{P+Q}{P+Q+R}\cdot V,$$

and the acceleration is

$$\frac{Q+R-P}{Q+R+P}\cdot g=f.$$

Hence R is back on the plane again after time

$$\frac{2V'}{f}=\frac{2V}{g}\cdot\frac{P+Q}{Q+R-P}.$$

The acceleration is now $\dfrac{P-Q}{P+Q}\cdot g$ upwards, so that Q is back in its original position after time

$$\frac{2V'}{g}\cdot\frac{P+Q}{P-Q}=\frac{2V}{g}\cdot\frac{(P+Q)^2}{(P-Q)(P+Q+R)}.$$

Another jerk then occurs, and R reaches the ground again after time

$$\frac{2V'}{g}\cdot\frac{P+Q}{Q+R-P},$$

and so on. Hence the time that elapses before the system comes to rest is

$$\frac{2V}{g}\cdot\frac{P+Q}{Q+R-P}\left[1+\frac{P+Q}{P+Q+R}+\left(\frac{P+Q}{P+Q+R}\right)^2+\dots\ ad\ inf.\right]$$
$$+\frac{2V}{g}\cdot\frac{(P+Q)^2}{(P-Q)(P+Q+R)}\ [\text{same series}]$$

$$=\frac{2V}{g}\cdot\frac{P+Q}{R}\left(\frac{P+Q+R}{Q+R-P}+\frac{P+Q}{P-Q}\right)$$

$$=\frac{4V}{g}\cdot\frac{P(P+Q)}{(P-Q)(Q+R-P)}.$$

12. We have

$$\frac{1}{x(x^2+1)}=\frac{1}{x}-\frac{x}{x^2+1}$$

$$=\frac{1}{x}-\frac{1}{2}\left(\frac{1}{x-i}+\frac{1}{x+i}\right),$$

and the nth differential coefficient is

$$(-1)^n\cdot n!\left[\frac{1}{x^{n+1}}-\frac{1}{2}\left\{\frac{1}{(x-i)^{n+1}}+\frac{1}{(x+i)^{n+1}}\right\}\right].$$

Now, if $x=\cot\theta$, then

$$(x\pm i)^{n+1}=\frac{1}{\sin^{n+1}\theta}\{\cos(n+1)\theta\pm i\sin(n+1)\theta\}.$$

Hence the above is

$$(-1)^n\cdot n!\left\{\frac{1}{x^{n+1}}-\sin^{n+1}\theta\cos(n+1)\theta\right\},$$

and since $\sin\theta=(x^2+1)^{-\frac{1}{2}}$, the result follows.

LXVIII.

1. Let AP meet BC in D. Then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1,$$

i.e. $$\frac{BD}{DC}\cdot\frac{1-p}{p}\cdot\frac{q}{1-q}=1 \qquad \text{(i)}.$$

Now $$\triangle PFE : BEF = PE : EB,$$

$$BEF : BEA = 1-q : 1,$$

$$BEA : ABC = p : 1.$$

Also $$\frac{CA}{EA}\cdot\frac{EP}{PB}\cdot\frac{BD}{DC}=1;$$

$$\therefore\quad \frac{EP}{PB}=\frac{(1-p)q}{1-q} \text{ by (i)};$$

$$\therefore\quad \frac{EP}{EB}=\frac{(1-p)q}{1-pq};$$

$$\therefore\quad \frac{\triangle PFE}{\triangle ABC}=p(1-q)\cdot\frac{PE}{EB}=\frac{pq(1-p)(1-q)}{1-pq}.$$

2. Let TQT' be the tangent at Q, meeting the parallel tangents at R, R' in T, T', and let CT, CT' cut the curve in P, D and QR, QR' in V, V'. Then we have

$$CV\,.\,CT=CP^2, \text{ and } CV'\,.\,CT'=CD^2;$$

$$\therefore\quad CT\,.\,CT':CP\,.\,CD=CP\,.\,CD:CV\,.\,CV',$$

i.e. the areas of the triangles CTT', CPD, CVV' are in proportion. But CP, CD are conjugate semi-diameters. Hence $\triangle CPD=\frac{1}{2}ab$. Also $\triangle CTT'=\frac{1}{4}\Delta$. Hence

$$\frac{\frac{1}{4}\Delta}{\frac{1}{2}ab}=\frac{\frac{1}{2}ab}{\triangle CVV'};$$

$$\therefore\quad \triangle CVV'=\frac{a^2b^2}{\Delta},$$

whence the result follows.

3. Since the equation is satisfied by $x=y$, if it has equal roots, they must both be equal to y. Hence, denoting

$$(u-\gamma)(u-\delta)-(u-\alpha)(u-\beta)$$

by P, we have the identity

$$(x-\alpha)(x-\beta)(u-\gamma)(u-\delta)-(x-\gamma)(x-\delta)(u-\alpha)(u-\beta)\equiv P(u-x)^2,$$

and a similar identity in v.

In this, put $x=\alpha$, $x=\gamma$ in succession, and divide. We then get

$$-\frac{(\alpha-\gamma)(\alpha-\delta)(u-\alpha)(u-\beta)}{(\gamma-\alpha)(\gamma-\beta)(u-\gamma)(u-\delta)}=\frac{(u-\alpha)^2}{(u-\gamma)^2},$$

$$\text{i.e.}\quad \frac{(u-\alpha)(u-\delta)}{(u-\beta)(u-\gamma)}=-\frac{\alpha-\delta}{\beta-\gamma},$$

and the same result must be true with v substituted for u: hence the second result. To get the first put $x=a$, $x=\delta$ and proceed similarly.

4. Suppose he obtains α, β, γ marks respectively in the three papers.

i. If he obtains over 50 in each paper, we have

$$\alpha+\beta+\gamma=200,$$

where α may have any value from 50 to 100. Taking a fixed value p for α, we have $\beta+\gamma=200-p$, and β may have any value from 50 to $150-p$, i.e. there are in this case $101-p$ solutions.

Hence the total number of solutions is

$$\sum_{p=50}^{p=100}(101-p)=\frac{51\times 52}{2}=1326.$$

ii. If he obtains less than 50 in the first paper, we have $\beta+\gamma=200$, and β may have any value from 50 to 150, giving rise to 101 solutions. Dealing similarly with the other two papers, the total number of possibilities of this kind is 303.

iii. If he obtains less than 50 in two papers, he must obtain 200 in the third, and there are three possibilities of this kind.

Hence the total number of possibilities is

$$1326+303+3=1632.$$

5. Writing the equation in the form

$$(x^2-\lambda x+\mu)(x^2-\lambda' x+\mu')=0,$$

we want the equation satisfied by $\theta=\dfrac{\lambda}{\lambda'}$.

Identifying the equations, and putting $\lambda=\theta\lambda'$, we get

$$\lambda'(1+\theta)=-p,\quad \theta\lambda'^2+\mu+\mu'=0,\quad \theta\mu'+\mu=0,\quad \mu\mu'=q,$$

whence
$$\frac{\mu}{\theta}=\frac{\mu'}{-1}=\frac{-\theta\lambda'^2}{\theta-1}=-\frac{\theta p^2}{(\theta-1)(\theta+1)^2}.$$

Hence the required equation is

$$q(\theta+1)^4(\theta-1)^2+p^4\theta^3=0.$$

6. From the intersections of the graph $y = \sec x$ with the line $y = x$, it is evident that the equation has a root in the neighbourhood $(2n+\frac{1}{2})\pi$, when n is large. To obtain a closer approximation put $x = \alpha + \theta$.

The equation $x \cos x = 1$ then becomes

$$-(\alpha+\theta)\sin\theta = 1 \quad \text{.....................(i).}$$

If we retain only the first power of θ, this gives

$$-\alpha\theta = 1, \quad \text{i.e. } \theta = -\frac{1}{\alpha}.$$

If we retain θ^2, the equation (i) becomes

$$-(\alpha+\theta)\,\theta = (1-\tfrac{1}{6}\theta^2)^{-1} = 1+\tfrac{1}{6}\theta^2,$$
$$\text{i.e. } \alpha\theta = -1-\tfrac{7}{6}\theta^2,$$

and putting $\theta = -\dfrac{1}{\alpha}$ in the term involving θ^2, this gives

$$\alpha\theta = -1-\frac{7}{6\alpha^2}, \quad \text{i.e. } \theta = -\frac{1}{\alpha}-\frac{7}{6\alpha^3}.$$

Thus the root is more nearly equal to

$$\alpha-\frac{1}{\alpha}-\frac{7}{6\alpha^3}.$$

7. The series

$$1+\frac{2}{1!}x+\frac{3}{2!}x^2+\ldots+\frac{n+1}{n!}x^n+\ldots$$

may be written

$$\sum_1^\infty \frac{x^n}{(n-1)!}+\sum_0^\infty \frac{x^n}{n!} = xe^x+e^x = (1+x)e^x.$$

Hence the given series is the real part of

$$(1+ae^{i\alpha})\,e^{a(\cos\alpha+i\sin\alpha)},$$

i.e. of

$$e^{a\cos\alpha}\{1+a(\cos\alpha+i\sin\alpha)\}\{\cos(a\sin\alpha)+i\sin(a\sin\alpha)\},$$

and the real part is

$$e^{a\cos\alpha}\{(1+a\cos\alpha)\cos(a\sin\alpha)-a\sin\alpha\sin(a\sin\alpha)\}$$
$$= e^{a\cos\alpha}\{\cos(a\sin\alpha)+a\cos(\alpha+a\sin\alpha)\}.$$

8. Any conic having four-point contact with $y^2 = 4ax$ at the point m is of the form

$$y^2 - 4ax + \lambda (x - my + am^2)^2 = 0.$$

This is a rectangular hyperbola if

$$1 + \lambda + \lambda m^2 = 0, \quad \text{i.e. } \lambda = -\frac{1}{1 + m^2},$$

and the conic is

$$(1 + m^2)(y^2 - 4ax) - (x - my + am^2)^2 = 0,$$

$$\text{i.e. } x^2 - 2mxy - y^2 + 2a(3m^2 + 2)x - 2am^3 y + a^2 m^4 = 0.$$

The centre is given by

$$x - my + a(3m^2 + 2) = 0, \quad mx + y + am^3 = 0,$$

whence $\qquad x = -am^2 - 2a, \quad y = 2am,$

and the locus is $\qquad y^2 = -4a(x + 2a).$

9. The area of the general conic is $\dfrac{\pi\Delta}{(ab - h^2)^{\frac{3}{2}}}$, where Δ is the discriminant, and this is

$$\frac{\pi}{C^{\frac{3}{2}}} \cdot \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}^{\frac{1}{2}} \quad \text{......................(i)},$$

where A, B, C, ... are the tangential coefficients.

Now the conic in question meets the line $lx + my + n = 0$ where

$$l(at^2 + 2bt + c) + m(a't^2 + 2b't + c') + n(At^2 + 2Bt + C) = 0,$$

and if this equation in t has equal roots, we have

$$(al + a'm + An)(cl + c'm + Cn) = (bl + b'm + Bn)^2,$$

or, say,

$$A'l^2 + \ldots + \ldots + 2F'mn + \ldots + \ldots = 0,$$

which is therefore the tangential equation to the conic, so that its area is the expression (i) with the letters accented.

Now if we multiply together the determinants

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ A, & B, & C \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} c, & -2b, & a \\ c', & -2b', & a' \\ C, & -2B, & A \end{vmatrix},$$

we obtain the determinant

$$\begin{vmatrix} 2A', & 2H', & 2G' \\ 2H', & 2B', & 2F' \\ 2G', & 2F', & 2C' \end{vmatrix}.$$

Hence

$$\begin{vmatrix} A', & H', & G' \\ H', & B', & F' \\ G', & F', & C' \end{vmatrix} = \tfrac{1}{4} \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ A, & B, & C \end{vmatrix}^2.$$

Also $C' = AC - B^2$. Hence the given result.

10. If each rod makes an angle θ with the vertical, and T is the thrust in BD, the equation of virtual work is

$$-T \,.\, \delta(2a\sin\theta) + W \,.\, \delta\left(2a\cos\theta - \frac{c}{2}\cot\theta\right) = 0,$$

i.e. $$-T \,.\, 2a\cos\theta \,.\, \delta\theta + W\left(-2a\sin\theta \,.\, \delta\theta + \frac{c}{2}\operatorname{cosec}^2\theta \,.\, \delta\theta\right) = 0,$$

whence $$T = W\left(\frac{c}{4a} \,.\, \frac{\operatorname{cosec}^2\theta}{\cos\theta} - \tan\theta\right),$$

and $\sin\theta = \dfrac{b}{2a}$;

$$\therefore \quad T = W\left(\frac{c}{4a} \,.\, \frac{4a^2}{b^2} \,.\, \frac{2a}{\sqrt{4a^2 - b^2}} - \frac{b}{\sqrt{4a^2 - b^2}}\right)$$

$$= \frac{W}{\sqrt{4a^2 - b^2}}\left(\frac{2a^2c}{b^2} - b\right).$$

11. Let the velocity of projection in any path passing through A and B be that due to height h. Then the directrix is at height h above the point of projection. Hence if S be the focus,

$$SA = h - h_1, \quad SB = h - h_2;$$

$$\therefore \quad 2h = h_1 + h_2 + SA + SB.$$

Now the least possible value of $SA + SB$ is AB, occurring when S is in AB. Therefore the least possible value of $2h$ is $h_1 + h_2 + d$.

12. Integrating by parts, we have

$$\int x^n \sin x\, dx = -x^n \cos x + \int nx^{n-1} \cos x\, dx$$

and $$\int x^{n-1} \cos x\, dx = x^{n-1} \sin x - \int (n-1)\, x^{n-2} \sin x\, dx,$$

whence $$u_n = n \int_0^{\frac{\pi}{2}} x^{n-1} \cos x\, dx \qquad \text{(i)},$$

and $$\int_0^{\frac{\pi}{2}} x^{n-1} \cos x\, dx = \left(\frac{\pi}{2}\right)^{n-1} - (n-1)\, u_{n-2};$$

$$\therefore\quad u_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\, u_{n-2}.$$

Hence also

$$u_3 = 3\left(\frac{\pi}{2}\right)^2 - 6u_1,$$

and from (i)

$$u_1 = \int_0^{\frac{\pi}{2}} \cos x\, dx = 1;$$

$$\therefore\quad u_3 = \tfrac{3}{4}\pi^2 - 6.$$

LXIX.

1. Let $ABCD$ be the parallelogram, P the given point, H, K the middle points of AD, BC. Divide HK in L in the given ratio. Then PL is the line required. Let it meet AB, CD in E, F. Then

$$\begin{aligned} \text{area } ED : \text{area } EC &= \tfrac{1}{2}(AE + DF) \,.\, AD : \tfrac{1}{2}(EB + CF) \,.\, BC \\ &= HL : KL. \end{aligned}$$

2. Let the tangents at P and Q to the hyperbola meet in T. Then $P\hat{Q}T = P\hat{R}Q = Q\hat{P}K$, where PK is the tangent at P to

the circle PQR. Hence QT and PK are parallel. Let CT meet PQ in V. Then V is the middle point of PQ. Therefore CV is parallel to PR. Hence if QT meets PR in U, we have $QT = TU$. Hence since QT is parallel to the ray PK, the pencil in question is harmonic.

3. Let $\beta - \gamma = p$, $\beta' - \gamma' = p'$, etc., so that $\Sigma p = \Sigma p' = 0$. Then subtracting the first row from each of the others, the given determinant is

$$\Delta = \begin{vmatrix} 1, & a, & a' \\ 0, & -r, & -r' \\ 0, & q, & q' \end{vmatrix} = \begin{vmatrix} q, & q' \\ r, & r' \end{vmatrix},$$

and similarly
$$\Delta = \begin{vmatrix} r, & r' \\ p, & p' \end{vmatrix}.$$

Hence

$$\Delta^2 = \begin{vmatrix} qr + q'r', & pq + p'q' \\ z^2 & rp + r'p' \end{vmatrix}.$$

Now
$$2qr = (q + r)^2 - q^2 - r^2 = p^2 - q^2 - r^2;$$

$$\therefore \quad 2(qr + q'r') = x^2 - y^2 - z^2;$$

$$\therefore \quad 4\Delta^2 = \begin{vmatrix} x^2 - y^2 - z^2, & z^2 - x^2 - y^2 \\ 2z^2, & y^2 - z^2 - x^2 \end{vmatrix}$$

$$= z^4 - (x^2 - y^2)^2 - 2z^2(z^2 - x^2 - y^2)$$

$$= 2\Sigma y^2 z^2 - \Sigma x^4.$$

4. Denoting the given sum by S, we have

$$\frac{u_r}{r!} = \text{coefficient of } x^r \text{ in } e^{nx} - n \, . \, e^{(n-1)x} + \frac{n(n-1)}{2!} \, . \, e^{(n-2)x} - \dots$$

$$= \text{coefficient of } x^r \text{ in } (e^x - 1)^n.$$

Hence

$$\frac{S}{r!} = \text{coefficient of } x^r \text{ in } (e^x - 1) - \tfrac{1}{2}(e^x - 1)^2 + \dots \pm \frac{1}{r}(e^x - 1)^r$$

$$= \text{coefficient of } x^r \text{ in } \log\{1 + (e^x - 1)\},$$

since x^r will not occur in $(e^x-1)^p$ if $p>r$;

$$\therefore \quad \frac{S}{r!} = \text{coefficient of } x^r \text{ in } x;$$

$$\therefore \quad S = 0 \text{ if } r > 1.$$

5. The equation whose roots are the squares of the roots of those of $x^3+px+q=0$ is

$$y(y+p)^2 = q^2,$$

$$\text{i.e} \quad y^3+2py^2+p^2y-q^2=0.$$

Writing $y+h$ for y, this becomes

$$y^3+(3h+2p)y^2+(3h^2+4ph+p^2)y+h^3+2ph^2+p^2h-q^2=0,$$

and putting $h=-\dfrac{2p}{3}$, this reduces to the given equation.

6. The equation may be written

$$\cos\theta = \frac{\sqrt{3}}{2} \cdot \frac{\sin\theta}{1-\sin\theta},$$

and putting
$$\frac{\sin\theta}{1-\sin\theta} = x,$$

so that
$$\sin\theta = \frac{x}{1+x},$$

this becomes
$$1-\left(\frac{x}{1+x}\right)^2 = \tfrac{3}{4}x^2,$$

$$\text{i.e.} \quad 3x^4+6x^3+3x^2-8x-4=0,$$

or
$$(x-1)(3x^3+9x^2+12x+4)=0.$$

Hence the equation can have no real positive root other than $x=1$, since the coefficients of the second factor are all positive. Therefore $\sin\theta$ can have no positive value other than $\frac{1}{2}$, so that the only possible values of θ between 0 and π are $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$, and of these only the former satisfies the equation.

7. Since
$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}},$$
we have
$$1-\tan^2\frac{x}{2} = \frac{2\tan\frac{x}{2}}{\tan x}.$$
Similarly
$$1-\tan^2\frac{x}{2^2} = \frac{2\tan\frac{x}{2^2}}{\tan\frac{x}{2}}, \text{ and so on.}$$
Hence
$$\prod_{r=1}^{r=n}\left(1-\tan^2\frac{x}{2^r}\right) = \frac{2\tan\frac{x}{2}}{\tan x}\cdot\frac{2\tan\frac{x}{2^2}}{\tan\frac{x}{2}}\cdots\frac{2\tan\frac{x}{2^n}}{\tan\frac{x}{2^{n-1}}}$$
$$= \frac{2^n\tan\frac{x}{2^n}}{\tan x}.$$
Now
$$\operatorname*{Lim}_{n\to\infty} 2^n\tan\frac{x}{2^n} = \operatorname{Lim} x\,.\,\frac{\tan\frac{x}{2^n}}{\frac{x}{2^n}} = x.$$
Hence the infinite product converges to the value $\dfrac{x}{\tan x}$.

8. Changing the axes to the tangent and normal at P, let the equation be
$$a'x^2+2h'xy+b'y^2=2y \quad \text{..................(i)}.$$
Then
$$\rho = \text{Lt.}\,\frac{x^2}{2y} = \frac{1}{a'}.$$
The lines joining the origin to the points where $lx+my=1$ meets (i) are
$$a'x^2+2h'xy+b'y^2=2y\,(lx+my),$$
and if these are perpendicular $a'+b'-2m=0$, shewing that the line passes through the fixed point $\left(0, \dfrac{2}{a'+b'}\right)$.

Now (i) referred to the centre is

$$a'x^2 + 2h'xy + b'y^2 = \frac{a'}{a'b' - h'^2};$$

$$\therefore \quad \frac{1}{a^2} + \frac{1}{b^2} = \frac{a'b' - h'^2}{a'}(a' + b'), \qquad \frac{1}{a^2b^2} = \frac{(a'b' - h'^2)^3}{a'^2},$$

whence $$\left(\frac{1}{a^2} + \frac{1}{b^2}\right)(a^2b^2)^{\frac{1}{3}} = \frac{a' + b'}{a'^{\frac{1}{3}}} = \frac{2\rho^{\frac{1}{3}}}{PF};$$

$$\therefore \quad PF = 2\,\frac{a^{\frac{4}{3}}\,b^{\frac{4}{3}}\,\rho^{\frac{1}{3}}}{a^2 + b^2}.$$

If the latus-rectum is $4p$, we have $b^2 = 2ap$;

$$\therefore \quad PF = 2\,\frac{2^{\frac{2}{3}}\,a^2p^{\frac{2}{3}}\,\rho^{\frac{1}{3}}}{a^2 + 2ap}.$$

Dividing by a^2, and making a infinite, we pass to the case of the parabola, and obtain

$$PF = 2^{\frac{5}{3}}\,p^{\frac{2}{3}}\,\rho^{\frac{1}{3}}.$$

9. Taking the three given points as the points of reference, let the equation of the conic be

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0,$$

and that of the fixed straight line $L\alpha + M\beta + N\gamma = 0$, so that we have the condition

$$\sqrt{lL} + \sqrt{mM} + \sqrt{nN} = 0 \quad \text{..................(i)}.$$

The centre is given by

$$\frac{m\gamma + n\beta}{a} = \frac{n\alpha + l\gamma}{b} = \frac{l\beta + m\alpha}{c},$$

whence $$\frac{l\beta\gamma}{b\beta + c\gamma - a\alpha} = \ldots = \ldots.$$

Hence from (i)

$$\Sigma\sqrt{L\alpha\,(b\beta + c\gamma - a\alpha)} = 0,$$

and the rational form of this is of the fourth degree.

10. If ABC is a triangle, then forces represented by AB, BC are equivalent to a force represented by AC, and a couple whose moment is $2\triangle ABC$, in the same sense as the direction ABC. Hence

forces GA, AA' are equivalent to a force GA' together with a couple of moment $2\triangle AGA'$ in direction GAA', and

forces GC, CC' to a force GC' and a couple of moment $2\triangle CGC'$ in the opposite direction.

Hence forces AA', CC' are equivalent to forces GA', GC' and a couple of moment $\triangle ACA' - \triangle ACC'$.

But the forces GA', GC' are equivalent to $2GG'$, and

$$\triangle ACA' - \triangle ACC'$$

$$= \tfrac{1}{2}AC \times (\text{difference of perpendiculars from } A', C' \text{ on } AC)$$

$$= \tfrac{1}{2}AC \times A'C' \sin\theta = \tfrac{1}{2}AC^2 \sin\theta.$$

Hence forces AA', CC' are equivalent to a force $2GG'$ and a couple of moment $\frac{1}{2}AC^2 \sin\theta$. Similarly for BB', DD'.

11. Let u_1, u_2 be the velocities of the sphere perpendicular to and along the face of the wedge, v that of the wedge immediately after impact, u the striking velocity. Then since the wedge is smooth

$$u_2 = u \sin\alpha,$$

and by Newton's Law

$$u_1 + v\sin\alpha = eu\cos\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Also the impulse perpendicular to the face is $m(u\cos\alpha + u_1)$;

$$\therefore\ m(u\cos\alpha + u_1)\sin\alpha = Mv \quad \ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) and (ii)

$$u_1 = u\cos\alpha \,.\, \frac{Me - m\sin^2\alpha}{M + m\sin^2\alpha};$$

$$\therefore\ \frac{u_1}{u_2} = \frac{Me - m\sin^2\alpha}{(M + m\sin^2\alpha)\tan\alpha}.$$

12. The co-ordinates of any point on the first curve may be written in the form

$$x = \frac{3am^2}{1+m^3}, \quad y = \frac{3am}{1+m^3}.$$

For the intersections with the parabola we have

$$9a^2m^2 = 12a^2m^2(1+m^3),$$

giving

$$m = 0 \text{ or } 1 + m^3 = \tfrac{3}{4},$$

$$\text{i.e. } m = -\frac{1}{2^{\frac{2}{3}}},$$

so that the intersection other than the origin is the point

$$(2^{\frac{2}{3}}a, \; -2^{\frac{4}{3}}a).$$

The area included will be the difference of the values of the integral $\int x\,dy$ taken between the origin and the other point of intersection for the curve and the parabola.

For the curve

$$\int x\,dy = 9a^2 \int \frac{m^2(1-2m^3)}{(1+m^3)^3}\,dm$$

$$= 3a^2 \int \frac{3-2t}{t^3}\,dt, \text{ where } t = 1 + m^3,$$

$$= 3a^2 \left(-\frac{3}{2} \cdot \frac{1}{t^2} + \frac{2}{t}\right),$$

and the limits for t being 1 and $\frac{3}{4}$, the value of this is $\frac{3a^2}{2}$.

For the parabola

$$\int x\,dy = \frac{1}{4a} \int y^2\,dy = \frac{y^3}{12a},$$

and the limits for y being 0 and $-2^{\frac{4}{3}}a$, the value of this is $\frac{4a^2}{3}$. Hence the area required is

$$\frac{3a^2}{2} - \frac{4a^2}{3} = \frac{a^2}{6}.$$

LXX.

1. The points in question are U, V, W, the middle points of AP, BQ, CR. Since M, N, P are collinear, we have

$$\frac{BP}{PC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = -1,$$

and two similar results for the other sets of points. Multiplying these together, and remembering that

$$\frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1,$$

since AL, BM, CN are concurrent, we get

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1.$$

Therefore P, Q, R are collinear. Hence U, V, W are the middle points of the diagonals of the complete quadrilateral $ACPR$, and are therefore collinear.

2. Let the tangents meet at T and let PG be the normal at P. Then, since the tangents subtend equal angles at S', therefore $PS'Q = 2PS'T$. But $PS'Q = SPS' = 2S'PG$.

Therefore $PS'T = S'PG$. Therefore $S'T$ is parallel to PG, i.e. $S'TP$ is a right angle. Therefore T is on the auxiliary circle.

3. Multiplying the first two equations and dividing by the third, we get $(x+y)^2 = c^2z^2$;

$$\therefore\ x + y \pm cz = 0.$$

Similarly

$$x \pm by + z = 0,$$

$$\pm ax + y + z = 0.$$

Hence, if x, y, z are not zero, we must have

$$\begin{vmatrix} \pm a, & 1, & 1 \\ 1, & \pm b, & 1 \\ 1, & 1, & \pm c \end{vmatrix} = 0.$$

4. We have to find the number of ways in which we can choose positive integers α, β, γ so that

$$\alpha + \beta + \gamma = n, \qquad (\beta \not> \alpha, \quad \beta \not> \gamma).$$

We may therefore write $\alpha = \beta + \alpha'$, $\gamma = \beta + \gamma'$, and we now have to find the number of solutions in positive integers (including zero) of $\alpha' + 3\beta + \gamma' = n$. The required number is evidently the coefficient of x^n in

$$(1 + x + x^2 + \ldots)(1 + x^3 + x^6 + \ldots)(1 + x + x^2 + \ldots),$$

i.e. in $$(1 - x)^{-2}(1 - x^3)^{-1}.$$

5. Suppose the equation equivalent to

$$(x^2 + 2ax + \lambda)^2 = (px + q)^2 \text{ ....(i).}$$

Then

$$4a^2 + 2\lambda - p^2 = 6(a^2 + b^2), \quad \text{i.e. } 2\lambda = p^2 + 2a^2 + 6b^2.$$

The discriminants of the two quadratics (i) are

$$(2a - p)^2 - 4(\lambda - q) \quad \text{and} \quad (2a + p)^2 - 4(\lambda + q),$$

and their sum is

$$8a^2 + 2p^2 - 8\lambda = -2p^2 - 24b^2,$$

and is therefore negative. Hence the discriminants cannot be both positive, i.e. at least one of the quadratics has imaginary roots.

6. We have

$$\Sigma(AP^2 \,.\, \triangle BNC) = PN^2 \,.\, \triangle ABC + \Sigma(AN^2 \,.\, \triangle BNC).$$

Now

$$\triangle BNC = \tfrac{1}{2}a \,.\, \tfrac{1}{2}(R\cos A + 2R\cos B\cos C) = \tfrac{1}{2}R^2 \sin A\cos(B - C).$$

Also

$$4AN^2 = 2AO^2 + 2AH^2 - OH^2$$

(where H is the orthocentre and O the circumcentre)

$$= 2R^2 + 8R^2\cos^2 A - R^2(1 - 8\cos A\cos B\cos C)$$

$$= R^2(1 + 8\cos A\sin B\sin C);$$

$$\therefore \ \Sigma(AN^2 . \triangle BNC)$$

$$= \tfrac{1}{4}R^2[\Delta + 4R^2 \sin A \sin B \sin C . \Sigma \cos A \cos(B-C)],$$

and

$$\Sigma \cos A \cos(B-C) = -\Sigma \cos 2A = 1 - 4\cos A \cos B \cos C.$$

Hence

$$\tfrac{1}{2}R^2 . \Sigma AP^2 \sin A \cos(B-C) = PN^2 . 2R^2 \sin A \sin B \sin C$$
$$+ \tfrac{1}{4}R^2(6R^2 \sin A \sin B \sin C$$
$$+ 16R^2 \sin A \sin B \sin C \cos A \cos B \cos C),$$

and dividing through by $\frac{1}{2}R^2 \sin A \sin B \sin C$, we obtain the result as given.

7. We have

$$me^{2ix} - \frac{m^2}{2}e^{4ix} + \ldots = \log(1 + me^{2ix})$$
$$= \log(1 + m\cos 2x + i . m \sin 2x).$$

Hence, equating coefficients of i,

$$m\sin 2x - \frac{m^2}{2}\sin 4x + \ldots = \tan^{-1}\left(\frac{m\sin 2x}{1 + m\cos 2x}\right),$$

and therefore by question

$$\tan(x-y) = \frac{m\sin 2x}{1+m\cos 2x} = \frac{2m\tan x}{1+\tan^2 x + m(1-\tan^2 x)}$$

$$= \frac{\dfrac{2m}{1+m}\tan x}{1 + \dfrac{1-m}{1+m}\tan^2 x}.$$

Hence evidently

$$\tan y = \frac{1-m}{1+m}\tan x = \cos\phi \tan x.$$

8. The equation

$$\frac{r}{l} = A\cos\theta + B\sin\theta$$

represents a circle through the origin. If this circle passes through the points $\alpha \pm \beta$ on the conic, we have

$$\frac{1}{1 + e\cos(\alpha \pm \beta)} = A\cos(\alpha \pm \beta) + B\sin(\alpha \pm \beta).$$

From these

$$2A \sin 2\beta = \frac{\sin(\alpha+\beta)}{1+e\cos(\alpha-\beta)} - \frac{\sin(\alpha-\beta)}{1+e\cos(\alpha+\beta)}$$

$$= \frac{2\cos\alpha\sin\beta + e\cos 2\alpha \sin 2\beta}{D},$$

where $D = \{1+e\cos(\alpha+\beta)\}\{1+e\cos(\alpha-\beta)\}$.

Thus $$A = \frac{\cos\alpha + e\cos 2\alpha\cos\beta}{D\cos\beta},$$

and similarly $$B = \frac{\sin\alpha + e\sin 2\alpha\cos\beta}{D\cos\beta}.$$

Hence the circle is

$$\frac{r}{l} \cdot D\cos\beta = \cos(\theta-\alpha) + e\cos\beta\cos(\theta-2\alpha),$$

and putting $\beta = 0$, this takes the required form.

9. Reciprocating for the polar circle, the triangle reciprocates into itself, and the parabola into the inscribed circle, which must therefore pass through the origin of reciprocation, i.e. the orthocentre, whose trilinear co-ordinates are sec A, sec B, and sec C. Thus, then, these co-ordinates must satisfy the equation to the inscribed circle, viz.:

$$\Sigma \cos \tfrac{1}{2}A \,.\, \sqrt{\alpha} = 0.$$

$$\therefore \quad \Sigma \cos \tfrac{1}{2}A \,.\, \sqrt{\sec A} = 0.$$

10. Let R be the pressure between the board and either sphere, F the friction at the point of contact.

Then, resolving vertically for the board, $2R = W$, and taking moments for either sphere about its point of contact with the plane

$$W' \,.\, a\sin\alpha + R \,.\, a\sin\alpha = F(a + a\cos\alpha),$$

where W, W' are the weights of the board and sphere respectively.

Hence $$F = (W'+R) \,.\, \frac{\sin\alpha}{1+\cos\alpha}.$$

$$\therefore \quad \frac{F}{R} = \left(\frac{W'}{\frac{1}{2}W} + 1\right)\tan\frac{\alpha}{2} = \left(\frac{1}{n} + 1\right)\tan\frac{\alpha}{2}.$$

Hence the coefficient of friction must be not less than this quantity.

11. Let V be the velocity and α the angle of projection. Then the horizontal velocity in the first trajectory is $V\cos\alpha$, and in the second $eV\cos\alpha$. Therefore the total time from H to K is

$$\frac{c}{V\cos\alpha} + \frac{c}{eV\cos\alpha}.$$

At the impact, the vertical velocity remains unchanged. Hence the vertical distance described during this time is

$$V\sin\alpha \,.\, \frac{c}{V\cos\alpha}\left(1+\frac{1}{e}\right) - \frac{1}{2}g \,.\, \frac{c^2}{V^2\cos^2\alpha} \,.\, \left(1+\frac{1}{e}\right)^2$$

$$= c\tan\alpha\left(1+\frac{1}{e}\right) - \frac{c^2}{4h}\left(1+\frac{1}{e}\right)^2(1+\tan^2\alpha)$$

$$= \frac{1}{4h}\left[4h^2 - c^2\left(1+\frac{1}{e}\right)^2 - \left\{2h - c\left(1+\frac{1}{e}\right)\tan\alpha\right\}^2\right],$$

and the greatest value of this expression for different values of α is

$$\frac{1}{4h}\left\{4h^2 - c^2\left(1+\frac{1}{e}\right)^2\right\}.$$

12. The radius of curvature at $P(x', y')$ of $xy = c^2$ is

$$\rho = \frac{CP^3}{2c^2} = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{2c^2}.$$

Now if $xy' + x'y = 2c^2$ coincides with $x\cos\psi + y\sin\psi = p$, we have

$$\frac{y'}{\cos\psi} = \frac{x'}{\sin\psi} = \frac{c}{\sqrt{\sin\psi\cos\psi}} = \frac{(x'^2+y'^2)^{\frac{1}{2}}}{1};$$

$$\therefore \quad \rho = \frac{\sqrt{2}c}{(\sin 2\psi)^{\frac{3}{2}}}.$$

$$\therefore\ \frac{d\rho}{ds}=\frac{1}{\rho}\cdot\frac{d\rho}{d\psi}=\frac{(\sin 2\psi)^{\frac{3}{2}}}{\sqrt{2c}}\cdot\left\{-\frac{\frac{3}{2}\sqrt{2c}}{(\sin 2\psi)^{\frac{5}{2}}}\right\}.\,2\cos 2\psi$$

$$=-3\cot 2\psi,$$

and $$\frac{d^2\rho}{ds^2}=\frac{1}{\rho}\cdot\frac{d}{d\psi}\left(\frac{d\rho}{ds}\right)=\frac{1}{\rho}.\,(6\operatorname{cosec}^2 2\psi).$$

$$\therefore\ \frac{1}{6}\rho\frac{d^2\rho}{ds^2}=\operatorname{cosec}^2 2\psi,$$

$$\text{i.e. }\ \frac{1}{6}\rho\frac{d^2\rho}{ds^2}=\left(\frac{1}{3}\frac{d\rho}{ds}\right)^2+1.$$

LXXI.

1. Let P be the orthocentre, O the circumcentre. Draw the chord AA' parallel to BC, and produce AP to meet the circumcircle in L. Then LA' is a diameter. Draw the diameter DOD', and the chord DL' perpendicular to BC. Then since the pedal line of D passes through the middle points of PD, PO, it is parallel to DD'. Also AL' is parallel to the pedal line ;

$$\therefore\ \text{arc } LL'=\text{arc } AD=\text{arc } L'D',$$

$$\text{i.e. arc } DA'=LD'=2AD.$$

Hence $$3AD=AA',$$

and similarly $$3AE=\text{circumference}+AA',$$

$$3AF=2\text{ circumference}+AA',$$

i.e. DE, EF, FD are each one-third of the circumference.

2. Let $ABCD$ be the cyclic quadrilateral, EF the third diagonal. Then S, the focus, lies on both the circles EBC and DCF. Hence

$$E\hat{S}C=180^\circ-E\hat{B}C$$

$$=A\hat{D}C=180^\circ-C\hat{D}F=180^\circ-C\hat{S}F.$$

$$\therefore\ E\hat{S}C+C\hat{S}F=180^\circ,$$

i.e. S lies on EF.

3. Let the number (as written) be $abcd$. Then

$$d \,.\, 7^3 + a \,.\, 7^2 + b \,.\, 7 + c = \tfrac{1}{2}(a \,.\, 7^3 + b \,.\, 7^2 + c \,.\, 7 + d),$$

whence $$49a + 7b + c = 137d,$$

$$\text{i.e.}\quad 7a + b = 19d + \frac{4d-c}{7} \quad\text{..................(i).}$$

Hence $\frac{4d-c}{7}$ must be an integer. But

$$2 \,.\, \frac{4d-c}{7} = d + \frac{d-2c}{7}.$$

Therefore $\frac{d-2c}{7}$ is an integer or zero, and remembering that a, b, c, d are all less than 7, the only possible cases are

$$d = 2,\ c = 1, \text{ and } d = 1,\ c = 4.$$

In the former case, from (i),

$$7a + b = 39,$$

whence $a = 5$, $b = 4$, so that the number is 5412.

In the latter case $$7a + b = 19,$$

whence $a = 2$, $b = 5$, and the number is 2541.

4. We have

$$a_r = \frac{(p-2)(p-3)\ldots(p-r-1)}{r!}$$

$$= \frac{M(p) + (-1)^r (r+1)!}{r!} = \frac{M(p)}{r!} + (-1)^r (r+1).$$

Hence $$a_r - (-1)^r (r+1) = \frac{M(p)}{r!}.$$

Now, since p is prime, it cannot be contained in $r!$ Further, since a_r is an integer, $\frac{M(p)}{r!}$ is an integer, and must therefore be an integral multiple of p.

5. We may suppose the quantities $a_1, a_2, \ldots a_n$ to be in descending order of magnitude. Consider

$$f(x) = A_1(x-a_2)(x-a_3)\ldots(x-a_n)$$
$$+ A_2(x-a_1)(x-a_3)\ldots(x-a_n) + \ldots.$$

Then

$$f(a_1) = A_1 (a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n) \text{ and is positive,}$$

$$f(a_2) = A_2 (a_2 - a_1)(a_2 - a_3) \ldots (a_2 - a_n) \text{ and is negative}$$

since one factor is negative,

$$f(a_3) = A_3 (a_3 - a_1)(a_3 - a_2) \ldots (a_3 - a_n) \text{ and is positive}$$

since two factors are negative,

and so on.

Hence the equation $f(x) = 0$ has at least one root between a_1 and a_2, another between a_2 and a_3, ..., another between a_{n-1} and a_n; i.e. the equation has at least $(n-1)$ real roots, and therefore all its roots are real.

6. Suppose O is in the arc AB away from C, and let OC make an acute angle θ with AB. Then

$$\frac{AP}{\sin B} = \frac{c}{\sin(\theta + 2B)}, \quad \frac{BQ}{\sin A} = \frac{c}{\sin(2A - \theta)}, \quad \frac{CR}{\sin B} = \frac{a}{\sin\theta},$$

whence

$$\frac{\cos A}{AP} = \frac{\cos A \sin(\theta + 2B)}{2R \sin B \sin C},$$

$$\frac{\cos B}{BQ} = \frac{\cos B \sin(2A - \theta)}{2R \sin C \sin A}, \quad \frac{\cos C}{CR} = \frac{\cos C \sin\theta}{2R \sin A \sin B},$$

and taking the second ratio negatively, we have

$$\Sigma \frac{\cos A}{AP} = \frac{\sin 2A \sin(\theta + 2B) - \sin 2B \sin(2A - \theta) + \sin 2C \sin\theta}{4R \sin A \sin B \sin C},$$

and the numerator is

$$\sin(2A + 2B) \sin\theta + \sin 2C \,.\, \sin\theta = 0.$$

7. The equation $\cos 3\theta = \frac{1}{2}$ gives $3\theta = 2n\pi \pm \frac{\pi}{3}$, and the distinct values of $\cos\theta$ are

$$\cos\frac{\pi}{9}, \; \cos\frac{5\pi}{9}, \; \cos\frac{7\pi}{9}, \quad \text{i.e. } \cos\frac{\pi}{9}, \; -\cos\frac{4\pi}{9}, \; -\cos\frac{2\pi}{9}.$$

Similarly the equation $\cos 3\theta = -\frac{1}{2}$ will give

$$\cos\frac{2\pi}{9},\quad \cos\frac{4\pi}{9},\quad -\cos\frac{\pi}{9}.$$

Hence the roots of $\cos^2 3\theta = \frac{1}{4}$, considered as an equation in $\cos\theta$, are $\pm\cos\frac{r\pi}{9}$ $(r = 1, 2, 4)$, so that, putting $\cos^2\theta = x$, the roots of

$$x(4x-3)^2 = \tfrac{1}{4}$$

are $\cos^2\frac{r\pi}{9}$ $(r = 1, 2, 4)$.

The factor corresponding to $\cos^2\frac{3\pi}{9}$ is $x - \frac{1}{4}$, and therefore the required equation is

$$(16x^3 - 24x^2 + 9x - \tfrac{1}{4})(x - \tfrac{1}{4}) = 0,$$

reducing to the given form.

8. The equation to the circle TPQ must be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda\left(\frac{xX}{a^2} + \frac{yY}{b^2} - 1\right)\left(\frac{xX}{a^2} - \frac{yY}{b^2} - k\right) = 0 \quad \text{...(i)},$$

where T is (X, Y), the condition for a circle being

$$\frac{1}{a^2} + \frac{\lambda X^2}{a^4} = \frac{1}{b^2} - \frac{\lambda Y^2}{b^4} \quad \text{..................(ii)}.$$

The co-ordinates of the centre are proportional to the coefficients of x and y, i.e. to $-(1+k)\frac{X}{a^2}$, $(1-k)\frac{Y}{b^2}$. Hence, by question,

$$(1+k)X + (1-k)Y = 0, \quad \text{i.e. } k = -\frac{X+Y}{X-Y} \quad \text{...(iii)}.$$

Also, since T is on (i), we have

$$1 + \lambda\left(\frac{X^2}{a^2} - \frac{Y^2}{b^2} - k\right) = 0 \quad \text{...............(iv)}.$$

Eliminating λ, k from (ii), (iii), (iv) we get the given locus.

9. The equation to any conic having three-point contact at α is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-1\right)$$
$$\left(\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right)=0,$$

the condition for a parabola being

$$\left(1+\lambda\cos\alpha\cos\frac{\alpha+\beta}{2}\right)\left(1+\lambda\sin\alpha\sin\frac{\alpha+\beta}{2}\right)=\frac{\lambda^2}{4}\sin^2\frac{3\alpha+\beta}{2}.$$

On reduction this is

$$1+\lambda\cos\frac{\alpha-\beta}{2}-\frac{\lambda^2}{4}\sin^2\frac{\alpha-\beta}{2}=0,$$

whence
$$\lambda=\frac{2}{\pm 1-\cos\frac{\alpha-\beta}{2}},$$

and the coefficients of $\frac{x^2}{a^2}$ and $\frac{y^2}{b^2}$ are

$$\frac{\pm 1+\cos\frac{3\alpha+\beta}{2}}{\pm 1-\cos\frac{\alpha-\beta}{2}} \text{ and } \frac{\pm 1-\cos\frac{3\alpha+\beta}{2}}{\pm 1-\cos\frac{\alpha-\beta}{2}}.$$

Hence the axes are parallel to the lines

$$\frac{x}{a}\cos\frac{3\alpha+\beta}{4}+\frac{y}{b}\sin\frac{3\alpha+\beta}{4}=0,$$

and
$$\frac{x}{a}\sin\frac{3\alpha+\beta}{4}-\frac{y}{b}\cos\frac{3\alpha+\beta}{4}=0.$$

Thus
$$\tan\phi=\frac{\frac{1}{ab}}{\frac{1}{2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin\frac{3\alpha+\beta}{2}},$$

$$\text{i.e. } \cot\phi=\frac{a^2-b^2}{2ab}\sin\frac{3\alpha+\beta}{2}.$$

10. Let A, B, C be the points of support, O the centre of the sphere. Draw ON perpendicular to the plane ABC: then N is the circumcentre of ABC. Let the plane OAN meet BC in K. Now the forces acting are the pressures P, Q, R at A, B, C and the weight W, and these are in equilibrium. Hence the resultant of Q and R must be in the plane of P and W, i.e. in the plane OAN. Therefore this resultant must act along OK. Therefore $Q\,.\,BK = R\,.\,CK$.

But from the triangle ABC,

$$\frac{BK}{c} = \frac{\cos C}{\sin AKB}, \quad \frac{CK}{c} = \frac{\cos B}{\sin AKC};$$

$$\therefore \quad \frac{BK}{CK} = \frac{c \cos C}{b \cos B}.$$

$$\therefore \quad \frac{Q}{b \cos B} = \frac{R}{c \cos C},$$

and similarly each is equal to $\dfrac{P}{a \cos A}$.

11. Suppose that during the accelerated, uniform and retarded motions, the train describes distances a, b, c respectively. Let the mass be n tons, and the resistance x lbs.-wt. per ton. Let f be the acceleration, and f' the retardation.

Then the pull of the engine during the first part is

$$2240nf + nxg \text{ poundals},$$

and during the second part nxg poundals, while the retarding force of the brakes in the third part is $2240nf' - nxg$ poundals. The difference of the works done by the first and third forces is

$$(2240nf + nxg)\,a - (2240nf' - nxg)\,c.$$

But $V^2 = 2fa = 2f'c$. Therefore this difference is $ngx\,(a+c)$. The work done by the resistance is $ngx\,(a+b+c)$, and the ratio of the works is $\dfrac{a+c}{a+b+c}$.

But

$$v = \frac{\text{total distance}}{\text{total time}} = \frac{a+b+c}{\dfrac{2a}{V} + \dfrac{b}{V} + \dfrac{2c}{V}} = V \cdot \frac{a+b+c}{2a+b+2c};$$

$$\therefore \quad \frac{a+c}{a+b+c} = \frac{V}{v} - 1.$$

12. The area of the loop is

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tfrac{1}{2} r^2 \, d\theta = a^2 \int_0^{\frac{\pi}{4}} \left(\frac{\cos 2\theta}{\cos\theta}\right)^2 d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} \left(2\cos\theta - \frac{1}{\cos\theta}\right)^2 d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} (4\cos^2\theta - 4 + \sec^2\theta)\, d\theta$$

$$= a^2 \Big[\sin 2\theta - 2\theta + \tan\theta\Big]_0^{\frac{\pi}{4}} = a^2\left(2 - \frac{\pi}{2}\right).$$

To find the volume generated, we use the Cartesian form of the equation, viz.,

$$y^2 = \frac{x^2(a-x)}{a+x}.$$

The volume required is then

$$\int_0^a \pi y^2\, dx = \pi \int_0^a \frac{x^2(a-x)}{a+x}\, dx$$

$$= \pi \int_0^a \left(-x^2 + 2ax - 2a^2 + \frac{2a^3}{x+a}\right) dx$$

$$= \pi \Big[-\tfrac{1}{3}x^3 + ax^2 - 2a^2x + 2a^3 \log(x+a)\Big]_0^a$$

$$= \pi\left(-\tfrac{4}{3}a^3 + 2a^3 \log 2\right).$$

LXXII.

1. Let EF, GH be the given parallels, P being on EF. Produce BC to meet EF in Q, and take R on BC, such that

$$(RBCQ) = -1.$$

Now produce RA to meet EF in P, which will be the required point. Then

$$P(\alpha\beta\gamma\infty) = P(RBCQ) = -1,$$

$$\therefore\ \alpha\beta = \beta\gamma.$$

2. Let A, B be the given points, C, D the other vertices of the square of which AB is a diagonal. Let a, b, c, d be the perpendiculars from A, B, C, D on the line. Then

$$a^2+b^2-2cd=\tfrac{1}{2}(a+b)^2+\tfrac{1}{2}(a-b)^2-\tfrac{1}{2}(c+d)^2+\tfrac{1}{2}(c-d)^2.$$

But $a+b=$ twice the perpendicular from the centre of the square $=c+d$.

Also $a-b$, $c-d$ are the projections of AB and CD on a line perpendicular to the line in question, and therefore since AB, CD are equal and perpendicular

$$(a-b)^2+(c-d)^2=AB^2,$$

$$\therefore\quad a^2+b^2-2cd=\text{area of the square.}$$

But, by datum, a^2+b^2 is constant, therefore cd is constant, i.e. the envelope of the line is a conic with foci C and D.

3. Consider the function

$$\frac{1}{x(u_1-x)(u_2-x)\dots(u_n-x)}.$$

By the ordinary rule of partial fractions it is equal to

$$\Sigma\frac{1}{u_r(u_1-u_r)(u_2-u_r)\dots(u_n-u_r)}\cdot\frac{1}{u_r-x}+\frac{1}{u_1u_2\dots u_n}\cdot\frac{1}{x}.$$

Hence $$\Sigma=\frac{1}{x}\left\{\frac{1}{(u_1-x)(u_2-x)\dots(u_n-x)}-\frac{1}{u_1u_2\dots u_n}\right\}$$

$$=\frac{\Sigma u_1u_2\dots u_{n-1}+\text{terms in }x}{(u_1-x)\dots(u_n-x)\,u_1u_2\dots u_n}.$$

Now putting $x=0$, we get the required formula.

4. Taking the inequality

$$x^m-y^m>my^{m-1}(x-y),$$

where $x>y$, and m is a positive integer, put

$$x=2m-1\ \text{ and }\ y=m.$$

Then $$(2m-1)^m-m^m>m^m(m-1),$$

$$\text{i.e. }\ (2m-1)^m>m^{m+1},$$

$$\text{i.e. }\ m<\left(2-\frac{1}{m}\right)^m$$

$$\text{i.e. }\ \sqrt[m]{m}+\frac{1}{m}<2$$

taking the real positive value of $\sqrt{m}$.

5. We have to find the value of the product

$$P = (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2).$$

We have

$$y^3 + (2q - p^2) y^2 + (q^2 - 2pr) y - r^2 \equiv (y - a^2)(y - b^2)(y - c^2).$$

Putting $$y = \tfrac{1}{2}(p^2 - 2q) = \tfrac{1}{2}(a^2 + b^2 + c^2),$$

we find

$$(p^2 - 2q)^3 + 2(2q - p^2)(p^2 - 2q)^2 + 4(q^2 - 2pr)(p^2 - 2q) - 8r^2 \equiv P.$$

Hence $$P = -(p^2 - 2q)^3 + 4(q^2 - 2pr)(p^2 - 2q) - 8r^2$$

$$= -p^6 + 6p^4q - 8p^2q^2 - 8p^3r + 16pqr - 8r^2$$

and the triangle is acute-, right- or obtuse-angled according as

$$P \gtreqless 0.$$

Also, since

$$x^3 - px^2 + qx - r \equiv (x - a)(x - b)(x - c),$$

we have, putting $$x = s = \tfrac{1}{2}p,$$

$$-\tfrac{1}{8}p^3 + \tfrac{1}{2}pq - r = (s - a)(s - b)(s - c),$$

$$\therefore \quad \Delta = \tfrac{1}{4}\sqrt{4p^2q - 8pr - p^4}$$

and the radius of the circumcircle is $abc/4\Delta$.

6. Since $$k \cos \alpha = \frac{\cos \frac{1}{2}(\beta + \gamma)}{\cos \frac{1}{2}(\beta - \gamma)},$$

$$\therefore \quad \frac{1 - k \cos \alpha}{1 + k \cos \alpha} = \tan \tfrac{1}{2}\beta \tan \tfrac{1}{2}\gamma.$$

Hence putting $\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta \tan \frac{1}{2}\gamma = p$, α and β satisfy the equation

$$\frac{1 - k \cos \theta}{1 + k \cos \theta} \cdot \tan \tfrac{1}{2}\theta = p.$$

Putting $\tan \frac{1}{2}\theta = t$, this becomes

$$\frac{(1 + t^2) - k(1 - t^2)}{(1 + t^2) + k(1 - t^2)} \cdot t = p;$$

or $$(1 + k)t^3 - (1 - k)pt^2 + (1 - k)t - (1 + k)p = 0.$$

The product of the roots of this equation is p, therefore the third root is $\tan \frac{1}{2}\gamma$. Further

$$\frac{1-k}{1+k} \cdot p = \Sigma \tan \tfrac{1}{2}\alpha.$$

$$\therefore \quad \left(1 + \frac{1}{k}\right)\left(\Sigma \tan \frac{\alpha}{2} - p\right) = -2p$$

and multiplying through by $\cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma$ we get the third result.

Again $$\frac{\Sigma \tan \frac{1}{2}\alpha}{\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta \tan \frac{1}{2}\gamma} = \frac{1-k}{1+k} = \Sigma \tan \tfrac{1}{2}\beta \tan \tfrac{1}{2}\gamma,$$

$$\text{i.e.} \quad \Sigma \cot \tfrac{1}{2}\beta \cot \tfrac{1}{2}\gamma = \Sigma \tan \tfrac{1}{2}\beta \tan \tfrac{1}{2}\gamma.$$

But
$$\begin{aligned} &\cot \tfrac{1}{2}\beta \cot \tfrac{1}{2}\gamma - \tan \tfrac{1}{2}\beta \tan \tfrac{1}{2}\gamma \\ &\qquad = \frac{\cos \frac{1}{2}(\beta+\gamma) \cos \frac{1}{2}(\beta-\gamma)}{\sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma \cos \frac{1}{2}\beta \cos \frac{1}{2}\gamma} \\ &\qquad = \frac{2(\cos\beta + \cos\gamma)}{\sin\beta \sin\gamma}. \end{aligned}$$

$$\therefore \quad \Sigma \sin\alpha (\cos\beta + \cos\gamma) = 0, \qquad \text{i.e.} \quad \Sigma \sin(\beta+\gamma) = 0.$$

7. We have

$$(p + iq) \log \sin(\alpha + i\beta) = \log(A + iB).$$

Now $$\log \sin(\alpha + i\beta) = \log(\sin\alpha \cosh\beta + i \cos\alpha \sinh\beta).$$

Also $$\log(x + iy) = \tfrac{1}{2} \log(x^2 + y^2) + i \tan^{-1} \frac{y}{x}.$$

Hence $$\log \sin(\alpha + i\beta) = \tfrac{1}{2} \log(\sin^2\alpha \cosh^2\beta + \cos^2\alpha \sinh^2\beta) + i \tan^{-1}(\tanh\beta \cot\alpha).$$

Again
$$\begin{aligned} &\sin^2\alpha \cosh^2\beta + \cos^2\alpha \sinh^2\beta \\ &\qquad = \sin^2\alpha (1 + \sinh^2\beta) + \sinh^2\beta (1 - \sin^2\alpha) \\ &\qquad = \sin^2\alpha + \sinh^2\beta; \end{aligned}$$

$$\therefore \quad (p + iq)\{\tfrac{1}{2} \log(\sin^2\alpha + \sinh^2\beta) + i \tan^{-1}(\tanh\beta \cot\alpha)\} = \tfrac{1}{2} \log(A^2 + B^2) + i \tan^{-1} \frac{B}{A}.$$

Equating coefficients of i, the result follows.

8. The centre of curvature at (x', y') on $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$ is the point

$$x=\frac{a^2-b^2}{(a^2+\lambda)^2}x'^3,\qquad y=-\frac{a^2-b^2}{(b^2+\lambda)^2}y'^3,$$

and if $x'=a\cos\phi$, $y'=b\sin\phi$, we have

$$\frac{a^2\cos^2\phi}{a^2+\lambda}+\frac{b^2\sin^2\phi}{b^2+\lambda}=1,$$

whence $$\lambda=0 \text{ or } -(a^2\sin^2\phi+b^2\cos^2\phi).$$

For the second value

$$a^2+\lambda=(a^2-b^2)\cos^2\phi,\qquad b^2+\lambda=-(a^2-b^2)\sin^2\phi.$$

Hence the centre of curvature is

$$x=\frac{a^2-b^2}{(a^2+\lambda)^2}\cdot a^3\cos^3\phi=\frac{a^3}{(a^2-b^2)\cos\phi},$$

$$y=-\frac{a^2-b^2}{(b^2+\lambda)^2}\cdot b^3\sin^3\phi=-\frac{b^3}{(a^2-b^2)\sin\phi},$$

and its locus is

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

9. Taking the self-conjugate triangle as the triangle of reference, the equation to the rectangular hyperbola is of the form $l\alpha^2+m\beta^2+n\gamma^2=0$, with the condition $\Sigma l=0$. But this is also the condition that the points $(1, \pm 1, \pm 1)$ should lie on the curve; and these points are the centres of the inscribed and escribed circles.

10. Let A and B be the points of contact, O the centre of the ring. The cross-section of the pole by the plane of the ring is an ellipse of semi-axes $a\sec\theta$ and a. Let C be its centre and let AB meet OC in N. Then since AO is also normal to the ellipse,

$$\therefore\ CO=CN\tan^2\theta.$$

Since the eccentricity of the ellipse is $\sin\theta$,

$$\therefore\ CN=\frac{CO}{\tan^2\theta}=\frac{ON}{\sec^2\theta}.$$

Also $\qquad AN^2\cos^2\theta = a^2 - CN^2$ and $AN^2 = r^2 - ON^2$.

From these we easily find

$$AN^2 = \frac{a^2 - r^2\cos^4\theta}{\sin^2\theta\cos^2\theta}, \qquad CN^2 = \frac{r^2\cos^4\theta - a^2\cos^2\theta}{\sin^2\theta}.$$

Now let the pressures at A and B be each R, making angles ϕ with the axis of the pole. Then

$$\tan\phi = CN \div \text{projection of } AN \text{ perpendicular to the axis}$$
$$= \frac{CN}{AN\cos\theta}.$$

Now resolving vertically $W = 2R\sin\phi$, and the moment of the required couple is

$$G = R\cos\phi\,(2AN\sin\theta)$$
$$= W\cot\phi\,.\,AN\sin\theta$$
$$= W\sin\theta\cos\theta\,.\,\frac{AN^2}{CN}$$

as given.

11. Let A be the point of projection, B, C, D the points of subsequent impact, O the centre, and let $ABO = \alpha$, $BCO = \beta$, $CDO = \gamma$, $DAO = \delta$. Let m, M be the masses of the particle and hoop respectively, u the striking velocity, v and V the velocities of the particle and hoop after the first impact.

Then, for the impact at B,

$$mu\cos\alpha = MV - mv\cos\beta,$$
$$eu\cos\alpha = V + v\cos\beta.$$

Hence V and $v\cos\beta$ are both proportional to $u\cos\alpha$.

Now we can stop the hoop by giving the system a velocity V along BO, and the relative motion is then the same as if the particle left B with radial velocity $\lambda u\cos\alpha$ and tangential velocity $u\sin\alpha$, where λ is a constant depending on M, m and e.

Hence $\tan\beta = \frac{1}{\lambda}\tan\alpha$, and similarly $\tan\gamma = \frac{1}{\lambda}\tan\beta$, $\tan\delta = \frac{1}{\lambda}\tan\gamma$, whence $\tan\alpha\tan\delta = \tan\beta\tan\gamma$.

But $\Sigma\alpha = \pi$;

$$\therefore \quad (\tan\alpha + \tan\delta)(1 - \tan\beta\tan\gamma) + (\tan\beta + \tan\gamma)(1 - \tan\alpha\tan\delta) = 0.$$

Hence either $\tan\beta\tan\gamma = 1$, in which case $\beta + \gamma = \alpha + \delta = \frac{\pi}{2}$, and BD is a diameter, or else $\Sigma\tan\alpha = 0$,

$$\text{i.e.}\quad \tan\alpha\left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3}\right) = 0,$$

i.e. $\tan\alpha = 0$, and then all the impacts take place at the ends of a diameter.

12. Considering the coefficients of x^2 and y^2, we see that two of the asymptotes are $bx + b' = 0$, and $ay + a' = 0$. If the other is $lx + my + 1 = 0$, the curve may be written

$$(bx + b')(ay + a')(lx + my + 1) + l'x + m'y + n' = 0.$$

Identifying this with the given equation, we have

$$ab'l + a'bm + ab = 0, \qquad a'b' + n' = 0,$$

$$bl = am = \frac{a'b'l + a'b + l'}{a''} = \frac{a'b'm + ab' + m'}{b''}.$$

From these

$$l = -\frac{a^2b}{a^2b' + a'b^2}, \qquad m = -\frac{ab^2}{a^2b' + a'b^2},$$

giving the other asymptote.

Further

$$l' = (a''b - a'b')\,l - a'b = -\frac{(a^2a'' + a'^2b)\,b^2}{a^2b' + a'b^2},$$

$$m' = (ab'' - a'b')\,m - ab' = -\frac{(b^2b'' + ab'^2)\,a^2}{a^2b' + a'b^2},$$

$$n' = -a'b',$$

giving the required equation for the line $l'x + m'y + n' = 0$.

LXXIII.

1. Let P be the middle point of AA', O the circumcentre, U the nine-point centre, and let AL, perpendicular to BC, meet the circumcircle in D. Then since OA, AP are perpendicular, the circle on AA' as diameter cuts the circumcircle orthogonally.

Again, if H is the orthocentre, $HU = UO$ and $HL = LD$. Therefore UL is parallel to OD;

$$\therefore \; U\hat{L}A = O\hat{D}A = O\hat{A}D = B - C.$$

Also $\quad C\hat{A}A' = B, \quad \therefore \; C\hat{A'}A = B - C, \quad$ i.e. $\; P\hat{L}A' = B - C.$

Therefore ULP is a right angle, i.e. the circle on AA' cuts the nine-point circle orthogonally. Hence since the circle on AA' cuts both circles orthogonally, its centre P lies on their radical axis.

2. Let PQR be the triangle formed by the tangents, E and F the points of contact of PR, PQ, and complete the parallelogram $PQUR$. Then

$$EP : PR = PF : FQ;$$

$$\therefore \; EP : ER = PF : PQ = PF : RU.$$

Hence E, F and U are collinear, i.e. EF passes through the fixed point U. Similarly for the other chords.

3. If the first two digits are a and b, the third may have any value from 0 to 9 except a and b. Hence the sum of all such numbers is

$$8\,(100a + 10b) + (45 - a - b) = 799a + 79b + 45.$$

But if the first digit is a, the second may have any value from 0 to 9 except a. Hence the sum of all numbers in which the first digit is a is

$$9 \,.\, 799a + 79\,(45 - a) + 9 \,.\, 45 = 7112a + 3960.$$

Finally a may have any value from 1 to 9, therefore the sum of all the numbers is

$$\begin{aligned} 45 \,.\, 7112 + 9 \,.\, 3960 &= 320040 + 35640 \\ &= 355680. \end{aligned}$$

4. If u_n stands for either p_n or q_n, we have

$$u_n = (na+1)\,u_{n-1} - (n-1)\,a\,.\,u_{n-2},$$

$$\text{i.e.}\quad u_n - na\,.\,u_{n-1} = u_{n-1} - (n-1)\,a\,.\,u_{n-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$u_3 - 3au_2 = u_2 - 2au_1.$$

Also $u_2 - 2au_1$ is 1 for the numerators and 0 for the denominators,

$$\therefore\quad u_n - na\,.\,u_{n-1} = 1 \quad\text{or}\quad 0,$$

$$\text{i.e.}\quad \frac{u_n}{n\,!\,a^n} - \frac{u_{n-1}}{(n-1)\,!\,a^{n-1}} = \frac{1}{n\,!\,a^n} \quad\text{or}\quad 0,$$

whence
$$\frac{p_n}{n\,!\,a^n} = \frac{1}{1\,!\,a} + \frac{1}{2\,!\,a^2} + \dots + \frac{1}{n\,!\,a^n},$$

$$\frac{q_n}{n\,!\,a^n} = 1\,;$$

$$\therefore\quad \underset{n=\infty}{\text{Lt}}\,\frac{p_n}{q_n} = e^{\frac{1}{a}} - 1.$$

5. Denote the left side by $f(x)$. Then

$$f'(x) = 1 + \frac{x}{1!} + \dots + \frac{x^{n-1}}{(n-1)!} = f(x) - \frac{x^n}{n!},$$

$$\text{i.e.}\quad \frac{f'(x)}{f(x)} = 1 - \frac{x^n}{n!}\cdot\frac{1}{f(x)} \quad \cdots\cdots\cdots\cdots\cdots\text{(i)}.$$

Now
$$\frac{f'(x)}{f(x)} = \Sigma\,\frac{1}{x-a_r} = -\,\Sigma\,\frac{1}{a_r}\left(1 - \frac{x}{a_r}\right)^{-1}$$

$$= -\sum_{p=0}^{p=\infty} s_{p+1}\,x^p \quad\cdots\cdots\cdots\cdots\text{(ii)}.$$

But from (i) $\dfrac{f'(x)}{f(x)} = 1 - \dfrac{x^n}{n!} +$ higher powers of x.

Hence comparing the expansions, we have

$$s_2 = s_3 = \dots = s_n = 0, \qquad s_{n+1} = \frac{1}{n!}.$$

6. Putting $\theta = \frac{\pi}{4} - \phi$, $X = (x+y)\sqrt{2}$, $Y = (x-y)\sqrt{2}$, the equations easily become

$$X \cos\phi + Y \sin\phi = 5 + 3\cos 4\phi,$$

$$-X \sin\phi + Y\cos\phi = -2\sin 4\phi,$$

whence, solving

$$X = 5\cos\phi + 3\cos 4\phi \cos\phi + 2\sin 4\phi \sin\phi$$

$$= 5\cos\phi + \cos 4\phi\cos\phi + 2\cos 3\phi,$$

$$\therefore\quad 2X = 10\cos\phi + 5\cos 3\phi + \cos 5\phi = 16\cos^5\phi$$

and similarly $2Y = 10\sin\phi - 5\sin 3\phi + \sin 5\phi = 16\sin^5\phi$,

$$\therefore\quad X^{\frac{2}{5}} + Y^{\frac{2}{5}} = 2^{\frac{6}{5}},$$

$$\therefore\quad (x+y)^{\frac{2}{5}} + (x-y)^{\frac{2}{5}} = 2.$$

7. If $\tan n\theta$ is infinite, we have

$$1 - {}^nC_2 \tan^2\theta + {}^nC_4\tan^4\theta - \dots + (-1)^{\frac{n}{2}}\tan^n\theta = 0.$$

Writing this in reverse order, and putting $\tan^2\theta = x$, the equation is

$$x^{\frac{n}{2}} - \frac{n(n-1)}{2!}\,.\,x^{\frac{n}{2}-1} + \frac{n(n-1)(n-2)(n-3)}{4!}\,.\,x^{\frac{n}{2}-2} - \dots = 0,$$

and the roots are

$$\tan^2\frac{(2r+1)\pi}{2n} \qquad \left(r = 0,\ 1,\ \dots \frac{n-2}{2}\right),$$

and the sum of their squares is

$$\left\{\frac{n(n-1)}{2}\right\}^2 - \frac{n(n-1)(n-2)(n-3)}{12}$$

$$= \frac{n(n-1)}{12}\left[3n(n-1) - (n-2)(n-3)\right]$$

$$= \frac{n(n-1)}{6}(n^2 + n - 3).$$

8. Taking the triangle as triangle of reference, let the rectangular hyperbola be $\Sigma l\beta\gamma = 0$, with the general condition $\Sigma l \cos A = 0$, which here becomes

$$\Sigma l = 0 \quad \text{..............................(i).}$$

The tangents at A and B are $\frac{\beta}{m} + \frac{\gamma}{n} = 0$, $\frac{\gamma}{n} + \frac{a}{l} = 0$, intersecting in the point

$$\frac{a}{l} = \frac{\beta}{m} = \frac{-\gamma}{n} = \frac{2\Delta/a}{l + m - n} = -\frac{\Delta}{a} \cdot \frac{1}{n} \text{ by (i),}$$

and similarly for the other intersections.

Now the area of the triangle formed by the points (a_1, β_1, γ_1) etc. is

$$\frac{R}{2\Delta} \cdot \begin{vmatrix} a_1, & \beta_1, & \gamma_1 \\ a_2, & \beta_2, & \gamma_2 \\ a_3, & \beta_3, & \gamma_3 \end{vmatrix},$$

where R is the radius of the circumcircle. Here this becomes

$$\frac{R}{2\Delta} \cdot \left(\frac{\Delta}{a}\right)^3 \left(-\frac{1}{lmn}\right) \cdot \begin{vmatrix} -l, & m, & n \\ l, & -m, & n \\ l, & m, & -n \end{vmatrix},$$

which, using (i), and neglecting sign, is

$$\frac{2R\Delta^2}{a^3} = \frac{2R\Delta^2}{4R\Delta} = \tfrac{1}{2}\Delta.$$

9. Since the similar conic only intersects S in two finite points, its equation (non-homogeneous) must be of the form

$$S + la + m\beta + n\gamma = 0.$$

Substituting the co-ordinates of the points, and eliminating l, m, n, the result follows.

10. Since the moment of the second system about A is

$$\frac{aL}{2\Delta} \cdot \frac{2\Delta}{a} = L,$$

and so for B and C, the systems have the same moments about three distinct points in the plane and are therefore equivalent, for if not, the resultant of either system and the other system reversed would have to pass through three points not in the same straight line.

To find the resultant, resolve along and perpendicular to BC. If X, Y are the resolved parts, we get

$$2\Delta X = aL - bM\cos C - cN\cos B,$$

$$2\Delta Y = bM\sin C - cN\sin B;$$

$$\therefore\quad 4\Delta^2 R^2 = \Sigma a^2L^2 - 2\Sigma bcMN\cos A$$

$$= \Sigma a^2L^2 - \Sigma MN(b^2+c^2-a^2)$$

$$= \Sigma a^2(L-M)(L-N).$$

11. The direction of motion at P is along the radius and must therefore make an angle $90° - \theta$ with PO. If then it reaches P after time t, since its velocities perpendicular and parallel to OP after that time are

$$u\sin(\alpha-\theta) - g\cos\theta\,.\,t \quad\text{and}\quad u\cos(\alpha-\theta) - g\sin\theta\,.\,t,$$

we must have

$$\frac{u\sin(\alpha-\theta) - g\cos\theta\,.\,t}{u\cos(\alpha-\theta) - g\sin\theta\,.\,t} = -\cot\theta,$$

whence
$$t = \frac{u\cos(\alpha-2\theta)}{2g\sin\theta\cos\theta}.$$

But in time t, the distance described perpendicular to OP is zero;

$$\therefore\quad u\sin(\alpha-\theta)\,t - \tfrac{1}{2}g\cos\theta\,.\,t^2 = 0,$$

giving
$$t = \frac{2u\sin(\alpha-\theta)}{g\cos\theta}.$$

$$\therefore\quad \cos(\alpha-2\theta) = 4\sin\theta\sin(\alpha-\theta) = 2\left[\cos(\alpha-2\theta) - \cos\alpha\right];$$

$$\therefore\quad 2\cos\alpha = \cos(\alpha-2\theta),$$

whence
$$\tan\alpha = \frac{2-\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta + 3\sin^2\theta}{2\sin\theta\cos\theta},$$

$$\text{i.e.}\quad 2\tan\alpha = \cot\theta + 3\tan\theta.$$

12. We have

$$\frac{\partial V}{\partial u} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= (1+v)\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y};$$

$$\therefore \quad u\frac{\partial V}{\partial u} = x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} \quad \text{......................(i)},$$

and similarly we find

$$\frac{1}{u}\frac{\partial V}{\partial v} = \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} \quad \text{......................(ii)}.$$

We have thus the equations of operators

$$u\frac{\partial}{\partial u} \equiv x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad \frac{1}{u}\frac{\partial}{\partial v} \equiv \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Operating with these on (i) and (ii) respectively, we get

$$u\frac{\partial V}{\partial u} + u^2\frac{\partial^2 V}{\partial u^2} = x^2\frac{\partial^2 V}{\partial x^2} + 2xy\frac{\partial^2 V}{\partial x \partial y} + y^2\frac{\partial^2 V}{\partial y^2} + x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y},$$

$$\text{i.e.} \quad u^2\frac{\partial^2 V}{\partial u^2} = x^2\frac{\partial^2 V}{\partial x^2} + 2xy\frac{\partial^2 V}{\partial x \partial y} + y^2\frac{\partial^2 V}{\partial y^2} \quad \text{......(iii)},$$

and

$$\frac{1}{u^2}\frac{\partial^2 V}{\partial v^2} = \frac{\partial^2 V}{\partial x^2} + 2\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y^2} \quad \text{..............(iv)}.$$

Multiplying (iv) by $xy\,[=u^2v\,(1+v)]$, and subtracting from (iii), the result follows, since $x-y=u$.

LXXIV.

1. Suppose O, the orthocentre, is inside the circle on AA'. Let U be the middle point of AA', and let the circle cut BC again in L, which is therefore the foot of the perpendicular from A.

Hence AL passes through O, so that the rectangle under the segments of any chord through O is $AO\,.\,OL$, and similarly for the other circles. But if BM, CN are the other perpendiculars, we know that

$$AO\,.\,OL = BO\,.\,OM = CO\,.\,ON,$$

therefore O is the radical centre.

2. Let VL be the perpendicular from the vertex on the section, AN, $A'N'$ the perpendiculars from A, A' on the axis. Then the volume of the portion of the cone is

$$\tfrac{1}{3} VL \times (\text{area of section}) = \tfrac{1}{3} VL \times \pi ab.$$

Now $VL \,.\, a =$ area of $\triangle VAA'$, which varies as $VA \,.\, VA'$.

Also $b^2 = AN \,.\, A'N'$, and since both the ratios

$$AN : AV \quad \text{and} \quad A'N' : A'V$$

are constant,

$$\therefore \quad b^2 \text{ varies as } VA \,.\, VA',$$

i.e. b varies as $(VA \,.\, VA')^{\frac{1}{2}}$. Hence the volume varies as

$$VL \,.\, ab, \quad \text{i.e. } (VA \,.\, VA')^{\frac{3}{2}}.$$

3. The rational form of $\Sigma\alpha^{\frac{1}{2}} = 0$ is

$$\Sigma\alpha^2 - 2\Sigma\beta\gamma = 0, \text{ or } \Sigma(\beta - \gamma)^2 = \Sigma\alpha^2 \ldots\ldots\ldots\ldots\text{(i)}.$$

But here, putting $\quad \Sigma x = p, \quad \Sigma x^2 - 2\Sigma xy = q,$

we have $\quad \alpha = 2px + q, \quad$ etc.,

so that $\quad \beta - \gamma = 2(y - z)p\,;$

$$\therefore \quad \Sigma(\beta - \gamma)^2 = 8p^2(\Sigma x^2 - \Sigma xy)$$

$$= 8p^2 \,.\, \frac{p^2 + 3q}{4} = 2p^2(p^2 + 3q) \ldots\ldots\ldots\text{(ii)}.$$

Also $\quad \Sigma\alpha^2 = 4p^2 \,.\, \Sigma x^2 + 4p^2q + 3q^2$

$$= 4p^2 \,.\, \frac{p^2 + q}{2} + 4p^2q + 3q^2$$

$$= 2p^4 + 6p^2q + 3q^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

From (i), (ii), and (iii), it appears that $q = 0$, so that the rational form required is $\Sigma x^2 = 2\Sigma xy$.

4. Adding the rows, we see that Σa_1 is a factor. Removing this, and then subtracting the first column from each of the others, the remaining determinant is equal to

$$\begin{vmatrix} 1, & 0, & 0, & 0 & \dots, & 0 \\ a_n, & -(n-1)r, & -(n-2)r, & -(n-3)r & \dots, & -r \\ a_{n-1}, & r, & -(n-2)r, & -(n-3)r & \dots, & -r \\ a_{n-2}, & r, & 2r, & -(n-3)r & \dots, & -r \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_2, & r, & 2r, & 3r & \dots, & -r \end{vmatrix}.$$

Now, suppressing the first row and column of this determinant, and then subtracting each row from the one above, we get a determinant in which every element on the right of the leading diagonal is zero, and every element of the leading diagonal, except the last, is $-nr$, the last being $-r$. Hence the value of the determinant is

$$(-nr)^{n-2}(-r) = \frac{(-nr)^{n-1}}{n}.$$

But $\Sigma a_1 = \frac{n}{2}\{2a_1 + (n-1)r\}$. Hence the result.

5. Let $a_1, a_2, \dots a_n$ be the roots of $f(x) = 0$. Then

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \dots + \frac{1}{x-a_n}.$$

Hence the function on the right will vanish when x is put equal to any root of $f'(x) = 0$. Hence, putting x equal to each of the roots of $f'(x) = 0$ in succession, and adding, we obtain the result.

6. It is evident that the triangles are equiangular. If, therefore, they are proved equal in area, they must be equal in all respects.

Now the area of ABC is $\Delta = \frac{1}{2}a^2 . \frac{\sin B \sin C}{\sin A}$, whence also

$$a^2 = 2\Delta(\cot B + \cot C);$$

$$\therefore\quad \Delta - \Delta' = \tfrac{1}{2}a^2 \cdot \frac{\sin\theta\sin(C-\theta)}{\sin C} + \tfrac{1}{2}b^2 \cdot \frac{\sin\theta\sin(A-\theta)}{\sin A} + \tfrac{1}{2}c^2 \cdot \frac{\sin\theta\sin(B-\theta)}{\sin B} \quad\ldots\ldots\text{(i)}$$

$$= \Delta\sin^2\theta\,[(\cot B + \cot C)(\cot\theta - \cot C) + (\cot C + \cot A)(\cot\theta - \cot A) + (\cot A + \cot B)(\cot\theta - \cot B)]$$

$$= \Delta\sin^2\theta\,[2\cot\theta \cdot \Sigma\cot A - \Sigma\cot^2 A - \Sigma\cot B\cot C].$$

Now $$\Sigma\cot B\cot C = 1,$$

$$\therefore\quad \operatorname{cosec}^2\theta = \cot^2\theta + 1 = \cot^2\theta + \Sigma\cot B\cot C,$$

$$\therefore\ \Delta' = \Delta\sin^2\theta[\cot^2\theta - 2\cot\theta \cdot \Sigma\cot A + \Sigma\cot^2 A + 2\Sigma\cot B\cot C]$$
$$= \Delta\sin^2\theta\,(\cot\theta - \Sigma\cot A)^2.$$

It is evident also that Δ'' is equal to the same expression, since the only difference is that in (i) the sides occur in the order b^2, c^2, a^2, and the ultimate result is the same.

7. Since $$\coth u = \frac{1}{u} + 2u \cdot \sum_1^\infty \frac{1}{u^2 + r^2\pi^2},$$

we have $$\coth 1 = 1 + 2S,$$

where S is the series in question;

$$\therefore\quad 2S = \frac{e + e^{-1}}{e - e^{-1}} - 1 = \frac{2}{e^2 - 1}.$$

8. Any conic passing through the extremities of the principal axes is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 2\lambda xy,$$

and the tangent at (x', y') is

$$x\left(\frac{x'}{a^2} - \lambda y'\right) + y\left(\frac{y'}{b^2} - \lambda x'\right) = 1 \quad\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The tangent to the given hyperbola at this point is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = \frac{a^2 - b^2}{a^2 + b^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

The lines (i) and (ii) are perpendicular if

$$\frac{x'}{a^2}\left(\frac{x'}{a^2}-\lambda y'\right)-\frac{y'}{b^2}\left(\frac{y'}{b^2}-\lambda x'\right)=0,$$

$$\text{i.e.}\quad \frac{x'^2}{a^4}-\frac{y'^2}{b^4}-\lambda x'y'\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0,$$

$$\text{i.e.}\quad 2\left(\frac{x'^2}{a^4}-\frac{y'^2}{b^4}\right)-\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=0,$$

$$\text{i.e.}\quad \left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{a^2}-\frac{1}{b^2}=0,$$

and this is true by virtue of the equation to the hyperbola.

9. If the tangential equation to a conic is

$$Al^2+Bm^2+Cn^2+2Fmn+\ldots+\ldots=0,$$

then its point equation is

$$\begin{vmatrix} A, & H, & G, & x \\ H, & B, & F, & y \\ G, & F, & C, & z \\ x, & y, & z, & 0 \end{vmatrix}=0.$$

Now the lines joining A to the points where the given conic cuts BC are $by^2+cz^2+2fyz=0$, or, say,

$$(my+nz)(m'y+n'z)=0.$$

If each of these lines touches the conic whose tangential equation is

$$A'l^2+\ldots+\ldots+2F'mn+\ldots+\ldots=0,$$

we have

$$B'm^2+C'n^2+2F'mn=0,$$

$$B'm'^2+C'n'^2+2F'm'n'=0,$$

whence

$$\frac{B'}{nn'}=\frac{C'}{mm'}=\frac{2F'}{-(mn'+m'n)},$$

$$\text{i.e.}\quad \frac{B'}{c}=\frac{C'}{b}=\frac{F'}{-f},$$

or

$$\frac{B'}{ca}=\frac{C'}{ab}=\frac{F'}{-af}.$$

Dealing with the other pairs of lines in the same way, we shall obtain the results

$$\frac{A'}{bc} = \frac{B'}{ca} = \frac{C'}{ab} = \frac{F'}{-af} = \frac{G'}{-bg} = \frac{H'}{-ch},$$

and the point equation will be that given.

10. Let c be the diameter of the circular hole, h the height of the cone, 2α the vertical angle. If the axis is not vertical, the cone will be in contact at A and B, the ends of a diameter of the hole. Let the axis of the cone meet AB in C, and let $VC = x$, V being the vertex, and suppose the axis makes an angle θ with the horizontal. Then

$$\frac{AC}{\sin\alpha} = \frac{x}{\sin(\theta+\alpha)}, \quad \frac{CB}{\sin\alpha} = \frac{x}{\sin(\theta-\alpha)};$$

$$\therefore \quad \frac{c}{\sin\alpha} = x \cdot \frac{2\sin\theta\cos\alpha}{\sin(\theta+\alpha)\sin(\theta-\alpha)}.$$

Hence, if z be the height of the centre of gravity above the hole,

$$z = (\tfrac{3}{4}h - x)\sin\theta = \tfrac{3}{4}h\sin\theta - \frac{c(\cos 2\alpha - \cos 2\theta)}{2\sin 2\alpha}.$$

For equilibrium this is a maximum or minimum,

$$\text{i.e.} \quad z' = \tfrac{3}{4}h\cos\theta - \frac{c\sin 2\theta}{\sin 2\alpha} = 0,$$

giving $$\cos\theta = 0, \quad \text{or} \quad \sin\theta = \frac{3h\sin 2\alpha}{8c},$$

so that the symmetrical position is not, or is, the only position of equilibrium, according as

$$3h\sin 2\alpha \lessgtr 8c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

To test for stability, consider

$$z'' = -\tfrac{3}{4}h\sin\theta - \frac{2c\cos 2\theta}{\sin 2\alpha}.$$

Its value when $\theta = \frac{\pi}{2}$ is

$$-\tfrac{3}{4}h + \frac{2c}{\sin 2\alpha},$$

which is positive or negative under the same conditions as (i), proving the result, since this position is stable or unstable according as z'' is positive or negative.

11. Since the horizontal distance described is the same in both cases, we must have

$$\frac{\cos\theta\sin(\theta-\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{\cos(\theta+\delta)\sin(\theta+\delta-\alpha)}{\cos\alpha}.$$

If β and δ are both small, this takes the form

$$\frac{\cos\theta\,(\sin\overline{\theta-\alpha}+\beta\cos\overline{\theta-\alpha})}{\cos\alpha+\beta\sin\alpha}$$

$$= \frac{(\cos\theta-\delta\sin\theta)\,(\sin\overline{\theta-\alpha}+\delta\cos\overline{\theta-\alpha})}{\cos\alpha},$$

and neglecting products $\beta\delta$ and δ^2, this takes the form

$$\beta\cos\theta\cos\alpha\cos(\theta-\alpha)$$

$$= \beta\cos\theta\sin\alpha\sin(\theta-\alpha) + \delta\cos\alpha\cos(2\theta-\alpha),$$

or $\quad \beta\cos^2\theta = \frac{1}{2}\delta\left[\cos 2\theta + \cos 2(\theta-\alpha)\right] = \delta\left[\cos^2\theta - \sin^2(\theta-\alpha)\right].$

12. The polar equation to the parabola, the vertex being the pole, is

$$r = \frac{4a\cos\theta}{\sin^2\theta},$$

and the area required is

$$\frac{1}{2}\int_\alpha^\beta r^2\,d\theta = 8a^2\int_\alpha^\beta \frac{\cos^2\theta}{\sin^4\theta}\,.\,d\theta$$

$$= -8a^2\int_\alpha^\beta \cot^2\theta\,.\,d(\cot\theta)$$

$$= -8a^2\left[\tfrac{1}{3}\cot^3\theta\right]_\alpha^\beta$$

$$= \tfrac{8}{3}a^2(\cot^3\alpha - \cot^3\beta).$$

LXXV.

1. Let the perpendiculars from E and F meet in O. Then

$$AO^2 - OB^2 = AF^2 - BF^2,$$

$$AO^2 - OC^2 = AE^2 - CE^2;$$

$$\begin{aligned}\therefore \quad OC^2 - OB^2 &= (AF^2 - AE^2) - (BF^2 - CE^2) \\ &= DF^2 - ED^2 - (BE^2 - CF^2) \\ &= CD^2 - BD^2,\end{aligned}$$

and therefore OD is perpendicular to BC.

2. Let AA', BB', CC' be the pairs of points of intersection. Then the three chords of intersection meet in the radical centre O, and we have

$$OA \, . \, OA' = OB \, . \, OB' = OC \, . \, OC'.$$

Hence if the six points lie on a conic, the diameters of this conic parallel to AA', BB', CC' are equal, and this is impossible unless the directions of two of them coincide, i.e. unless BB', CC' are parallel, which is only possible when the centres of the circles are collinear.

3. The number of ways of dividing p things into groups containing α, β, γ, ... respectively, where $\alpha + \beta + \gamma + \ldots = n$, is

$$\frac{p!}{\alpha! \, \beta! \, \gamma! \ldots}.$$

Putting $p = 3n$, and $\alpha = \beta = \gamma = \ldots$, this becomes $\dfrac{3n!}{(3!)^n}$.

But this regards as different all the possible orders in which the n groups can occur in any one method of subdivision, and the number of these orders is $n!$.

Hence the number of distinct ways of distribution is

$$\frac{3n!}{(3!)^n \, n!} = \frac{3(3n-1)!}{6^n (n-1)!}.$$

4. By the method of differences, we easily find the $(n+1)$th numerator to be $2n^2+2n+1$, and the $(n+1)$th denominator $2n^2+3n+1$, so that the coefficient of x^n is

$$\frac{2n^2+2n+1}{2n^2+3n+1}=1-\frac{n}{2n^2+3n+1}$$

$$=1-\left(\frac{1}{n+1}-\frac{1}{2n+1}\right),$$

so that the sum is

$$\sum_0^\infty x^n-\sum_0^\infty \frac{1}{n+1}x^n+\sum_0^\infty \frac{1}{2n+1}x^n$$

$$=\frac{1}{1-x}+\frac{1}{x}\log(1-x)+\frac{1}{2\sqrt{x}}\log\frac{1+\sqrt{x}}{1-\sqrt{x}}.$$

5. Suppose the roots of the equation in question are $\tan\alpha$, $\tan\beta$, $\tan\gamma$, $\tan\delta$, and let T_r denote the sum of their products r together. Then

$$\tan(\alpha+\beta+\gamma+\delta)=\frac{T_1-T_3}{1-T_2+T_4}$$

$$=\frac{p-r}{1-q+s}.$$

But if $\alpha+\beta+\gamma=\pi$, then

$$\tan(\alpha+\beta+\gamma+\delta)=\tan(\pi+\delta)=\tan\delta.$$

Hence the fourth root, viz. $\tan\delta$, has the value stated.

6. Suppose the circle cuts AB, AC in D, E respectively. Then

$$AB\,.\,AD=AC\,.\,AE=bc\cos^2\frac{A}{2}=\sqrt{s(s-a)}=AL\,.\,AL',$$

where L, L' are the points of contact of the inscribed circle and the escribed circle opposite A with AB or AC. Hence if we invert the figure from A, the constant of inversion being $\sqrt{s(s-a)}$, the circle ADE will invert into the line BC, and the inscribed and escribed circles will invert into each other. But BC touches both these circles; therefore its inverse touches both their inverses, i.e. the circle ADE touches the same two circles.

7. We have

$$\frac{AD}{AB}=\frac{\sin\dfrac{3\pi}{7}}{\sin\dfrac{\pi}{7}}=\frac{\sin\dfrac{3\pi}{7}}{\sin\dfrac{6\pi}{7}}=\frac{1}{2\cos\dfrac{3\pi}{7}}.$$

Now if $\sin 4\theta=\sin 3\theta$, then either $\theta=r\pi$ or $7\theta=(2r+1)\pi$. Hence if we divide by $\sin\theta$, and put $\cos\theta=x$, the roots of the equation in x are $\cos\dfrac{\pi}{7}$, $\cos\dfrac{3\pi}{7}$, $\cos\dfrac{5\pi}{7}$.

But this equation is

$$4\cos\theta\cos 2\theta=3-4\sin^2\theta,$$

$$\text{i.e.}\quad 4x(2x^2-1)=4x^2-1,$$

and the substitution $y=\dfrac{1}{2x}$ makes this

$$1-2y^2=y-y^3,$$

which is the equation in question.

8. Since $\rho=\dfrac{b'^3}{ab}$, therefore $(\rho ab)^{\frac{2}{3}}=a^2\sin^2\phi+b^2\cos^2\phi$, where ϕ is the eccentric angle of the point.

Now if the circle of curvature at θ intersects the curve again in δ, we have $3\theta+\delta=2n\pi$, i.e. $\sin 3\theta=-\sin\delta$,

$$\text{i.e.}\quad 4\sin^3\theta-3\sin\theta-\sin\delta=0\ \ldots\ldots\ldots\ldots\ldots(\text{i}).$$

Hence if α, β, γ are the three points, the circles of curvature at which pass through δ, then $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are the roots of the equation (i) in $\sin\theta$;

$$\therefore\quad \Sigma\sin\alpha=0,\quad \Sigma\sin\alpha\sin\beta=-\tfrac{3}{4},$$

whence $\Sigma\sin^2\alpha=\frac{3}{2}$, and therefore also $\Sigma\cos^2\alpha=\frac{3}{2}$.

Hence $\Sigma(\rho ab)^{\frac{2}{3}}=a^2\,.\,\Sigma\sin^2\alpha+b^2\,.\,\Sigma\cos^2\alpha=\frac{3}{2}(a^2+b^2)$.

9. The equation to the system of conics is

$$ax^2+2hxy+(b+\lambda)y^2+2fy=0,$$

and the equation to their director-circles, formed by calculating the tangential coefficients, is

$$(ab-h^2+\lambda a)(x^2+y^2)-2hfx+2afy-f^2=0.$$

Putting $ab - h^2 + \lambda a = p$, this becomes

$$\left(x - \frac{hf}{p}\right)^2 + \left(y + \frac{af}{p}\right)^2 = \frac{f^2}{p} + \frac{(a^2 + h^2) f^2}{p^2}.$$

For the limiting points the right side must vanish, giving $p = \infty$ or $p = -(a^2 + h^2)$, and the co-ordinates being $\left(\frac{hf}{p}, -\frac{af}{p}\right)$, these are

$$(0, 0) \text{ and } \left(-\frac{hf}{a^2 + h^2}, \frac{af}{a^2 + h^2}\right).$$

10. Taking the vertical section in the plane of the string, let α be the angle which the line joining the centre of the upper circle to the centre of one of the lower ones makes with the vertical, θ the angle which the same line makes with that part of the string which is a common tangent to the same two circles. Let R be the reaction between the upper cylinder and either of the lower ones.

Then resolving vertically for the upper cylinder,

$$2T \cos(\alpha + \theta) + W = 2R \cos \alpha,$$

and horizontally for the lower,

$$T + T \sin(\alpha + \theta) = R \sin \alpha.$$

From these we get

$$2T(\sin \theta + \cos \alpha) = W \sin \alpha.$$

But $\sin \alpha = \dfrac{a}{a + b}$, $\sin \theta = \dfrac{a - b}{a + b}$, and substituting these we obtain the value of T given.

11. Let θ be the angle which the direction of projection makes with the side a, α the inclination of the direction of motion to the side a after the first impact, x the side of the rhombus. Then the necessary equations are

$$\cot \alpha = e \cot \theta \quad \text{......................(i)},$$

$$x(\cos \theta + \cos \alpha) = a \quad \text{....................(ii)},$$

$$x(\sin \theta + \sin \alpha) = b \quad \text{...................(iii)}.$$

From (ii) and (iii), we get

$$\tan\frac{\theta+\alpha}{2}=\frac{b}{a}; \qquad \therefore\ \tan(\theta+\alpha)=\frac{2ab}{a^2-b^2}.$$

But

$$\tan(\theta+\alpha)=\frac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}=\frac{e+1}{e\cot\theta-\tan\theta} \quad \text{from (i).}$$

Equating these values, the result follows.

12. The curve has one real asymptote, viz. the axis of y, and is symmetrical in opposite quadrants. If we transfer the axes by writing

$$X=\frac{x+y}{\sqrt{2}}, \quad Y=\frac{y-x}{\sqrt{2}},$$

the equation becomes

$$X^3-Y^3+2a^2Y=0,$$

or

$$X^3=Y(Y^2-2a^2)\dots\dots\dots\dots\dots\text{(i)}.$$

The equation (i) shews that there is also symmetry in opposite quadrants referred to the new axes, and therefore, since the tangent at the origin is now $Y=0$, the origin must be a point of inflexion.

Further, differentiating (i), we find that $\frac{dY}{dX}$ vanishes when $X=0$, i.e. at the origin and the points $(0, \pm\sqrt{2}a)$. Also from (i), if Y is a little greater than $\sqrt{2}a$, X is small and positive, while if Y is a little less than $\sqrt{2}a$, X is small and negative. Therefore the point $(0, \sqrt{2}a)$ must be a point of inflexion, and similarly for the other point. Hence the points of inflexion lie on $X=0$, or, referring to the original axes,

$$x+y=0.$$

LXXVI.

1. Let PQ and $P'Q'$ be the two given pairs of points, supposed outside the given circle, and suppose A and B are the points required, so that AB is the radical axis of any pair of the three circles. Let C be the centre of the given circle, C' that of the circle through P and Q. Draw PM, QN perpendicular to AB. Then the square of the tangent from P to the given circle equals $2PM \cdot CC'$, and similarly for Q. Hence $PM : QN$ is a given ratio. But if PQ, AB produced meet in O, then $PM : QN = OP : OQ$. Therefore O is a given point. Similarly, if $P'Q'$ meets AB in O', then O' is a given point, and OO' is the common radical axis.

2. Let QQ' be any chord subtending a right angle at the centre, V its middle point. Let CV meet the ellipse in P, and let CD be the diameter conjugate to CP. Draw CH perpendicular to QQ', meeting the tangent at P in K. Then

$$\begin{aligned} CD^2 : CP^2 &= QV^2 : CP^2 - CV^2 \\ &= CV^2 : CP^2 - CV^2, \quad \text{since } QV = CV, \\ &= CH^2 : CK^2 - CH^2; \end{aligned}$$

$$\therefore \quad CD^2 : a^2 + b^2 = CH^2 : CK^2.$$

Hence if A, B be extremities of the axes, $CH \cdot AB = CD \cdot CK$. But $CD \cdot CK = ab$. Therefore $CH \cdot AB = ab$, shewing that CH is equal to the radius of the circle in question.

3. If in the identity

$$(1 + x_1) - x_1(1 + x_2) + x_1x_2(1 + x_3) - \dots$$
$$+ (-1)^{n-1} x_1x_2 \dots x_{n-1}(1 + x_n) \equiv 1 + (-1)^{n-1} x_1x_2 \dots x_n,$$

we put

$$x_1 = \frac{n-1}{k}, \; x_2 = \frac{n-2}{k+1} \dots x_{n-1} = \frac{1}{k+n-2}, \; x_n = 0,$$

we obtain

$$\frac{n+k-1}{k} - \frac{n-1}{k} \cdot \frac{n+k-1}{k+1} + \frac{(n-1)(n-2)}{k(k+1)} \cdot \frac{n+k-1}{k+2} + \dots = 1,$$

or, denoting the given series by S,

$$(n+k-1)\left(\frac{1-S}{n}\right)=1,$$

$$\text{i.e.}\quad S=\frac{k-1}{n+k-1}.$$

4. The problem is that of dividing n units into three groups, blanks being allowed. When there are only two groups, the number of ways is $1+\frac{n}{2}$ or $1+\frac{n-1}{2}$, according as n is even or odd, i.e. in either case $I\left(\frac{n+2}{2}\right)$.

Now, suppose that in any division into three groups the smallest group contains x units, then the other two groups must each contain at least x, and we have to distribute the remaining $(n-3x)$ units between these two groups. This can be done in $I\left(\frac{n-3x+2}{2}\right)$ ways.

Now the greatest and least possible values of x are $I\left(\frac{n}{3}\right)$ and zero. Hence the number of ways required is

$$\sum_0^k I\left(\frac{n-3x+2}{2}\right),\quad \text{where } k=I\left(\frac{n}{3}\right)\ldots\ldots\ldots\ldots\text{(i)}.$$

Now if m is an integer, $I\left(\frac{m}{2}\right)$ is either $\frac{m}{2}$ or $\frac{m}{2}-\frac{1}{2}$, according as m is even or odd. Thus the sum (i) takes one of the forms

$$S-\frac{1}{2}\cdot\frac{k}{2},\quad S-\frac{1}{2}\cdot\frac{k+1}{2}\quad\text{or}\quad S-\frac{1}{2}\cdot\frac{k+2}{2},$$

where

$$\begin{aligned}S&=\sum_0^k\left(\frac{n-3x+2}{2}\right)\\&=\frac{n+2}{2}(k+1)-\frac{3}{2}\cdot\frac{k(k+1)}{2}\\&=\frac{k+1}{2}\left(n+2-\frac{3}{2}k\right).\end{aligned}$$

Now $n = 3k + s$, where s is 0, 1 or 2. Hence the sum (i) is

$$\frac{n-s+3}{6} \cdot \frac{n+s+4}{2} - \frac{n-s}{12} - f \quad (f = 0, \tfrac{1}{4} \text{ or } \tfrac{1}{2})$$

$$= \frac{n^2+6n}{12} + 1 - \frac{s^2}{12} - f,$$

and $\dfrac{s^2}{12} + f$ is always a proper fraction, so that this is

$$I\left(\frac{n^2+6n}{12}\right) + 1.$$

5. If the equation has equal roots, they must be real, for otherwise there would be two pairs of equal roots, and the expression in x would be a perfect square, which is impossible in this case, since the term in x^3 is wanting, and the term in x is not.

If the equation has two roots equal to a, its form must be

$$(x^2 - 2ax + a^2)\left(x^2 + 2ax + \frac{s}{a^2}\right) = 0,$$

whence by comparison

$$-3a^2 + \frac{s}{a^2} = q, \qquad \text{i.e. } 3a^4 + qa^2 - s = 0.$$

This equation in a^2 must have real roots. Therefore $q^2 + 12s$ must be positive. Again

$$q^2 - 4s = \left(\frac{s}{a^2} - 3a^2\right)^2 - 4s = \left(\frac{s}{a^2} - 9a^2\right)\left(\frac{s}{a^2} - a^2\right).$$

If this is negative, $\dfrac{s}{a^2} - a^2$ must be positive, and therefore the roots of the second quadratic in x are imaginary.

6. The denominator of the first fraction is

$$\sin\left(\frac{B+C}{2} - \frac{A}{2}\right) + \sin\frac{A}{2} - \sin\frac{B}{2} - \sin\frac{C}{2}$$

$$= 2\sin\frac{B+C}{4}\cos\left(\frac{B+C}{4} - \frac{A}{2}\right) - 2\sin\frac{B+C}{4}\cos\frac{B-C}{4}$$

$$= 4\sin\frac{B+C}{4}\sin\frac{B-A}{4}\sin\frac{A-C}{4}.$$

We have therefore to prove that

$$\Sigma \sin A \sin \frac{C+A}{4} \sin \frac{A+B}{4} \sin \frac{B-C}{4} = 0.$$

Now

$$4 \sin \frac{C+A}{4} \sin \frac{A+B}{4} \sin \frac{B-C}{4}$$

$$= 2\left[\cos \frac{B-C}{4} - \cos\left(\frac{A}{2} + \frac{B+C}{4}\right)\right] \sin \frac{B-C}{4}$$

$$= \sin \frac{B-C}{2} - \cos \frac{C}{2} + \cos \frac{B}{2},$$

and $\Sigma \sin A \left(\sin \frac{B-C}{2} - \cos \frac{C}{2} + \cos \frac{B}{2}\right)$

$$= \Sigma \cos \frac{A}{2} (\sin B - \sin C) + \Sigma \sin A \left(\cos \frac{B}{2} - \cos \frac{C}{2}\right)$$

$$\equiv 0.$$

7. Calling the series S, the number of terms is $\frac{n-1}{2}$;

$$\therefore \quad 2S = \frac{n-1}{2} + \cos \frac{2\pi}{n} + \cos \frac{6\pi}{n} + \dots + \cos \frac{2(n-2)\pi}{n}$$

$$= \frac{n-1}{2} + \frac{\cos \frac{(n-1)\pi}{n} \sin \frac{(n-1)\pi}{n}}{\sin \frac{2\pi}{n}}$$

$$= \frac{n-1}{2} + \frac{\sin \frac{2(n-1)\pi}{n}}{2 \sin \frac{2\pi}{n}} = \frac{n-1}{2} - \frac{1}{2};$$

$$\therefore \quad S = \frac{n-2}{4}.$$

8. The tangents from (x', y') to $y^2 = 4ax$ are

$$(y^2 - 4ax)(y'^2 - 4ax') = [yy' - 2a(x + x')]^2,$$

and the parallel lines through the origin are

$$ax^2 - y'xy + x'y^2 = 0.$$

The bisectors of the angles between these are

$$\frac{x^2 - y^2}{a - x'} = \frac{2xy}{-y'},$$

and these are also the bisectors of the angles between the lines

$$xyy' + y^2 (a - x') = 0,$$

$$\text{i.e.}\quad y = 0 \quad \text{and} \quad xy' + y (a - x') = 0,$$

and these are respectively the axis and the parallel through the origin to the line joining (x', y') to the focus.

9. If the triangle of reference is equilateral, the equation

$$u\alpha^2 + \ldots + \ldots + 2u'\beta\gamma + \ldots + \ldots = 0$$

will represent a circle if

$$v + w - 2u' = w + u - 2v' = u + v - 2w'.$$

Now the tangents from $(\alpha', \beta', \gamma')$ to $\beta^2 + \gamma^2 - 3\alpha^2 = 0$ are

$$(\beta^2 + \gamma^2 - 3\alpha^2)(\beta'^2 + \gamma'^2 - 3\alpha'^2) = (\beta\beta' + \gamma\gamma' - 3\alpha\alpha')^2,$$

and these satisfy the conditions for a circle if

$$(\beta' + \gamma')^2 - 6\alpha'^2 = -2\beta'^2 - 3(\gamma' + \alpha')^2 = -2\gamma'^2 - 3(\alpha' + \beta')^2,$$

and these equations are evidently satisfied by the point $(-1, 1, 1)$, which is therefore a focus, and the corresponding directrix is

$$\beta\beta' + \gamma\gamma' - 3\alpha\alpha' = 0,$$

$$\text{i.e.}\quad \beta + \gamma + 3\alpha = 0.$$

10. Let O be the centre of the sphere, G the centre of gravity of ABC, P its circumcentre. Produce AG to meet BC in L and join OL. Then since the forces acting along AO, BO, CO and OG are in equilibrium, the resultant of BO and CO must act along LO. Hence, since L is the middle point of BC, the forces along BO and CO must be equal, and similarly each is equal to the force along AO. Again, since G is the centre of mean position for ABC, the resultant of the pressures is $\frac{3X}{R}\cdot OG$.

But $OG^2 = OP^2 + GP^2 = R'^2 - R^2 + \frac{1}{9}HP^2$, where H is the orthocentre,

$$\text{i.e.}\quad OG^2 = R'^2 - R^2 + \tfrac{1}{9}R^2(1 - 8\cos A\cos B\cos C)$$
$$= R'^2 - \tfrac{8}{9}R^2(1 + \cos A\cos B\cos C);$$

$$\therefore\quad \frac{W^2}{X^2} = \frac{9}{R'^2}\cdot OG^2 = 9 - \frac{8R^2}{R'^2}(1 + \cos A\cos B\cos C).$$

11. Let T be the impulsive tension of the string, v and v' the velocities of either particle m along and perpendicular to the original direction of motion immediately after the jerk, V the velocity of M.

Then $$2T\cos\frac{\alpha}{2} = MV,\quad T\sin\frac{\alpha}{2} = mv';$$

$$\therefore\quad MV\sin\frac{\alpha}{2} = 2mv'\cos\frac{\alpha}{2}\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Also, since the velocities of m and M along the string must be the same,

$$\therefore\quad V\cos\frac{\alpha}{2} = v\cos\frac{\alpha}{2} - v'\sin\frac{\alpha}{2}.$$

Hence, by (i),

$$2mv'\cos^2\frac{\alpha}{2} = Mv\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} - Mv'\sin^2\frac{\alpha}{2},$$

$$\text{i.e.}\quad 2mv'(1 + \cos\alpha) = Mv\sin\alpha - Mv'(1 - \cos\alpha),$$

$$\text{or}\quad \frac{v'}{v} = \frac{M\sin\alpha}{(2m - M)\cos\alpha + 2m + M}.$$

12. The point of contact is the intersection with $\alpha'x + \beta'y = 0$, and its co-ordinates are given by

$$\frac{x}{\beta'} = \frac{y}{-\alpha'} = \frac{1}{\alpha\beta' - \alpha'\beta}.$$

Now

$$\frac{dy}{dx} = -\frac{\alpha}{\beta};\quad \therefore\quad \frac{d^2y}{dx^2} = \frac{1}{x'}\cdot\frac{d}{dt}\left(-\frac{\alpha}{\beta}\right) = -\frac{\alpha'\beta - \alpha\beta'}{\beta^2 x'}.$$

Also

$$x' = \frac{\beta''(\alpha\beta' - \alpha'\beta) - \beta'(\alpha\beta'' - \alpha''\beta)}{(\alpha\beta' - \alpha'\beta)^2} = \frac{\beta(\alpha''\beta' - \alpha'\beta'')}{(\alpha\beta' - \alpha'\beta)^2};$$

$$\therefore\quad \frac{d^2y}{dx^2} = \frac{(\alpha\beta' - \alpha'\beta)^3}{\beta^3(\alpha''\beta' - \alpha'\beta'')},$$

and $$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(\alpha^2 + \beta^2)^{\frac{3}{2}}(\alpha''\beta' - \alpha'\beta'')}{(\alpha\beta' - \alpha'\beta)^3}.$$

LXXVII.

1. Let O be the radical centre, OT the tangent to any one of the three circles. Then if we invert from O with radius of inversion OT, the three circles invert into themselves, and therefore the two circles touching them must invert into each other. Hence the line joining their centres passes through the origin of inversion, i.e. PQ passes through O.

2. The locus of the centre of a rectangular hyperbola passing through three given points A, B, C is the nine-point circle of the triangle ABC, the radius of which is half that of the circumcircle. Now if A, B, C coincide, the circumcentre becomes the centre of curvature, which in this case is given. Hence the locus of the centre of the hyperbola is the circle described on the radius of curvature at the given point as diameter.

3. Calling the given determinant Δ_n and expanding according to the elements of the first column, we have

$$\Delta_n = D + \Delta_{n-1},$$

where D is a determinant of $n-1$ rows and columns, in which each element in the leading diagonal is 2, and each element to the right of the leading diagonal is zero, so that $D = 2^{n-1}$, i.e.

$$\Delta_n = 2^{n-1} + \Delta_{n-1},$$

whence it follows that

$$\begin{aligned}\Delta_n &= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1\\ &= 2^n - 1.\end{aligned}$$

4. If n is a positive integer, we have

$$\frac{(1+x)^n-(1-x)^n}{2x}={}^nC_1+{}^nC_3\,.\,x^2+{}^nC_5\,.\,x^4+\dots.$$

Hence, writing $\sqrt{3}i$ for x, the first series in question is equal to

$$\frac{(1+\sqrt{3}i)^n-(1-\sqrt{3}i)^n}{2\sqrt{3}i}.$$

Now $1+\sqrt{3}i=-2\omega$, $1-\sqrt{3}i=-2\omega^2$, where ω and ω^2 are the imaginary cube roots of unity. Also, if n is a multiple of 6, then $\omega^n=\omega^{2n}=1$, and the above expression vanishes.

Again, the second series is equal to

$$\frac{\left(1+\frac{1}{\sqrt{3}}i\right)^n-\left(1-\frac{1}{\sqrt{3}}i\right)^n}{\frac{2}{\sqrt{3}}i},$$

and $\left(1+\frac{1}{\sqrt{3}}i\right)^2=\frac{2}{3}+\frac{2}{\sqrt{3}}i=-\frac{4}{3}\omega$ and $\left(1-\frac{1}{\sqrt{3}}i\right)^2=-\frac{4}{3}\omega^2$.

Hence, since $\frac{n}{2}$ is a multiple of 3, the expression vanishes.

5. By the substitution $z=ax+b$, the equation becomes

$$z^3+3Hz+G=0\;\dots\dots\dots\dots\text{(i)},$$

where $H=ac-b^2$.

We then have

$$z=a(\lambda y+\mu)+b=a\lambda y+a\mu+b$$
$$=\lambda' y+\mu' \text{ suppose.}$$

Since this substitution renders the equation (i) reciprocal, we have

$$\lambda'^3=\mu'^3+3H\mu'+G,$$
$$3\lambda'^2\mu'=3\lambda\mu'^2+3H\lambda', \text{ i.e. } \lambda'\mu'=\mu'^2+H.$$

Hence $$\mu'^3(\mu'^3+3H\mu'+G)=(\mu'^2+H)^3,$$

or $$G\mu'^3-3H\mu'^2-H^3=0.$$

But this equation can be derived from (i) by the substitution

$$\mu'=-\frac{H}{z},$$

and since the roots of (i) are $a\alpha+b$, etc., the result follows.

6. The given expression is

$$4R^2(\sin A + \sin B - 2\sin C)^2 \sec^2 \frac{C}{2} + 4R^2(\sin A - \sin B)^2 \operatorname{cosec}^2 \frac{C}{2}$$

$$= 16R^2\left(\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\right)^2 + 16R^2\sin^2\frac{A-B}{2}$$

$$= 16R^2\left(1 - 4\cos\frac{A-B}{2}\sin\frac{C}{2} + 4\sin^2\frac{C}{2}\right)$$

$$= 16R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 16(R^2 - 2Rr).$$

7. The roots of $\tan 4\theta = 1$, considered as an equation in $\tan\theta$, are

$$\tan\frac{(4r+1)\pi}{16}, \qquad (r = 0, 1, 2, 3)$$

$$\text{i.e. } \tan\frac{\pi}{16}, \ \cot\frac{3\pi}{16}, \ -\cot\frac{\pi}{16}, \ -\tan\frac{3\pi}{16}.$$

Similarly the roots of $\tan 4\theta = -1$ are

$$-\tan\frac{\pi}{16}, \ -\cot\frac{3\pi}{16}, \ \cot\frac{\pi}{16}, \ \tan\frac{3\pi}{16}.$$

Hence if, in the equation $\tan^2 4\theta = 1$, we put $x = \tan^2\theta$, the roots of the equation in x are

$$\tan^2\frac{\pi}{16}, \ \cot^2\frac{\pi}{16}, \ \tan^2\frac{3\pi}{16}, \ \cot^2\frac{3\pi}{16}.$$

Now this equation is

$$16x(1-x)^2 = (1 - 6x + x^2)^2,$$

$$\text{i.e. } x^4 - 28x^3 + 70x^2 - \ldots = 0,$$

and the sum of the squares of the roots is $28^2 - 2 . 70 = 644$.

Again, if in $\tan^2 2\theta = 1$ we put $x = \tan^2\theta$, the values of x are

$$\tan^2\frac{\pi}{8}, \ \cot^2\frac{\pi}{8}.$$

But the equation is $x^2 - 6x + 1 = 0$, and the sum of the squares of the roots is 34. Hence the expression in question is

$$644 + 34 = 678.$$

8. If the axes are turned through an angle θ, the equation $y^2 = 4ax$ becomes

$$(x \sin\theta + y\cos\theta)^2 = 4a(x\cos\theta - y\sin\theta).$$

Considered as a quadratic in x, this has equal roots if

$$\cos^2\theta\,(y\sin\theta - 2a)^2 = \sin^2\theta\,(y^2\cos^2\theta + 4ay\sin\theta),$$

whence $\quad y = a\dfrac{\cos^2\theta}{\sin\theta}$ is the tangent parallel to the axis of x.

Similarly $x = -a\dfrac{\sin^2\theta}{\cos\theta}$ is the tangent parallel to the axis of y.

Eliminating θ between these, we get the locus.

9. Any conic circumscribing the triangle is (as in trilinears) of the form

$$\lambda(bx + cy - 1)^{-1} + \mu(cx + ay - 1)^{-1} + \nu(ax + by - 1)^{-1} = 0 \ldots \text{(i)},$$

or $$\Sigma\lambda(cx + ay - 1)(ax + by - 1) = 0.$$

The conditions for a circle are

$$\lambda ca + \mu ab + \nu bc = \lambda ab + \mu bc + \nu ca,$$

$$\text{i.e. } \lambda a(b - c) + \mu b(c - a) + \nu c(a - b) = 0 \ldots\ldots\ldots \text{(ii)},$$

and $$\lambda(bc + a^2) + \mu(ca + b^2) + \nu(ab + c^2) = 0 \ldots\ldots\ldots \text{(iii)}.$$

Eliminate λ, μ, ν between (i), (ii) and (iii).

10. Let R be the reaction at B making an angle θ with AB, and let AB make an angle α with the vertical. Then taking moments about A for AB,

$$R \,.\, a\sin\theta = wa \,.\, \tfrac{1}{2}a\sin\alpha;$$

$$\therefore\ R\sin\theta = \tfrac{1}{2}w \,.\, \frac{ab}{\sqrt{a^2 + b^2}}, \text{ since } \tan\alpha = \frac{b}{a}.$$

Similarly by taking moments about C for BC, we find

$$R\cos\theta = \tfrac{1}{2}w \,.\, \frac{ab}{\sqrt{a^2 + b^2}}.$$

Hence squaring and adding

$$R = \frac{wab}{\sqrt{2(a^2 + b^2)}}.$$

Again, the equation of virtual work for a small vertical displacement is

$$-T\,.\,\delta\,(AC) + 2w\,(a+b)\,.\,\delta\,(\tfrac{1}{2}AC) = 0,$$

$$\therefore\ T = w\,(a+b).$$

11. Let T, T' be the tensions of the upper and lower strings, γ the inclination of the lower string to the plane of the circle, v the speed at the highest point. Then

$$T\cos\beta + T'\cos\gamma + mg\sin\alpha = \frac{mv^2}{a},$$

$$T\sin\beta - T'\sin\gamma = mg\cos\alpha,$$

whence $$T'\sin(\beta+\gamma) = -\,mg\cos(\alpha-\beta) + \frac{mv^2}{a}\sin\beta.$$

Now T' is positive since the strings remain taut, and $\sin(\beta+\gamma)$ is positive. Hence we must have

$$\frac{mv^2}{a}\sin\beta > mg\cos(\alpha-\beta),$$

$$\text{i.e. } v^2 > ga\,(\cos\alpha\cot\beta + \sin\alpha).$$

12. Suppose the tangent makes angles ϕ_1, ϕ_2 in the same sense with the radii vectores. Then differentiating the equation

$$r_1 r_2 = \tfrac{1}{2}a^2$$

with respect to the arc, we get

$$r_2\cos\phi_1 + r_1\cos\phi_2 = 0 \quad\text{.....................(i)}.$$

Now if θ_1, θ_2 are the vectorial angles, ψ the angle the tangent makes with the initial line, we have

$$\psi = \theta_1 + \phi_1 = \theta_2 + \phi_2;$$

$$\therefore\ \frac{d\phi_1}{ds} = \frac{1}{\rho} - \frac{\sin\phi_1}{r_1},\qquad \frac{d\phi_2}{ds} = \frac{1}{\rho} - \frac{\sin\phi_2}{r_2}.$$

Hence differentiating (i) again, we get

$$2\cos\phi_1\cos\phi_2 - r_2\sin\phi_1\left(\frac{1}{\rho} - \frac{\sin\phi_1}{r_1}\right) - r_1\sin\phi_2\left(\frac{1}{\rho} - \frac{\sin\phi_2}{r_2}\right) = 0,$$

whence

$$\frac{1}{\rho}(r_2 \sin \phi_1 + r_1 \sin \phi_2) = \frac{(r_2 \sin \phi_1 - r_1 \sin \phi_2)^2 + 2r_1 r_2 \cos(\phi_1 - \phi_2)}{r_1 r_2} \quad \text{......(ii).}$$

Now if $\dfrac{\sin \phi_1}{r_1} + \dfrac{\sin \phi_2}{r_2} = X$, we get, using (i),

$$X^2 = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{2 \cos(\phi_1 - \phi_2)}{r_1 r_2}.$$

Also from the figure

$$2r_1 r_2 \cos(\phi_1 - \phi_2) = r_1^2 + r_2^2 - 2a^2 = r_1^2 + r_2^2 - 4r_1 r_2,$$

whence

$$X^2 = 2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^2 \quad \text{......................(iii).}$$

Also, from (i), $X\left(\dfrac{\sin \phi_1}{r_1} - \dfrac{\sin \phi_2}{r_2}\right) = \dfrac{1}{r_1^2} - \dfrac{1}{r_2^2}$;

$$\therefore \left(\frac{\sin \phi_1}{r_1} - \frac{\sin \phi_2}{r_2}\right)^2 = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)^2.$$

Hence from (ii)

$$\frac{X}{\rho} = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)^2 + \frac{r_1^2 + r_2^2 - 4r_1 r_2}{r_1^2 r_2^2}$$

$$= \frac{3}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)^2 = \tfrac{3}{4}X^2;$$

$$\therefore X = \frac{4}{3\rho} \quad \text{..........................(iv).}$$

Hence, putting $\sin \phi_1 = \dfrac{p_1}{r_1}$, etc., in the original value of X, the results given follow from (iii) and (iv).

LXXVIII.

1. Since the tetrahedra $OBCD$, $ABCD$ stand on the same base, their volumes are proportional to their altitudes, and these latter are clearly proportional to OA' and AA'. We thus have

$$\frac{OA'}{AA'} = \frac{\text{vol. } OBCD}{\text{vol. } ABCD}.$$

Writing down the other three similar relations and adding, the result follows.

2. Let $2a$ be the vertical angle of the cone, β the angle the section makes with the axis of the cone. Let V be the vertex of the cone, A that of the section, and let the focal sphere touch VA in L. Then if X is the foot of the directrix, LX is perpendicular to the axis of the cone, and

$$e = \frac{AL}{AX} = \frac{\cos\beta}{\cos a},$$

and this quantity ranges in value from 1 to $\frac{1}{\cos a}$. Hence we must have

$$e \ngtr \frac{1}{\cos a}, \qquad \text{i.e. } \cos a \ngtr \frac{1}{e}.$$

Hence the least possible value of the vertical angle is $2\sec^{-1} e$.

3. The cases may be exhausted as follows:

we may select 5 from the 1st class in

$$\frac{n(n+1)(n+2)(n+3)(n+4)}{120} \text{ ways};$$

we may select 4 from the 2nd class and 1 from the 1st in

$$\frac{n(n+1)}{2} \,.\, n \text{ ways};$$

we may select 2 from the 2nd class and 3 from the 1st in

$$n \,.\, \frac{n(n+1)(n+2)}{6} \text{ ways};$$

we may select 3 from the 3rd class and 2 from the 2nd in

$$n \,.\, n \text{ ways};$$

we may select 3 from the 3rd class and 2 from the 1st in

$$n \,.\, \frac{n(n+1)}{2} \text{ ways};$$

we may select 4 from the 4th class and 1 from the 1st in

$$n \,.\, n \text{ ways};$$

we may select 5 from the 5th class in n ways.

Adding these numbers we obtain the expression given.

4. The expression in the curved bracket is $n!$ times the coefficient of x^n in

$$e^{rx} - re^{(r-1)x} + \frac{r(r-1)}{2!}e^{(r-2)x} - \ldots,$$

$$\text{i.e. in } (e^x - 1)^r.$$

Hence the given series is $n!$ times the coefficient of x^n in

$$\sum_{r=1}^{r=n} \frac{n!}{r!(n-r)!}(e^x - 1)^r,$$

$$\text{i.e. in } \{1 + (e^x - 1)\}^n, \text{ i.e. in } e^{nx},$$

and the series is therefore equal to n^n.

5. Suppose x, y, z the roots of $t^3 - pt^2 + qt - r = 0$, and let $s_n = \Sigma x^n$. Then, using the given equations, we have

$$s_3 + qs_1 - 3r = 0, \quad -ps_3 - rs_1 = 0, \quad 2 + qs_3 = 0.$$

From these, since $s_1 = p$, we easily get $q(3r - pq) + 2 = 0$, $qr = 2$, whence $pq^2 = 8$, and therefore since $p^2 - 2q = 0$, we have $p^5 = 32$, so that $p = 2\theta$, where $\theta^5 = 1$. Hence $q = 2\theta^2$, $r = \theta^3$ so that the equation is

$$t^3 - 2\theta \,.\, t^2 + 2\theta^2 \,.\, t - \theta^3 = 0,$$

$$\text{i.e. } (t - \theta)(t^2 - \theta t + \theta^2) = 0,$$

whence $t = \theta$, $-\omega\theta$, $-\omega^2\theta$, where ω is either of the imaginary cube roots of unity, and these are the general values of x, y and z, taken in any order.

6. Let the side $B'C'$ through A make an angle θ with AB. Then

$$\frac{AC'}{\sin(60° + \theta)} = \frac{c}{\sin 60°}, \quad \frac{AB'}{\sin(A + \theta - 60°)} = \frac{b}{\sin 60°},$$

$$\therefore \; B'C' = \frac{2}{\sqrt{3}}[c \sin(60° + \theta) + b \sin(A + \theta - 60°)].$$

Now the maximum value of $\alpha \cos\theta + \beta \sin\theta$ is $(\alpha^2 + \beta^2)^{\frac{1}{2}}$. Hence the maximum value of $B'C'^2$ is

$$\tfrac{4}{3}\{[c \sin 60° + b \sin(A - 60°)]^2 + [c \cos 60° + b \cos(A - 60°)]^2\}$$

$$= \tfrac{4}{3}\{b^2 + c^2 + 2bc \cos(A - 120°)\}$$

$$= \tfrac{4}{3}(b^2 + c^2 - bc \cos A + bc\sqrt{3} \sin A)$$

$$= \tfrac{4}{3}\{\tfrac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}\,\Delta\}$$

and the area of the triangle is $\frac{\sqrt{3}}{4} \,.\, B'C'^2$.

7. Let O be the centre and θ the angle AOP. Then

$$PA^2 = a^2 + c^2 - 2ac\cos\theta, \quad PB^2 = a^2 + c^2 - 2ac\cos\left(\theta + \frac{2\pi}{n}\right), \text{ etc.}$$

Now if $f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$, then $f^r(x)$, the rth derived function, is the sum of the products of the factors $x - a_1$, $x - a_2$ etc. taken r together.

But since

$$y^{2n} - 2y^n \cos n\theta + 1 = \prod_{r=0}^{r=n-1} \left\{y^2 - 2y\cos\left(\theta + \frac{2r\pi}{n}\right) + 1\right\},$$

if we divide by y^n and put $y + \frac{1}{y} = x$, we get

$$(x^n + \text{lower powers of } x) - 2\cos n\theta = \prod_{s=0}^{s=n-1} \left\{x - 2\cos\left(\theta + \frac{2s\pi}{n}\right)\right\},$$

and taking the rth derived, the result is independent of θ. Hence putting $x = \frac{a^2 + c^2}{2ac}$ we obtain the theorem in question provided c is constant.

8. The conditions that the four lines in question should touch the general conic are

$$Am^2 + B - 2Hm = 0, \text{ and a similar equation in } m' \dots\text{(i)},$$

$$A\mu^2 + B + C\mu^2c^2 + 2F\mu c - 2G\mu^2 c - 2H\mu = 0,$$

and a similar equation in μ' ...(ii),

i.e. $A\mu^2 - 2H\mu + B = \mu^2 c(2G - Cc) - 2F\mu c$,

or by (i) $\quad A(\mu - m)(\mu - m') = \mu^2 c(2G - Cc) - 2F\mu c$,

and a similar equation in μ'...(iii).

Now the co-ordinates of the centre are $x = \frac{G}{C}$, $y = \frac{F}{C}$. Hence, from the equations (ii), we get

$$\frac{(\mu - m)(\mu - m')}{(\mu' - m)(\mu' - m')} = \frac{\mu^2(2x - c) - 2\mu y}{\mu'^2(2x - c) - 2\mu' y},$$

reducing to the form given.

9. Changing to Cartesians, the equation becomes

$$\Sigma l\,(x\cos\alpha + y\sin\alpha - p_1)^2 = 0,$$

in which the coefficients of x^2, y^2 and $2xy$ are

$$\Sigma l\cos^2\alpha,\quad \Sigma l\sin^2\alpha \text{ and } \Sigma l\sin\alpha\cos\alpha,$$

respectively.

Also $$(\Sigma l\cos^2\alpha)(\Sigma l\sin^2\alpha) - (\Sigma l\sin\alpha\cos\alpha)^2$$
$$= \Sigma mn\sin^2(\beta-\gamma) = \Sigma mn\sin^2 A.$$

Hence, using the ordinary Cartesian formula, the eccentricity is given by

$$\frac{e^4}{1-e^2} + 4 = \frac{(\Sigma l)^2}{\Sigma mn\sin^2 A},$$

which is equivalent to the form given.

10. Let $B\hat{A}C = \alpha$, $B\hat{Q}P = \theta$, $BD = 4x$. Considering the whole system, the C. of G. must be vertically below A, i.e. AC must be vertical.

If T is the tension in the bar, then taking moments about A for AB, BC we have

$$2W.x = -T.AP\sin\theta - R.AC\sin\phi \quad \text{.........(i)},$$

where R is the reaction at C, making an angle ϕ with AC.

Also taking moments about B for BC, and D for DC, we get

$$W.x = T.BQ\sin\theta + R.BC\sin(\phi+\alpha) \text{.........(ii)},$$

$$W.x = \qquad\qquad R.BC\sin(\phi-\alpha) \quad \text{......(iii)}.$$

But $AC = 2BC\cos\alpha$. Hence adding (i), (ii), (iii), we get

$$4W.x = T(BQ - AP)\sin\theta.$$

But $$2x = AB\sin\alpha, \text{ and } \frac{\sin\alpha}{\sin\theta} = \frac{PQ}{AC}.$$

Hence $$T = \frac{2W.AB.PQ}{AC(BQ-AP)}.$$

11. Let the enveloping parabola of trajectories from the fort A meet the ground in B, and draw AN vertical to the ground. Then since the latus-rectum of the parabola is $2r$, we have

$$BN^2 = 2r\left(h + \frac{r}{2}\right) = 2hr + r^2.$$

Now suppose the ship S is distant x from N, and that the enveloping parabola of trajectories from S cuts AN in C.

Then $$x^2 = 2r\left(\frac{r}{2} - CN\right) = r^2 - 2r\,.\,CN.$$

Now for the fort to be in range, we must have $CN \not< h$;

$$\therefore\ x^2 \not> r^2 - 2hr.$$

Hence the area in question is that included between two circles, centre N, and radii BN and the maximum value of x.

This area is

$$\pi(2hr + r^2) - \pi(r^2 - 2hr) = 4\pi hr.$$

12. For points at which $x_1 \equiv \dfrac{dx}{dy} = 0$, we have

$$4y^3 - 2axy = 0,$$

or, excluding the origin, $2y^2 - ax = 0$, whence

$$x = \pm\frac{a}{2} \quad\text{.............................(i).}$$

Also differentiating

$$4y^3 - 2axy - (ay^2 - 4x^3)\,x_1 = 0$$

and putting $x_1 = 0$ in the result, we get

$$12y^2 - 2ax - (ay^2 - 4x^3)\,x_2 = 0,$$

whence, for points (i),

$$x_2 = -\frac{8}{a}; \qquad \therefore\ \rho = \frac{(1 + x_1^2)^{\frac{3}{2}}}{x_2} = -\frac{a}{8}.$$

Also, at the origin, the tangent to one branch is the axis of y. To find the curvature there, we have

$$\rho = \mathrm{Lt}\,\frac{y^2}{2x}.$$

But from the equation,

$$\frac{y^4}{x^2} - a \cdot \frac{y^2}{x} + x^2 = 0;$$

$$\text{i.e.} \quad (2\rho)^2 - a\,(2\rho) = 0, \quad \text{whence } \rho = \frac{a}{2}.$$

Hence the radii of curvature are numerically as 4 : 1.

LXXIX.

1. Let the circle touch AB in L, and let the escribed circle opposite A touch in L'. Now invert the system from A, the constant of inversion being $\sqrt{AL \cdot AL'}$. Let B', C' be the points inverse to B, C. Then since the given circle inverts into the escribed circle, and the circumcircle into $B'C'$, the line $B'C'$ must touch the escribed circle. Also since $AB \cdot AB' = AC \cdot AC'$, the triangles ABC, $A'B'C'$ are similar, and therefore we must have $B'C' = BC$, i.e. the triangles are equal in all respects, and therefore $AB' = AC$. Hence $AB \cdot AC = AL \cdot AL'$. But $AL' = s$;

$$\therefore \quad AL = \frac{bc}{s}.$$

2. The circle and the hyperbola belong to a four-point system of conics, which have double contact with them at the same two points, Q and Q', say. Two of the opposite sides of the four-point are the tangents at Q and Q', and the other two coincide in QQ'.

Now the transversal PM meets the system in an involution, and since the second point at which it meets the hyperbola is at infinity, P is the centre of the involution. Also M is a point of coincidence of a pair, i.e. a double point of the involution. Hence if PM meets the circle in R and R', then $PR \cdot PR' = PM^2$. But $PR \cdot PR' = PT^2$. $\quad \therefore \ PM = PT$.

3. Suppose n is even, and let u_n denote the number of triangles. If we add another line of length $n+1$, the additional triangles thus formed will be those in which one side is $(n+1)$.

If the second side is n, the third may have any value from $(n-1)$ to 2, giving $(n-2)$ triangles; if the second side is $(n-1)$, the third may have any value from $(n-2)$ to 3, giving $(n-4)$ triangles, and so on. Hence the number of additional triangles is

$$(n-2)+(n-4)+\ldots+2=\frac{n(n-2)}{4},$$

while if n is odd, it is evident by the same reasoning that the number of additional triangles is

$$(n-2)+(n-4)+\ldots+1=\frac{(n-1)^2}{4}.$$

Hence, reverting to the case where n is even, we have

$$u_{n+1}-u_n=\frac{n(n-2)}{4},\quad u_n-u_{n-1}=\frac{(n-2)^2}{4}=\frac{(n-1)(n-3)}{4}+\frac{1}{4}.$$

In each of these, writing $n-2$, $n-4$, ..., etc. for n, and adding, we get

$$\begin{aligned} u_{n+1} &= \sum_1^n \frac{n(n-2)}{4}+\frac{n}{8} \\ &= \frac{1}{4}\left\{\frac{n(n+1)(2n+1)}{6}-n(n+1)\right\}+\frac{n}{8} \\ &= \frac{n(n-2)(2n+1)}{24} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}; \end{aligned}$$

$$\therefore\ u_n=\frac{n(n-2)(2n+1)}{24}-\frac{n(n-2)}{4}=\frac{n(n-2)(2n-5)}{24}.$$

If n is odd, we have merely to write $n-1$ for n in (i).

4. Let $a_1, a_2, \ldots a_{n-1}$ be the remainders. Then $\dfrac{n-a_r}{r}$ is exactly the number of integers less than n divisible by r.

Consider then the sum

$$1\,.\,\frac{n-a_1}{1}+2\,.\,\frac{n-a_2}{2}+\ldots+(n-1)\,.\,\frac{n-a_{n-1}}{n-1}+n\,.\,\frac{n}{n}.$$

This is the sum of the factors of all numbers $\not> n$, unity and the number itself included. But this is equal to

$$(n + n + n + \ldots \text{ to } n \text{ terms}) - (a_1 + a_2 + \ldots + a_{n-1}) = n^2 - \Sigma a_r;$$

$$\therefore \text{ sum of all the factors} + \text{sum of all the remainders} = n^2.$$

5. If the roots a, β, γ, δ determine a harmonic range, there must be a relation of the form

$$\frac{\gamma - a}{\beta - \gamma} = \frac{\delta - a}{\delta - \beta},$$

$$\text{i.e.} \quad \Sigma a\beta = 3(a\beta + \gamma\delta); \quad \therefore a\beta + \gamma\delta = 0.$$

Hence the required condition is

$$(a\beta + \gamma\delta)(\beta\gamma + a\delta)(\gamma a + \beta\delta) = 0.$$

Now if

$$ax^4 - bx^3 + dx - e = 0 \equiv a(x^2 + px + q)(x^2 + p'x - q),$$

then $\quad a(p + p') = -b, \quad aq(p' - p) = d, \quad aq^2 = e, \quad app' = 0,$

whence $\quad \dfrac{b^2}{a^2} - \dfrac{d^2}{a^2 q^2} = 0, \quad \text{i.e.} \ \dfrac{b^2}{a^2} - \dfrac{d^2}{ae} = 0, \quad \text{or} \quad ad^2 = b^2 e.$

6. If AB and CD are at right angles, we have from a figure

$$a = b \sin C + d \cos A,$$

$$c = b \cos C - d \sin A,$$

the angle A being supposed obtuse.

Putting $C = a - A$, these equations take the form

$$(b \sin a + d) \cos A - b \cos a \sin A = a,$$

$$b \cos a \cos A + (b \sin a - d) \sin A = c;$$

whence

$$(b^2 - d^2) \cos A = ab \sin a - ad + bc \cos a,$$

$$(b^2 - d^2) \sin A = bc \sin a + cd - ab \cos a.$$

Squaring and adding, we obtain the relation required.

7. We have

$$\cos^3\frac{2\pi}{7}\sec\frac{6\pi}{7}=\frac{\cos^2\frac{2\pi}{7}}{4\cos^2\frac{2\pi}{7}-3}=\frac{1}{1-3\tan^2\frac{2\pi}{7}}.$$

Now if in the equation $\dfrac{\tan 7\theta}{\tan\theta}=0$ we put $\tan^2\theta=y$, the values of y will be

$$\tan^2\frac{2\pi}{7},\quad \tan^2\frac{4\pi}{7}\quad\text{and}\quad\tan^2\frac{6\pi}{7}.$$

The equation in y is

$$7-35y+21y^2-y^3=0,$$

and the required equation is obtained from this by the substitution

$$x=\frac{1}{1-3y}\quad\text{or}\quad y=\frac{x-1}{3x}.$$

8. Let m' be the other extremity of the normal at m, so that

$$m'=-\frac{m^2+2}{m}.$$

Then the middle point of the normal is

$$X=\frac{a\,(m^2+m'^2)}{2}=a\left(m^2+2+\frac{2}{m^2}\right),$$

$$Y=a\,(m+m')=-\frac{2a}{m}.$$

For the perpendicular normal we write $-\dfrac{1}{m}$ for m, and the middle point is

$$X'=a\left(2m^2+2+\frac{1}{m^2}\right),\qquad Y'=2am.$$

Forming the equation to the line joining these two points, and dividing through by $m+\dfrac{1}{m}$, we find

$$-2x+y\left(m-\frac{1}{m}\right)+2a\left(m^2+\frac{1}{m^2}+3\right)=0,$$

and putting $m-\dfrac{1}{m}=\theta$, this becomes

$$-2x+y\theta+2a(\theta^2+5)=0$$

or $$2a\theta^2+y\theta+2(5a-x)=0$$

of which the envelope is $y^2=16a(5a-x)$.

9. The equation to any conic having double contact must be of the form

$$ax^2+2hxy+by^2+2y+k(lx+my)^2=0 \quad \text{.........(i)},$$

and if this is a parabola

$$(a+kl^2)(b+km^2)=(h+klm)^2,$$

whence $$k=-\frac{ab-h^2}{bl^2+am^2-2hlm},$$

$$a+kl^2=\frac{(am-hl)^2}{bl^2+am^2-2hlm}.$$

Now if ρ is the radius of curvature of (i) at the origin, then

$$\rho=\text{Lt}\,\frac{x^2}{2y}$$

when $x=0$, $y=0$.

Hence from (i) $\qquad (a+kl^2)\rho+1=0.$

10. Let L, M be the points of contact of the rod with the hemispheres, the corresponding radii meeting in N, which is therefore vertically above G, the centre of the rod. Suppose the radii make acute angles θ, ϕ with the horizontal.

Then projecting horizontally and vertically, we have

$$-a\cos\theta+ka\cos\lambda+a\cos\phi=2a, \quad \text{i.e. } \cos\phi-\cos\theta=2-k\cos\lambda \quad \text{............(i)}$$

and $ka\sin\lambda=a\sin\theta+a\sin\phi$, i.e. $\sin\theta+\sin\phi=k\sin\lambda$......(ii).

Also, since

$$\frac{GN}{LG}=\frac{GN}{GM}; \quad \therefore \frac{\sin(\theta-\lambda)}{\cos\theta}=\frac{\sin(\phi+\lambda)}{\cos\phi},$$

whence $$\tan\theta-\tan\phi=2\tan\lambda \quad \text{..................(iii)}.$$

Now from (i) and (ii)

$$2 \cos(\theta + \phi) = -2 + 4k \cos\lambda - k^2 \quad \text{............(iv)},$$

$$\tan \frac{\theta - \phi}{2} = \frac{2 - k \cos\lambda}{k \sin\lambda},$$

and from the latter

$$\left.\begin{aligned} \cos(\theta - \phi) &= \frac{k^2(1 - 2\cos^2\lambda) + 4k\cos\lambda - 4}{k^2 - 4k\cos\lambda + 4} \\ \sin(\theta - \phi) &= \frac{2k\sin\lambda\,(2 - k\cos\lambda)}{k^2 - 4k\cos\lambda + 4} \end{aligned}\right\} \text{......(v)}.$$

But from (iii),

$$\frac{\cos(\theta + \phi) + \cos(\theta - \phi)}{\sin(\theta - \phi)} = \frac{\cos\lambda}{\sin\lambda}.$$

Hence, substituting from (iv) and (v), we get

$$\tfrac{1}{2}(k^2 - 4k\cos\lambda + 4)(-2 + 4k\cos\lambda - k^2) + k^2(1 - 2\cos^2\lambda)$$
$$+ 4k\cos\lambda - 4 = 2k\cos\lambda\,(2 - k\cos\lambda)$$

which reduces to

$$8k^2\cos^2\lambda - 4k(k^2 + 3)\cos\lambda + \tfrac{1}{2}(k^4 + 4k^2 + 16) = 0,$$

giving the required values of $\cos\lambda$.

11. Let I be the impulse of the forces of compression. Then since at the end of the period of compression the balls are instantaneously at rest, we have

$$mv = 2I\cos 30^\circ = \sqrt{3}I.$$

The forces of restitution on B will be $e'I$ along AB, and eI along CB. Hence if V be the velocity of B in the direction AB, we have

$$\begin{aligned} mV &= e'I + eI\cos 60^\circ \\ &= \frac{2e' + e}{2} \cdot I = \frac{2e' + e}{2} \cdot \frac{mv}{\sqrt{3}}, \end{aligned}$$

and similarly for A. Hence A and B will separate with velocity $2V$,

$$\text{i.e. } \frac{(2e' + e)\,v}{\sqrt{3}}.$$

12. The integral may be written in the form

$$\int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}\left(2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}\right)}$$

$$=\frac{1}{4}\int\frac{\sec^4\frac{x}{2}}{\tan\frac{x}{2}\left(1+\tan\frac{x}{2}\right)}\,dx.$$

Putting $\tan\frac{x}{2}=t$, this becomes

$$\frac{1}{2}\int\frac{(1+t^2)\,dt}{t\,(1+t)}=\frac{1}{2}\int\left(1+\frac{1}{t}-\frac{2}{1+t}\right)dt$$

$$=\frac{1}{2}\left[t+\log t-\log(1+t)^2\right]=\frac{1}{2}\left[t+\log\frac{t}{(1+t)^2}\right].$$

Adding the constant $\frac{1}{2}\log 2$, the latter term becomes

$$\log\frac{2t}{(1+t)^2}=\log\frac{\sin x}{1+\sin x}.$$

LXXX.

1. Let O be the circumcentre, I the incentre, P the orthocentre, U the nine-point centre. Then

$$OI^2=R^2-2Rr,\quad IU=\tfrac{1}{2}R-r,$$

i.e. OI and IU are both known. But if we produce OI to K, making $IK=OI$, then K is a fixed point, and

$$PK=2IU=R-2r.$$

Hence the locus of P is a circle with centre K.

2. The pairs of tangents drawn from any point O to a system of conics touching four given straight lines are in involution. Hence if O be the intersection of two of the conics, the tangents at O to these two must be the double lines of the involution. Also, if A and C be two opposite vertices of the quadrilateral, then OA, OC are a pair belonging to the involution,

since the line AC is the limiting form of a conic touching the four lines. Hence if the tangents at O cut AC in K and K' we must have, since K and K' are the double points of the line-involution of which A and C are a pair of points,

$$(KAK'C) = (KCK'A),$$

i.e. the range is harmonic.

3. Adding the columns, we see that $1 + a + a^2 + a^3$ is a factor; also multiplying the columns by $1, \theta, \theta^2, \theta^3, \theta^4$ where θ is any imaginary fifth root of unity, it appears that $1 + \theta a + \theta^2 a^2 + \theta^3 a^3$ is a factor.

Hence the value of the determinant is

$$\frac{1-a^4}{1-a} \cdot \frac{1-(\theta a)^4}{1-\theta a} \cdot \frac{1-(\theta^2 a)^4}{1-\theta^2 a} \cdot \frac{1-(\theta^3 a)^4}{1-\theta^3 a} \cdot \frac{1-(\theta^4 a)^4}{1-\theta^4 a},$$

and since $(1-x)(1-\theta x)(1-\theta^2 x)(1-\theta^3 x)(1-\theta^4 x) \equiv 1 - x^5$, this is

$$\frac{1-a^{20}}{1-a^5} = 1 + a^5 + a^{10} + a^{15}.$$

4. Let $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, etc. Then we have to prove that

$$1 - a_1 + a_1 a_2 - a_1 a_2 a_3 + \ldots + a_1 a_2 \ldots a_{2n}$$
$$\nless (1-a_2)(1-a_4)\ldots(1-a_{2n}).$$

Calling the expression on the left $f(n)$, we have

$$f(n) - f(n+1) = a_1 a_2 \ldots a_{2n+1} - a_1 a_2 \ldots a_{2n+2}.$$

Now $a_r - a_{r+1} \equiv a_r a_{r+1}$, and we therefore have

$$\begin{aligned} f(n) - f(n+1) &= a_{2n+1}(a_1 - a_2)(a_3 - a_4)\ldots(a_{2n-1} - a_{2n}) \\ &\qquad - (a_1 - a_2)(a_3 - a_4)\ldots(a_{2n+1} - a_{2n+2}) \\ &= a_{2n+2}(a_1 - a_2)(a_3 - a_4)\ldots(a_{2n-1} - a_{2n}) \\ &< a_{2n+2}(1-a_2)(1-a_4)\ldots(1-a_{2n}), \end{aligned}$$

since the a's are all $\ngtr 1$;

$$\therefore \quad f(n+1) > f(n) - a_{2n+2}(1-a_2)(1-a_4)\ldots(1-a_{2n}).$$

Hence assuming $f(n) \nless (1-a_2)(1-a_4)\ldots(1-a_{2n})$, we have, *à fortiori*,

$$f(n+1) \nless (1-a_2)(1-a_4)\ldots(1-a_{2n}) - a_{2n+2}(1-a_2)\ldots(1-a_{2n})$$
$$\nless (1-a_2)(1-a_4)\ldots(1-a_{2n})(1-a_{2n+2}),$$

and the theorem follows by induction.

5. Writing the equation in the form

$$(x^2-2px+\lambda)^2-(\mu x+\mu')^2=0,$$

it is equivalent to the quadratics

$$x^2-2px+\lambda \pm (\mu x+\mu')=0.$$

If α, β be the roots of the first, and γ, δ of the second, then

$$\alpha\beta=\lambda+\mu', \quad \gamma\delta=\lambda-\mu',$$
$$\text{i.e. } \tfrac{1}{2}(\alpha\beta+\gamma\delta)=\lambda,$$

so that θ_1, θ_2, θ_3 are the possible values of λ.

But on identifying the equations, we get

$$2\lambda+4p^2-\mu^2=6q, \quad 2p\lambda+\mu\mu'=2r, \quad \lambda^2-\mu'^2=s,$$

whence

$$(2\lambda+4p^2-6q)(\lambda^2-s)=(2p\lambda-2r)^2;$$

$$\therefore \ (\lambda+2p^2-3q)(\lambda^2-s)-2(p\lambda-r)^2 \equiv (\lambda-\theta_1)(\lambda-\theta_2)(\lambda-\theta_3).$$

Putting $\lambda=q$, the left side becomes

$$(2p^2-2q)(p^2-s)-2(pq-r)^2,$$

which is twice the determinant in question.

6. We have

$$r_1^{(n)}-r_1^{(n+1)}=\{r_1^{(n)}+r_1^{(n+1)}\}\sin\frac{A}{2};$$

$$\therefore \ r_1^{(n+1)}=\frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}\cdot r_1^{(n)},$$

whence evidently

$$r_1^{(n)}=\left(\frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}\right)^n\cdot r=\left(\frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}}\right)^{2n}\cdot r,$$

so that
$$\sqrt[2n]{\frac{\rho_1}{r}} = \frac{\cos\frac{A}{2}}{1+\sin\frac{A}{2}} = \tan\left(\frac{\pi}{4}-\frac{A}{4}\right).$$

But the sum of the angles $\frac{\pi}{4}-\frac{A}{4}$, etc., is $\frac{\pi}{2}$;

$$\therefore \quad \Sigma \tan\left(\frac{\pi}{4}-\frac{B}{4}\right)\tan\left(\frac{\pi}{4}-\frac{C}{4}\right)=1,$$

$$\text{i.e.} \quad \Sigma\left(\sqrt[2n]{\rho_2\rho_3}\right) = \sqrt[n]{r}.$$

7. The series is the real part of

$$e^{i\theta}\sin\theta - he^{2i\theta}\sin^2\theta + h^2e^{3i\theta}\sin^3\theta - \ldots = e^{i\theta}\sin\theta\,(1+he^{i\theta}\sin\theta)^{-1},$$

and is therefore equal to

$$\frac{1}{2}\left\{\frac{e^{i\theta}\sin\theta}{1+he^{i\theta}\sin\theta}+\frac{e^{-i\theta}\sin\theta}{1+he^{-i\theta}\sin\theta}\right\}$$

$$=\sin\theta\,.\,\frac{\cos\theta+h\sin\theta}{1+2h\sin\theta\cos\theta+h^2\sin^2\theta}$$

$$=\frac{\cot\theta+h}{\operatorname{cosec}^2\theta+2h\cot\theta+h^2}=\frac{x+h}{1+x^2+2hx+h^2}.$$

8. Let A, B, C, D be the four points, TP, TQ the tangents to the ellipse parallel to AB and CD. Then if AB, CD meet in O, we have

$$TP^2 : TQ^2 = OA\,.\,OB : OC\,.\,OD.$$

But the squares of the tangents to the parabola parallel to AB, CD are also in the latter ratio. Hence the tangents to the parabola parallel to TP, TQ are in the ratio $TP : TQ$. Hence PQ must be parallel to the axis of the parabola. Similarly, if PP' is the diameter of the ellipse, $P'Q$ is parallel to the axis of the other parabola.

But P and Q are the points $\frac{\alpha+\beta}{2}$, $\frac{\gamma+\delta}{2}$, and therefore PQ is parallel to the line

$$\frac{x}{a}\cos\frac{\alpha+\beta+\gamma+\delta}{4}+\frac{y}{b}\sin\frac{\alpha+\beta+\gamma+\delta}{4}=0.$$

Similarly, $P'Q$ is parallel to

$$\frac{x}{a} \sin \frac{\alpha+\beta+\gamma+\delta}{4} - \frac{y}{b} \cos \frac{\alpha+\beta+\gamma+\delta}{4} = 0,$$

and if θ be the angle between these lines,

$$\tan\theta = \frac{-\dfrac{1}{ab}}{\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}\right)\sin\dfrac{\Sigma\alpha}{4}\cos\dfrac{\Sigma\alpha}{4}} = \frac{2ab}{(a^2-b^2)\sin\dfrac{\Sigma\alpha}{2}}.$$

9. If the line $\dfrac{x\cos\alpha}{a} + \dfrac{y\sin\alpha}{b} = 1$ touches the parabola whose tangential equation is

$$Al^2 + Bm^2 + 2Fmn + 2Gnl + 2Hlm = 0 \quad \text{.........(i)},$$

we have

$$A\,.\,\frac{\cos^2\alpha}{a^2} + B\,.\,\frac{\sin^2\alpha}{b^2} - 2F\,.\,\frac{\sin\alpha}{b} - 2G\,.\,\frac{\cos\alpha}{a} + 2H\,.\,\frac{\sin\alpha\cos\alpha}{ab} = 0 \quad \text{......(ii)}.$$

Also, if the normal at α passes through (f, g), then

$$\frac{f\sin\alpha}{b} - \frac{g\cos\alpha}{a} = \frac{a^2-b^2}{ab}\sin\alpha\cos\alpha \quad \text{............(iii)}.$$

Now the equations (ii) and (iii) must give the same values of α;

$$\therefore\quad A = B = 0, \quad \frac{F}{f} = \frac{G}{-g} = \frac{H}{a^2-b^2}.$$

But the Cartesian equation corresponding to (i) is now

$$\sqrt{Fx} + \sqrt{Gy} + \sqrt{H} = 0,$$

$$\text{i.e.}\quad \sqrt{fx} + \sqrt{-gy} + \sqrt{a^2-b^2} = 0.$$

10. Let $AP = x$, $AP' = x'$, so that

$$\left.\begin{aligned}(x+x')\cos\alpha &= l\cos\theta + l'\cos\theta' \\ x\sin\alpha + l\sin\theta &= x'\sin\alpha + l'\sin\theta'\end{aligned}\right\} \text{............(i)}.$$

The equation of virtual work is

$$W\,.\,\delta(x\sin\alpha + \tfrac{1}{2}l\sin\theta) + W'\,.\,\delta(x'\sin\alpha + \tfrac{1}{2}l'\sin\theta') = 0,$$

i.e. $(W\,.\,\delta x + W'\,.\,\delta x')\sin\alpha + \frac{1}{2}(W\,.\,l\cos\theta\,.\,\delta\theta + W'\,.\,l'\cos\theta'\,.\,\delta\theta') = 0$ ......(ii).

Now from the equations (i),

$$(\delta x + \delta x') \cos \alpha = -l \sin \theta \,.\, \delta\theta - l' \sin \theta' \,.\, \delta\theta',$$

$$(\delta x - \delta x') \sin \alpha = -l \cos \theta \,.\, \delta\theta + l' \cos \theta' \,.\, \delta\theta'.$$

Solving these, we find

$$2\delta x = -\frac{l \cos(\alpha - \theta)}{\sin \alpha \cos \alpha} \,.\, \delta\theta + \frac{l' \cos(\alpha + \theta')}{\cos \alpha \sin \alpha} \,.\, \delta\theta',$$

$$2\delta x' = \frac{l \cos(\alpha + \theta)}{\sin \alpha \cos \alpha} \,.\, \delta\theta - \frac{l' \cos(\alpha - \theta')}{\sin \alpha \cos \alpha} \,.\, \delta\theta'.$$

Substituting in (ii) and reducing, this leads to

$$\left[-W \,.\, \frac{\sin \alpha \sin \theta}{\cos \alpha} + W' \,.\, \frac{\cos(\alpha + \theta)}{\cos \alpha}\right] l\delta\theta$$
$$+ \left[-W' \,.\, \frac{\sin \alpha \sin \theta'}{\cos \alpha} + W \,.\, \frac{\cos(\alpha + \theta')}{\cos \alpha}\right] l'\delta\theta' = 0.$$

Since this must be satisfied for all values of $\delta\theta : \delta\theta'$, we must have the coefficients of $l\delta\theta$ and $l'\delta\theta'$ both zero, leading to the results given.

11. Let U_1, U_2 be the velocities of the falling sphere along and perpendicular to the line of impact just after impact, u that of the other sphere horizontally, θ the angle the line of impact makes with the vertical. Then

$$U_2 = V \sin \theta \quad \text{.......................(i)},$$

and by Newton's Law

$$U_1 + u \sin \theta = eV \cos \theta \quad \text{..................(ii)}.$$

Also the force of impact is $m(V \cos \theta + U_1)$ and the horizontal resolved part of this is the momentum communicated to the other sphere;

$$\therefore \quad (V \cos \theta + U_1) \sin \theta = u \quad \text{...............(iii)}.$$

Further, since the sphere moves horizontally after impact,

$$\therefore \quad U_1 \cos \theta = U_2 \sin \theta \quad \text{..................(iv)}.$$

From (ii) and (iii) we find

$$U_1 = \frac{V \cos \theta \, (e - \sin^2 \theta)}{1 + \sin^2 \theta}.$$

Hence from (i) and (iv),

$$\frac{\cos^2\theta\,(e-\sin^2\theta)}{1+\sin^2\theta}=\sin^2\theta\,;$$

$$\therefore\ \tan^2\theta=\frac{e}{2}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)}.$$

Now the horizontal velocity is

$$U_1\sin\theta+U_2\cos\theta=V\sin\theta\cos\theta\left(\frac{e-\sin^2\theta}{1+\sin^2\theta}+1\right)$$

$$=V\sin\theta\cos\theta\,\frac{e+1}{1+\sin^2\theta}$$

$$=V\,.\,\frac{e^{\frac{1}{2}}}{2^{\frac{1}{2}}},\ \text{from (v)}.$$

Also if t is the time before the impact on the table, then

$$\tfrac{1}{2}gt^2=2c\cos\theta,$$

$$\text{i.e.}\ \ t=\frac{2c^{\frac{1}{2}}}{g^{\frac{1}{2}}}\cdot\frac{2^{\frac{1}{4}}}{(2+e)^{\frac{1}{4}}},$$

and the required distance is

$$V\,.\,\frac{e^{\frac{1}{2}}}{2^{\frac{1}{2}}}\,.\,t=\frac{2^{\frac{3}{4}}}{g^{\frac{1}{2}}}\cdot\frac{e^{\frac{1}{2}}c^{\frac{1}{2}}}{(2+e)^{\frac{1}{4}}}\,.\,V.$$

12. Taking the co-ordinates of any point on the curve in the form $x=a\cos^3\theta$, $y=a\sin^3\theta$, we easily obtain $ds=3a\cos\theta\sin\theta\,d\theta$, and $\int 2\pi y\,ds$ thus takes the form

$$2\pi\int_0^{\frac{\pi}{2}} a\sin^3\theta\,.\,3a\cos\theta\sin\theta\,d\theta$$

$$=6\pi a^2\int_0^{\frac{\pi}{2}}\sin^4\theta\cos\theta\,d\theta$$

$$=6\pi a^2\int_0^1 t^4\,dt,\ \text{putting}\ \sin\theta=t,$$

$$=\tfrac{6}{5}\pi a^2.$$

LXXXI.

1. If A, B, C, D be four points on a circle, and O, O' the orthocentres of the triangles ACB, ADB, then OO' is equal and parallel to CD, for, drawing the perpendiculars AOL and $BO'L'$, we have from similar triangles

$$OC : AB = CL : AL, \quad O'D : AB = DL' : BL'.$$

But since $A\hat{C}B = A\hat{D}B$, therefore $CL : AL = DL' : BL'$, therefore $OC = O'D$. Hence OC, $O'D$ being equal and parallel, so also are OO' and CD.

Now it is easy to see from cyclic quadrilaterals that the pedal lines of A and D make equal angles with BC. Hence if the pedal line of A meets the perpendicular from D on BC in M, their intersection will be the middle point of LM.

But if AL meets the circle in E, $DMLE$ is a parallelogram, and therefore, since $OL = LE$, it follows that OL and DM are equal and parallel. Hence, similarly, if on the perpendicular from C on AD we take $CM' = O'L'$, the intersection of the other two pedal lines will be the middle point of $L'M'$. Hence we have to shew that $LM'ML'$ is a parallelogram. But since OO' is equal and parallel to CD, OL to DM, and $O'L'$ to CM', it follows that LL' is equal and parallel to MM'.

2. Let Q' be the other extremity of the diameter through P': then PQ' is parallel to the axis. Therefore $P'PQ'$ is a right angle. But the angle between PP' and the tangent at P is equal to the angle $PQ'P'$. Therefore the tangent at P is perpendicular to $P'Q'$.

3. Let $\dfrac{x}{a} = X$, etc. Then the equations are

$$\Sigma aX = 0, \quad \Sigma aX^2 = 0, \quad \Sigma X^{-1} = 0.$$

From the first two

$$\frac{a}{YZ(Y-Z)} = \ldots = \ldots = \frac{k}{XYZ}, \text{ suppose.}$$

Then $aX = k(Y-Z)$, etc., whence

$$(a+k)X + (b-k)Y = 0, \quad (b+k)Y + (c-k)Z = 0,$$
$$(c+k)Z + (a-k)X = 0,$$

and from these

$$(a+k)(b+k)(c+k) = -(a-k)(b-k)(c-k); \therefore k^2 = -\frac{abc}{a+b+c}.$$

Further, since $\Sigma X^{-1} = 0$, we have

$$1 - \frac{b-k}{a+k} - \frac{c+k}{a-k} = 0,$$

$$\text{i.e. } a^2 - 3k^2 - a(b+c) + k(b-c) = 0,$$

and two similar equations. Adding these we get

$$\Sigma a^2 - 9k^2 - 2\Sigma bc = 0;$$

$$\therefore (\Sigma a^2 - 2\Sigma bc)\Sigma a + 9abc = 0,$$

$$\text{i.e. } \Sigma a^2 b = \Sigma a^3 + 3abc,$$

or
$$(b+c)(c+a)(a+b) = \Sigma a^3 + 5abc.$$

4. Let $bx + cy + az = X, \quad cx + ay + bz = Y.$

Then the numerator is

$$(x + \omega y + \omega^2 z)(a + \omega^2 b + \omega c)(x + \omega^2 y + \omega z)(a + \omega b + \omega^2 c)$$
$$= (Y\omega + X\omega^2)(X\omega + Y\omega^2) = X^2 + Y^2 - XY,$$

and the denominator is $(X+Y)^2$.

Hence we have to shew that

$$\frac{X^2 + Y^2 - XY}{(X+Y)^2} \nless \tfrac{1}{4},$$

which is the same as

$$3(X-Y)^2 \nless 0.$$

5. Consider the equation

$$(ax^2 + bx + c)(p_0 x^3 + p_1 x^2 + p_2 x + p_3) = 0.$$

If the p's be so chosen that

$$\left.\begin{aligned} p_0 b + p_1 a &= 0 \\ p_0 c + p_1 b + p_2 a &= 0 \\ p_1 c + p_2 b + p_3 a &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(\text{A}),$$

then the terms in x^4, x^3, x^2 will be wanting, and therefore the equation has at least two imaginary roots, and these must belong to the cubic

$$p_0x^3 + p_1x^2 + p_2x + p_3 = 0 \text{ ....(B).}$$

But, eliminating the p's between (A) and (B), this cubic is that given in the question.

6. If the radii of two circles are r_1 and r_2, and the distance between their centres d, they intersect at an angle θ given by

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}.$$

Now, for the circumcircle and the polar circle,

$$r_1 = R, \quad r_2^2 = -4R^2 \cos A \cos B \cos C,$$

$$d^2 = R^2(1 - 8\cos A \cos B \cos C).$$

Hence

$$\cos\theta = \frac{4\cos A \cos B \cos C}{4(-\cos A \cos B \cos C)^{\frac{1}{2}}} = -(-\cos A \cos B \cos C)^{\frac{1}{2}}.$$

For the nine-point circle and the polar circle,

$$r_1 = \tfrac{1}{2}R, \quad r_2^2 = -4R^2 \cos A \cos B \cos C,$$

$$d^2 = \tfrac{1}{4}R^2(1 - 8\cos A \cos B \cos C),$$

and the result is obtained as before.

7. We have

$$chd\,\theta = 2\sin\frac{\theta}{2} = \theta - \frac{1}{3!}\cdot\frac{\theta^3}{4} + \frac{1}{5!}\cdot\frac{\theta^5}{16} - \dots.$$

Hence $A\,chd\,\theta + B\,chd\,2\theta + C\,chd\,4\theta$

$$= A\left(\theta - \frac{1}{3!}\cdot\frac{\theta^3}{4} + \frac{1}{5!}\cdot\frac{\theta^5}{16} - \dots\right)$$

$$+ B\left(2\theta - \frac{1}{3!}\cdot 2\theta^3 + \frac{1}{5!}\cdot 2\theta^5 - \dots\right)$$

$$+ C\left(4\theta - \frac{1}{3!}\cdot 16\theta^3 + \frac{1}{5!}\cdot 64\theta^5 - \dots\right),$$

and the coefficients of θ^3 and θ^5 will both vanish, provided

$$\frac{A}{4} + 2B + 16C = 0, \quad \text{and} \quad \frac{A}{16} + 2B + 64C = 0,$$

$$\text{i.e.} \quad \frac{A}{256} = \frac{B}{-40} = \frac{C}{1} = \frac{A + 2B + 4C}{180}.$$

Hence the equation given is correct if θ^7 and higher powers can be disregarded.

8. The equation to the second circle is

$$(x - p\cos\alpha)^2 + (y - p\sin\alpha)^2 = r^2 - p^2 \,(\equiv r'^2),$$

and the external centre of similitude is $\dfrac{pr\cos\alpha}{r - r'}$, $\dfrac{pr\sin\alpha}{r - r'}$.

The line

$$x\cos(\alpha + \theta) + y\sin(\alpha + \theta) = r$$

is a tangent to $x^2 + y^2 = r^2$. If this passes through the centre of similitude, we must have

$$\frac{pr\cos\theta}{r - r'} = r, \qquad \text{i.e.} \quad \cos\theta = \frac{r - r'}{p}$$

(as is obvious geometrically).

Hence

$$\sin^2\theta = \frac{p^2 - (r - r')^2}{p^2} = \frac{2p^2 - 2r^2 + 2rr'}{p^2},$$

and the common tangents are

$$(x\cos\alpha + y\sin\alpha)(r - r') \pm (y\cos\alpha - x\sin\alpha)\sqrt{2p^2 - 2r^2 + 2rr'} = pr.$$

9. The conic $\Sigma l\beta\gamma = 0$ is a rectangular hyperbola if

$$\Sigma l\cos A = 0 \quad \text{..........................(i).}$$

The normals at A and B are (XLIV. 9)

$$(m - n\cos A)\beta - (n - m\cos A)\gamma = 0,$$

$$(n - l\cos B)\gamma - (l - n\cos B)\alpha = 0,$$

or

$$m(\beta + \gamma\cos A) - n(\gamma + \beta\cos A) = 0,$$

$$n(\gamma + \alpha\cos B) - l(\alpha + \gamma\cos B) = 0.$$

Hence, if (α, β, γ) be their intersection,

$$\frac{l}{n} = \frac{\gamma + \alpha \cos B}{\alpha + \gamma \cos B}, \quad \frac{m}{n} = \frac{\gamma + \beta \cos A}{\beta + \gamma \cos A},$$

and, substituting in (i), we get the locus.

10. Draw AL, DM, EN perpendicular to BC. Then the sum of the resolved parts of the forces along BC is

$$-BL + BC - CM - LM + LC - BM$$
$$= LC - BM = MC - BL = NC - BN = 2NF.$$

The sum of the resolved parts perpendicular to BC is

$$AL + DM = 2EN.$$

Hence the resultant is $2EF$ in magnitude and direction.

Suppose the line of action cuts AD in X, and let EF make an angle θ with AD. Then, taking moments about A,

$$2EF \, . \, AX \sin\theta = 2 \triangle BCD,$$

$$\text{i.e.} \quad EN \, . \, AX = \tfrac{1}{2} DM \, . \, BC; \qquad \therefore \quad AX = \tfrac{1}{2} BC.$$

11. Let A be the point of projection of one of the paths, O the foot of the perpendicular h, ON the distance d. Let angle $OAN = \alpha$, and let the angle of projection be θ.

The equation to the path is

$$y = x \tan\theta - \frac{1}{2} g \cdot \frac{x^2}{u^2}.$$

This passes through the point $\left(\dfrac{d}{\sin\alpha}, h\right)$, and we therefore have

$$h = \frac{d}{\sin\alpha} \, . \tan\theta - \frac{1}{2} g \cdot \frac{d^2}{u^2 \sin^2\alpha},$$

whence
$$\frac{2u^2 \tan\theta}{g} - \frac{d}{\sin\alpha} = \frac{2hu^2}{gd} \sin\alpha.$$

But the range is $\dfrac{2u^2 \tan\theta}{g}$. Hence, if the particle strikes the plane again at B,

$$OB = \frac{2u^2 \tan\theta}{g} - \frac{d}{\sin\alpha};$$

$$\therefore \quad OB = \frac{2hu^2}{gd}\sin\alpha.$$

Now draw BC perpendicular to OB, to meet NO produced in C. Then $OB = OC \sin\alpha$. Therefore $OC = \frac{2hu^2}{gd}$, so that C is a fixed point, and the locus of B is the circle on OC as diameter.

12. The curve is symmetrical about the axis of x, and since the tangents at the origin are $x^2 + y^2 = 0$, the origin is a conjugate point. The line $x + a = 0$ is an asymptote, and since for large values of x

$$\begin{aligned} y &= \pm x\left(1 - \frac{a}{x}\right)^{\frac{1}{2}}\left(1 + \frac{a}{x}\right)^{-\frac{1}{2}} \\ &= \pm x\left(1 - \frac{1}{2}\cdot\frac{a}{x} - \ldots\right)\left(1 - \frac{1}{2}\cdot\frac{a}{x} + \ldots\right) \\ &= \pm x\left(1 - \frac{a}{x} + \ldots\right), \end{aligned}$$

therefore the lines $y = x - a$ and $y = -x + a$ are asymptotes. Also from the equation, if $-a < x < a$, y is imaginary.

Differentiating and reducing, we find

$$y_1^2 = \frac{(x^2 + ax - a^2)^2}{(x - a)(x + a)^3}, \quad \left(y_1 \equiv \frac{dy}{dx}\right) \quad \ldots\ldots\ldots\ldots(\text{i}).$$

Differentiating again, and putting $y_2 = 0$, this gives

$$2(2x + a)(x^2 - a^2) - (x^2 + ax - a^2)(x + a + 3\overline{x - a}) = 0$$

reducing to $a^2x - 2a^3 = 0$, whence $x = 2a$.

For this value we find

$$y = \pm\frac{2}{\sqrt{3}}a, \qquad y_1 = \pm\frac{5}{3\sqrt{3}} \text{ from (i)},$$

and the tangents at these points are

$$y \mp \frac{2}{\sqrt{3}}a = \pm\frac{5}{3\sqrt{3}}(x - 2a),$$

or

$$5x \mp 3\sqrt{3}y - 4a = 0.$$

LXXXII.

1. The inverse of the circumcircle of a triangle with regard to the incircle is the nine-point circle of the triangle formed by joining the points of contact of the incircle. Hence the centres of these three circles are collinear. But for the triangle formed by the points of contact, two of these points are the circumcentre and the nine-point centre. Hence the line joining them passes through the centroid and orthocentre.

2. Let V be the middle point of PQ, and let TV meet the parabola in P'. Draw SY perpendicular to TP, and draw $P'Z$ the tangent at P', meeting TP in Z. Then Z is the middle point of TP, and therefore ZV is parallel to TQ. Hence

$$T\hat{V}Z = V\hat{T}Q = S\hat{T}P,$$

and
$$S\hat{P}T = S\hat{T}Q = V\hat{T}Z.$$

Hence the triangles STP, VZT are similar;

$$\therefore \quad TV : TZ = TP : SP,$$

$$\text{i.e. } 2SP \,.\, TV = TP^2.$$

Also
$$SP : SY = TV : PV;$$

$$\therefore \quad \rho = \frac{2SP^2}{SY} = 2SP \,.\, \frac{TV}{PV} = \frac{TP^2}{PV} = \frac{2TP^2}{PQ}.$$

3. We have $x_2 = -\dfrac{bx_1 + d}{ax_1 + c}$, and so on.

Hence if $\theta(x) = -\dfrac{bx + d}{ax + c}$, we have

$$x_2 = \theta(x_1), \quad x_3 = \theta(x_2) = \theta[\theta(x_1)] = \theta^2(x_1) \text{ say},$$

$$x_4 = \theta^3(x_1), \quad x_1 = \theta(x_4) = \theta^4(x_1),$$

and it is evident that the equation $\theta^4(x) = x$ is satisfied by each of the quantities x_1, x_2, x_3, x_4.

Now the values of x for which $\theta(x) = x$ are the roots of the quadratic

$$ax^2 + (b + c)x + d = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Denoting these by x', x'', we have

$$\theta(x) - x' = \theta(x) - \theta(x') = \frac{(ad-bc)(x-x')}{(ax+c)(ax'+c)}.$$

Hence

$$\frac{\theta(x) - x'}{\theta(x) - x''} = \frac{x-x'}{x-x''}\,.\,\lambda, \quad \text{where} \quad \lambda = \frac{ax''+c}{ax'+c}.$$

From this
$$\frac{\theta^4(x) - x'}{\theta^4(x) - x''} = \frac{x-x'}{x-x''}\,.\,\lambda^4,$$

and therefore if $\theta^4(x) = x$, we must have $\lambda^4 = 1$, i.e. $\lambda^2 = \pm 1$. Now λ^2 cannot be 1, for then we should have $\theta^2(x) = x$, which is not the case for the values of x in question. We must therefore have $\lambda^2 = -1$, i.e. $(ax'+c)^2 + (ax''+c)^2 = 0$. Hence, from quadratic (i),

$$a^2\left[\frac{(b+c)^2}{a^2} - \frac{2d}{a}\right] - 2ac\,.\,\frac{b+c}{a} + 2c^2 = 0,$$

reducing to $b^2 + c^2 = 2ad$.

4. Since $\dfrac{1}{r(p+r)} = \dfrac{1}{p}\left(\dfrac{1}{r} - \dfrac{1}{p+r}\right)$, the series is equal to

$$\frac{1}{p}\left\{\sum_1 \frac{x^r}{r} - \frac{1}{x^p}\,.\,\sum_1 \frac{x^{p+r}}{p+r}\right\}$$

$$= \frac{1}{p}\left\{-\log(1-x) - \frac{1}{x^p}\left[-\log(1-x) - x - \frac{x^2}{2} - \dots - \frac{x^p}{p}\right]\right\}$$

$$= \frac{1}{px^p}\left\{(1-x^p)\log(1-x) + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^p}{p}\right\}.$$

When $x = 1$, we may proceed independently, as follows: Multiplying the series by p, the sum of the first p terms is

$$\left(1 + \frac{1}{2} + \dots + \frac{1}{p}\right) - \left(\frac{1}{p+1} + \frac{1}{p+2} + \dots + \frac{1}{2p}\right),$$

the sum of the next p is

$$\left(\frac{1}{p+1} + \frac{1}{p+2} + \dots + \frac{1}{2p}\right) - \left(\frac{1}{2p+1} + \frac{1}{2p+2} + \dots + \frac{1}{3p}\right),$$

and, generally, the sum of the $(r+1)$th group of p is

$$\left(\frac{1}{rp+1} + \frac{1}{rp+2} + \dots + \frac{1}{\overline{r+1}p}\right) - \left(\frac{1}{\overline{r+1}p+1} + \dots + \frac{1}{\overline{r+2}p}\right).$$

Hence the sum of the first $(r+1)p$ terms is

$$\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p}\right)-\left(\frac{1}{\overline{r+1}\,p+1}+\ldots+\frac{1}{r+2p}\right).$$

The latter bracket is

$$<\frac{p}{(r+1)p+1}<\frac{1}{r+1+p^{-1}},$$

and therefore, since p is finite, may be made as small as we please by sufficiently increasing r. Hence the result.

5. Suppose x a root of $x^3-px^2+qx-r=0$. Then

$$\begin{aligned} x^3 &= px^2-qx+r, \\ x^4 &= px^3-qx^2+rx \\ &= p\,(px^2-qx+r)-qx^2+rx \\ &= (p^2-q)x^2-(pq-r)x+pr, \end{aligned}$$

and by continuing this process, any integral power of x can be expressed in the form $\lambda x^2+\mu x+\nu$. Thus any rational function of x can be reduced to the form

$$\frac{Ax^2+Bx+C}{A'x^2+B'x+C'},$$

where A, B, C, A', B', C' are rational functions of p, q and r.

This can be further reduced to $\dfrac{ax+b}{cx+d}$, if we can make

$$\frac{Ax^2+Bx+C}{A'x^2+B'x+C'}=\frac{ax+b}{cx+d}$$

identical with the given equation.

The necessary conditions are

$$\begin{aligned} Ac-A'a &= \frac{Bc+Ad-B'a-A'b}{-p} \\ &= \frac{Cc+Bd-C'a-B'b}{q}=\frac{Cd-C'b}{-r}, \end{aligned}$$

and these will in general give unique determinate values for the ratios $a:b:c:d$, and these values are evidently rational functions of p, q and r.

6. We may suppose a positive. Then considering the graph

$$y = a + b\sin x - c\tan x$$

between 0 and $-\frac{\pi}{2}$, when x is near $-\frac{\pi}{2}$, y is very large and of the same sign as c. Hence if the graph is to cut the axis of x twice, and only twice, between 0 and $-\frac{\pi}{2}$, then c must also be positive. The turning value of the function is given by

$$b\cos x - c\sec^2 x = 0, \quad \text{i.e.} \quad \cos x = \left(\frac{c}{b}\right)^{\frac{1}{3}}.$$

This will give a real value of x between 0 and $-\frac{\pi}{2}$ if b is positive and $> c$. The value of y is then (since $\sin x$ and $\tan x$ are both negative)

$$a - b\,.\,\frac{\sqrt{b^{\frac{2}{3}} - c^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + c\,.\,\frac{\sqrt{b^{\frac{2}{3}} - c^{\frac{2}{3}}}}{c^{\frac{1}{3}}} = a - (b^{\frac{2}{3}} - c^{\frac{2}{3}})^{\frac{3}{2}}.$$

If the graph cuts the axis of x in *two* real points, this must be *negative*, i.e. $b^{\frac{2}{3}} - c^{\frac{2}{3}} > a^{\frac{2}{3}}$.

7. We have

$$\log(1+i) = \log\left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}$$

$$= \tfrac{1}{2}\log 2 + \frac{i\pi}{4};$$

$$\therefore\quad i^{\log(1+i)} = \left(e^{\frac{i\pi}{2}}\right)^{\frac{1}{2}\log 2 + \frac{i\pi}{4}}$$

$$= e^{\frac{i\pi}{4}\log 2 - \frac{\pi^2}{8}}$$

$$= e^{-\frac{\pi^2}{8}}\left\{\cos\left(\frac{\pi}{4}\log 2\right) + i\sin\left(\frac{\pi}{4}\log 2\right)\right\}.$$

8. Let P be the point m, then Q is $-\frac{1}{m}$; let R be m'. Then the equation to QR is

$$2x - y\left(m' - \frac{1}{m}\right) - 2a\frac{m'}{m} = 0 \text{ .... (i)},$$

and the circle is of the form

$$y^2 - 4ax + \lambda(x - my + am^2)\left[2x - \left(m' - \frac{1}{m}\right)y - 2a\frac{m'}{m}\right] = 0.$$

The conditions for a circle are

$$2\lambda = 1 + \lambda m\left(m' - \frac{1}{m}\right) \text{ and } -2m - \left(m' - \frac{1}{m}\right) = 0.$$

From this latter, $m' = \dfrac{1 - 2m^2}{m}$. Hence (i) becomes

$$x + my - a\,.\,\frac{1 - 2m^2}{m^2} = 0,$$

or
$$m^3y + (x + 2a)m^2 - a = 0 \quad\text{.................(ii).}$$

For the envelope

$$3m^2y + 2(x + 2a)m = 0,$$

$$\text{i.e.}\quad m = -\frac{2(x + 2a)}{3y},$$

and substituting in (ii), the envelope is $27ay^2 = 4(x + 2a)^3$.

9. Since the conic passes through B and C, its equation is of the form

$$\alpha^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0 \quad\text{...............(i).}$$

Calling the foci $(\alpha', 0, 0)$ and $(0, \beta'', \gamma'')$; if $l\alpha + m\beta + n\gamma = 0$ be any tangent to (i), the product of the perpendiculars from the foci on it is

$$\frac{l\alpha'(m\beta'' + n\gamma'')}{\Sigma l^2 - 2\Sigma mn \cos A} = k^2, \text{ a constant.}$$

This equation must be the same as the tangential equation of (i), which is

$$\Sigma f^2l^2 - 2(gh - f)mn - 2hfnl - 2fglm = 0.$$

Comparing, we get

$$\frac{k^2}{f^2} = \ldots = \ldots = \frac{k^2 \cos A}{gh - f} = \frac{2k^2 \cos B + \alpha'\gamma''}{2hf} = \frac{2k^2 \cos C + \alpha'\beta''}{2fg}.$$

From the first three, $f = \pm g = \pm h$. Now, if $f = -h$, we get

$$-2k^2 = 2k^2 \cos B + a'\gamma'' \quad \text{or} \quad 4k^2 \cos^2 \frac{B}{2} + a'\gamma'' = 0,$$

an impossible equation, since k^2, a', γ'' are all positive. Hence $f = h$, and similarly $f = g$. This being so,

$$\frac{1}{f^2} = \frac{\cos A}{f^2 - f}, \quad \text{i.e. } f = \frac{1}{1 - \cos A} = \frac{1}{2 \sin^2 \frac{A}{2}}.$$

10. Let R be the pressure of the hoop on either rod. Then, resolving vertically for the system, we have

$$w = R \cos \beta,$$

and taking moments about A for either rod, supposed of length $2l$,

$$R \,.\, 2l \sin(\alpha - \beta) = w \,.\, l \sin \alpha + T \,.\, l \cos \alpha,$$

$$\text{i.e.} \quad T = \frac{2R \sin(\alpha - \beta) - w \sin \alpha}{\cos \alpha}$$

$$= \frac{2w \sin(\alpha - \beta) - w \sin \alpha \cos \beta}{\cos \alpha \cos \beta}$$

$$= w(\tan \alpha - 2 \tan \beta).$$

11. The axle must be in contact with its bearings at some point behind A, the point of contact with the ground. Let R be the pressure at this point, supposed at angular distance θ from A. Then taking moments about A,

$$c \,.\, R \sin \theta = \mu R (c \cos \theta - r),$$

whence $$\sin(\lambda - \theta) = \frac{r}{c} \sin \lambda \quad \text{........ ....(i).}$$

Also if P be the pull on either shaft, we have, since there is no acceleration, resolving horizontally and vertically,

$$P = \mu R \cos \theta - R \sin \theta$$

$$= R \,.\, \frac{\sin(\lambda - \theta)}{\cos \lambda},$$

and $$\tfrac{1}{2}W = R\cos\theta + \mu R\sin\theta$$

$$= R \, . \, \frac{\cos(\lambda - \theta)}{\cos\lambda};$$

$$\therefore \; P = \tfrac{1}{2}W\tan(\lambda - \theta).$$

But from (i) $$\tan(\lambda - \theta) = \frac{r\sin\lambda}{\sqrt{c^2 - r^2\sin^2\lambda}},$$

and the horse is working at rate $2P \, . \, v$.

12. For the curve $r^n = a^n \cos n\theta$, we have

$$\phi = \frac{\pi}{2} + n\theta,$$

$$\psi = \theta + \phi = \frac{\pi}{2} + (n+1)\,\theta,$$

ϕ and ψ having their usual meanings.

Also $$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2\sec^2 n\theta;$$

$$\therefore \; \rho = \frac{ds}{d\psi} = \frac{1}{n+1} \cdot \frac{ds}{d\theta} = \frac{1}{n+1} \, . \, r\sec n\theta$$

$$= \frac{a}{n+1}(\cos n\theta)^{\frac{1}{n}-1}.$$

The radius of curvature of the evolute is

$$\frac{d\rho}{d\psi} = \frac{1}{n+1} \cdot \frac{d\rho}{d\theta}$$

$$= \frac{1}{n+1} \cdot \frac{a}{n+1}\left(\frac{1}{n} - 1\right)(\cos n\theta)^{\frac{1}{n}-2} \, . \, (-n\sin n\theta)$$

$$= \frac{n-1}{(n+1)^2} \, . \, a\sin n\theta \, . \, (\cos n\theta)^{\frac{1}{n}-2}.$$

LXXXIII.

1. In such a tetrahedron, the four perpendiculars from the vertices $ABCD$ on the opposite faces meet in a point, say P.

Let α, β, γ, δ be the centroids of the four faces, A' a point on PA such that $PA' = \frac{1}{3}PA$, E the middle point of BC. Then since

$$PA' = \tfrac{1}{3}PA \text{ and } E\delta = \tfrac{1}{3}EA,$$

therefore $\delta A'$ is parallel to PE. Also $\alpha\delta$ is parallel to AD.

But since AD is at right angles to the plane PBC, AD is at right angles to PE; therefore $\alpha\delta A'$ is a right angle. Similarly $\alpha\beta A'$ and $\alpha\gamma A'$ are right angles, i.e. A' lies on the sphere $\alpha\beta\gamma\delta$ and $\alpha A'$ is a diameter. But if L is the foot of the perpendicular AP, then $\alpha LA'$ is a right angle; therefore L lies on the sphere, and similarly for the other feet of perpendiculars. Hence the twelve points, viz.: the feet of perpendiculars, the centroids of the four faces, and the points of trisection of PA, PB, PC, PD nearer P, all lie on the same sphere.

2. Let AR, BR' meet in P, and AB, RR' in V. Then $A\hat{B}R' = A\hat{R}R'$. Hence

$$P\hat{A}B - P\hat{B}A = R\hat{A}B - A\hat{R}V = R\hat{V}A.$$

Thus in the triangle PAB, the base AB is given, and the difference of the base angles is constant. Therefore the locus of the vertex P is a rectangular hyperbola.

3. We may write A_r in the form

$$a_1 a_2 \,.\, S_{r-2} + (a_1 + a_2)\, S_{r-1} + S_r,$$

where S_r is the sum of the products of $a_3, a_4 \ldots a_n$ taken r together. Now consider the effect of replacing a_1 and a_2 by the equal quantities $\dfrac{a_1 + a_2}{2}$. The first term will be increased, since

$$\left(\frac{a_1 + a_2}{2}\right)^2 > a_1 a_2,$$

while the other two remain unaltered. Hence the effect will be to increase A_r, the sum $a_1 + a_2 + \ldots + a_n$ remaining unaltered. It therefore follows that A_r can be increased as long as any two of the quantities $a_1, a_2 \ldots a_n$ are unequal, their sum remaining constant; therefore A_r must be greatest when they are all equal, i.e. each equal to $\frac{A_1}{c_1}$, and in this case the value of A_r is $c_r\left(\frac{A_1}{c_1}\right)^r$. Hence, in general,

$$c_r\left(\frac{A_1}{c_1}\right)^r > A_r,$$

$$\text{i.e. } \frac{A_1}{c_1} > \left(\frac{A_r}{c_r}\right)^{\frac{1}{r}}.$$

4. Suppose A wins at the nth game. Then he must have won *one* of the preceding games, and the remaining $(n-2)$ games are either all drawn, or else B has won one only, the rest being drawn. Also the first set of circumstances could arise in $n-1$ ways, since A could win any one of the first $(n-1)$ games: and the second set could arise in $(n-1)(n-2)$ ways, this being the number of permutations of $(n-1)$ things of which $(n-3)$ are alike.

Hence the probability that A wins at the nth game is

$$p_n = (n-1)\,a^2 b^{n-2} + (n-1)(n-2)\,a^2 b^{n-3} c,$$

and therefore his chance of winning the match is

$$\sum_{n=2}^{n=\infty} p_n = a^2(1-b)^{-2} + 2a^2 c\,(1-b)^{-3}$$

$$= \frac{a^2}{(1-b)^3}(1-b+2c),$$

and since $a+b+c=1$, this is $\dfrac{a^2(a+3c)}{(a+c)^3}$.

5. Let α, β, γ be the roots. Then

$$\Sigma(\alpha-\beta)^2 = 2\Sigma\alpha^2 - 2\Sigma\alpha\beta$$

$$= 2(9p^2 - 6q) - 6q$$

$$= 18(p^2 - q).$$

Hence since $(\alpha-\beta)^2$ etc. are all positive, no one of them can exceed $\sqrt{18(p^2-q)}$.

Also if α be the greatest root and γ the least, then $\alpha-\gamma$ is greater than either $\alpha-\beta$ or $\beta-\gamma$. Hence

$$3(\alpha-\gamma)^2 > \Sigma(\alpha-\beta)^2$$
$$> 18(p^2-q),$$
$$\text{i.e. } \alpha-\gamma > \sqrt{6(p^2-q)},$$

and therefore, *à fortiori*, greater than the quantity in question.

6. Suppose the line does not cut any of the sides internally, and take the figure in which $\alpha > \gamma > \beta$ and α falls between β and γ. If, in this case, the sides make acute angles θ, ϕ, ψ with the given line, we have

$$\sin\theta = \frac{\gamma-\beta}{a}, \quad \sin\phi = \frac{\alpha-\gamma}{b}, \quad \sin\psi = \frac{\alpha-\beta}{c}.$$

Also $\phi = C-\theta$, $\psi = B+\theta$, so that $\phi+\psi = B+C$. Hence, denoting the sines of θ, ϕ, ψ by λ, μ, ν, we have

$$\cos\phi\cos\psi - \mu\nu = -\cos A,$$
$$\text{i.e. } (1-\mu^2)(1-\nu^2) = (\mu\nu-\cos A)^2,$$

whence $$\mu^2+\nu^2-2\mu\nu\cos A = \sin^2 A \quad\text{..................(i).}$$

We also have $\lambda\sin A + \mu\sin B - \nu\sin C = 0$, whence

$$\mu^2 = \frac{\mu\nu\sin C - \lambda\mu\sin A}{\sin B}, \quad \nu^2 = \frac{\nu\lambda\sin A + \mu\nu\sin B}{\sin C}.$$

Substituting in (i) and reducing, we get

$$\mu\nu\sin A + \nu\lambda\sin B - \lambda\mu\sin C = \sin A\sin B\sin C,$$

whence $$\frac{(\alpha-\gamma)(\alpha-\beta)}{bc}\cdot\frac{a}{2R} + \frac{(\alpha-\beta)(\gamma-\beta)}{ca}\cdot\frac{b}{2R}$$
$$-\frac{(\gamma-\beta)(\alpha-\gamma)}{ab}\cdot\frac{c}{2R} = \frac{abc}{8R^3},$$

giving the required equation. Modified forms will arise in the three cases in which the line cuts pairs of the sides of the triangle internally.

7. Denoting the given sum by S, we have, since

$$2\cos^2\theta = 1 + \cos 2\theta,$$

$$2S = \sum_{r=0}^{r=n-1} \left\{1 + \cos\left(2x + \frac{4r\pi}{n}\right)\right\}$$

$$= n + \sum_{r=0}^{r=n-1} \cos\left(2x + \frac{4r\pi}{n}\right).$$

The latter sum has a factor $\sin\left(n \cdot \frac{2\pi}{n}\right) = 0$;

$$\therefore\ S = \frac{n}{2}.$$

8. The chord θ, α is

$$\frac{x}{a}\cos\frac{\theta+\alpha}{2} + \frac{y}{b}\sin\frac{\theta+\alpha}{2} = \cos\frac{\theta-\alpha}{2},$$

or $\left(\frac{x}{a}\cos\frac{\theta}{2} + \frac{y}{b}\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)\cos\frac{\alpha}{2}$

$$= \left(\frac{x}{a}\sin\frac{\theta}{2} - \frac{y}{b}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\sin\frac{\alpha}{2},$$

and putting the quantities in the brackets equal to X and Y respectively, this is $X = Y\tan\frac{\alpha}{2}$.

Similarly the chords θ, β, etc. are $X = Y\tan\frac{\beta}{2}$, etc. Hence one of the anharmonic ratios is

$$\frac{\left(\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}\right)\left(\tan\frac{\gamma}{2} - \tan\frac{\delta}{2}\right)}{\left(\tan\frac{\gamma}{2} - \tan\frac{\beta}{2}\right)\left(\tan\frac{\alpha}{2} - \tan\frac{\delta}{2}\right)} = \frac{\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma-\delta}{2}}{\sin\frac{\gamma-\beta}{2}\sin\frac{\alpha-\delta}{2}}.$$

9. If $\lambda x + \mu y + \nu = 0$ be one of the chords, the equation to the lines joining its extremities to x', y' is

$$S + (\lambda x + \mu y + \nu)\,T = 0 \quad \text{..................(i)},$$

where $T = 0$ is the tangent at $(x'y')$ to $S = 0$.

To make the conic (i) a pair of lines, we must make (x', y') the centre. This gives the equations

$$2(ax' + hy' + g) + (ax' + hy' + g)(\lambda x' + \mu y' + \nu) = 0,$$
$$2(hx' + by' + f) + (hx' + by' + f)(\lambda x' + \mu y' + \nu) = 0.$$

Hence, since (x', y') is not the centre of S, we must have

$$2 + \lambda x' + \mu y' + \nu = 0,$$

and the equation to the chord is

$$\lambda(x - x') + \mu(y - y') - 2 = 0 \quad \text{..............(ii)}.$$

Further, since the lines (i) are perpendicular, we have

$$a + b + \lambda(ax' + hy' + g) + \mu(hx' + by' + f) = 0.$$

Hence (ii) may be written in the form

$$(a + b)\{\lambda(x - x') + \mu(y - y')\}$$
$$+ 2\{\lambda(ax' + hy' + g) + \mu(hx' + by' + f)\} = 0,$$

shewing that the chord passes through the fixed point in question.

10. When the bowl is about to slip, let B be the point of contact. Let A be the highest point of the sphere, A' the extremity of the central radius OA' of the hemisphere, so that A and A' are originally in contact. Let D be the middle point of OA', L the point of attachment of the scale-pan, W the weight in it: then the centre of gravity of the whole is at a point G in LD, such that $W \,.\, LG = w \,.\, GD$.

Now in the limiting position, the friction at B balances the weight, i.e. the direction of limiting friction at B is vertical. Hence if C be the centre of the sphere, $ACB = \lambda$. Also, from the data, BCO is a diameter of the sphere, therefore $\angle BOA = \frac{\lambda}{2}$, and OAA' is a straight line.

Now the distance of G from OA is $2kr$, where $k = \frac{W}{W + w}$, and r is the radius of the sphere, and the distance of B from OA is

$$2r \sin \frac{\lambda}{2}.$$

Also the projection of BG on OA is

$$2r\cos\frac{\lambda}{2}-(1-k)\,r.$$

But BG makes an angle $\frac{\lambda}{2}$ with OA. Hence

$$\frac{2kr-2r\sin\frac{\lambda}{2}}{2r\cos\frac{\lambda}{2}-(1-k)\,r}=\tan\frac{\lambda}{2}.$$

Substituting for k, this gives

$$\frac{2W-2(W+w)\sin\frac{\lambda}{2}}{2(W+w)\cos\frac{\lambda}{2}-w}=\tan\frac{\lambda}{2},$$

i.e. $W\left(2\cos\frac{\lambda}{2}-\sin\lambda\right)-w\sin\lambda=W\sin\lambda+w\left(\sin\lambda-\sin\frac{\lambda}{2}\right),$

$$\text{i.e. } W=\frac{w\left(2\sin\lambda-\sin\frac{\lambda}{2}\right)}{2\left(\cos\frac{\lambda}{2}-\sin\lambda\right)}.$$

11. Let v be the velocity of the ring along the rod, ω the angular velocity, when the string makes an angle θ with the rod. Then m' has two velocities, viz. v parallel to the rod, and $l\omega$ perpendicular to the string.

Hence the equations of momentum and energy are

$$mv+m'(v+l\omega\sin\theta)=m'V \quad\text{..................(i)},$$

$$\tfrac{1}{2}mv^2+\tfrac{1}{2}m'(v^2+l^2\omega^2+2vl\omega\sin\theta)=\tfrac{1}{2}m'V^2 \quad\text{......(ii)}.$$

From (i) $v=\dfrac{m'V-m'l\omega\sin\theta}{m+m'}$. Hence from (ii)

$$m'V^2=\frac{(m'V-m'l\omega\sin\theta)^2}{m+m'}+m'l^2\omega^2+2m'l\omega\sin\theta\,.\,\frac{m'V-m'l\omega\sin\theta}{m+m'},$$

$$\text{i.e. } m'V^2=\frac{m'^2V^2}{m+m'}+m'l^2\omega^2-\frac{m'^2l^2\omega^2\sin^2\theta}{m+m'};$$

$$\therefore\ mV^2=ml^2\omega^2+m'l^2\omega^2\cos^2\theta,$$

$$\text{i.e. } \omega^2=\frac{mV^2}{(m+m'\cos^2\theta)\,l^2}.$$

12. Let DC meet OA in E, and suppose $OE = c$. Let P be any point on the circle between C and D, and draw PN perpendicular to OA cutting CD in Q. Then if $P\hat{O}B = \theta$, we have

$$QN = NE = c - a\sin\theta.$$

Hence
$$PN^2 - QN^2 = a^2\cos^2\theta - (c - a\sin\theta)^2$$
$$= a^2\cos 2\theta + 2ac\sin\theta - c^2,$$

and the volume required is

$$\int \pi\,(PN^2 - QN^2)\,dx = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (a^2\cos 2\theta + 2ac\sin\theta - c^2)\,a\cos\theta\,d\theta$$

$$= \pi a\left[\tfrac{1}{2}a^2\left(\tfrac{1}{3}\sin 3\theta + \sin\theta\right) - \tfrac{1}{2}ac\cos 2\theta - c^2\sin\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \pi a\left[\left(\frac{\sqrt{3}}{4} - \frac{1}{6} - \frac{1}{4}\right)a^2 + \frac{1}{2}ac - \frac{\sqrt{3}-1}{2}\cdot c^2\right],$$

and since $c = \dfrac{\sqrt{3}+1}{2}a$, this reduces to $\pi a^3\left(\dfrac{\sqrt{3}}{4} - \dfrac{5}{12}\right)$.

LXXXIV.

1. Let A be the area of the base. Then the portion left after the steps have been cut away consists of n prisms, with cross-sections of areas

$$\frac{n^2}{(n+1)^2}\cdot A,\quad \frac{(n-1)^2}{(n+1)^2}\cdot A,\ \ldots\ldots\ \frac{1}{(n+1)^2}\cdot A$$

(counting from the base), each of height $\dfrac{h}{n+1}$. The sum of their volumes is therefore

$$\frac{h}{n+1}\cdot\frac{A}{(n+1)^2}\cdot\Sigma n^2 = \frac{hA}{(n+1)^3}\cdot\frac{n(n+1)(2n+1)}{6}.$$

But if V is the volume of the pyramid, $V = \frac{1}{3}hA$ and therefore the volume remaining is

$$\frac{1}{2} \cdot \frac{n(2n+1)}{(n+1)^2} \cdot V,$$

so that the volume removed is

$$\frac{3n+2}{2(n+1)^2} \cdot V.$$

2. Let UVW be the self-conjugate triangle, C the centre. Let CU cut VW in K. Then if we take P and P' on CU, such that $CP^2 = CU \,.\, CK$, P and P' are on the ellipse.

Four other points on the ellipse can be obtained in a similar manner. Thus, six points being given, the construction can be carried out by Pascal's Theorem.

3. Suppose

$$\frac{ax_1 + hy_1 + gz_1}{x_1} = \ldots = \ldots = k_1.$$

Then $ax_1 + hy_1 + gz_1 = k_1 x_1$, etc. Mutiplying these equations by x_2, y_2, z_2 respectively and adding we have

$$\Sigma a x_1 x_2 + \Sigma f(y_1 z_2 + y_2 z_1) = k_1 (x_1 x_2 + y_1 y_2 + z_1 z_2),$$

and similarly if

$$\frac{ax_2 + hy_2 + gz_2}{x_2} = \ldots = \ldots = k_2,$$

the expression on the left is equal to $k_2 (x_1 x_2 + y_1 y_2 + z_1 z_2)$. Hence since $k_1 \neq k_2$, we must have $x_1 x_2 + y_1 y_2 + z_1 z_2 = 0$.

4. Putting $\log_e x = y$, we have to shew that

$$y > 1 - e^{-y}, \text{ i.e. } e^{-y} > 1 - y$$

for all values of y. This is obvious in two cases, viz. when y is negative (by the Exponential Series), and when y is positive and > 1 (in which case the left-hand side is positive and the right-hand side negative).

If $1 > y > 0$, we have

$$e^y = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \dots$$

$$< 1 + y + y^2 + \dots < \frac{1}{1-y};$$

$$\therefore\ e^{-y} > 1 - y.$$

Hence the result stated is true in all cases.

Now put x equal to $\frac{n+1}{n}, \frac{n+2}{n+1}, \dots \frac{2n}{2n-1}$ in succession, and add the resulting inequalities. We then get $\log_e 2 > S$, where S is the given series.

Again putting x equal to $\frac{n+1}{n+2}, \frac{n+2}{n+3}, \dots \frac{2n}{2n+1}$, and adding, we find $\qquad \log_e\left(\frac{n+1}{2n+1}\right) > -S.$

Hence S lies between $\log_e 2$ and $\log_e\left(2 - \frac{1}{n+1}\right)$, so that clearly, when n is infinite, the limiting value of S is $\log_e 2$.

5. This equation may be written

$$(x-1)(x-2)(x-3) = -p - 6,$$

and putting $x - 2 = y$ this becomes

$$y^3 - y + (p+6) = 0$$

which has three real roots if

$$27(p+6)^2 - 4 \leqslant 0,$$

$$\text{i.e. } (p+6)^2 \leqslant \tfrac{4}{27},$$

i.e. the absolute value of $p + 6$ is $\leqslant \frac{2}{3\sqrt{3}}$.

6. If D divides BC so that $\lambda . BD = \mu . CD$, then

$$\lambda . PB^2 + \mu . PC^2 = (\lambda + \mu) PD^2 + \lambda . BD^2 + \mu . CD^2.$$

Now if D is the foot of the perpendicular from A, then $\lambda = \cot C$, $\mu = \cot B$. Hence

$$PB^2 \cot C + PC^2 \cot B = (\cot C + \cot B) PD^2$$
$$+ \cot C (2R \sin C \cos B)^2 + \cot B (2R \sin B \cos C)^2$$
$$= \frac{\sin A}{\sin B \sin C} . PD^2 + 4R^2 \sin A \cos B \cos C \ldots \text{(i)}.$$

Again, by the above theorem, since $AK : KD = \cos A : \cos B \cos C$, we have

$$PD^2 \cos A + PA^2 \cos B \cos C = PK^2 \sin B \sin C$$
$$+ \cos A (2R \cos B \cos C)^2 + \cos B \cos C (2R \cos A)^2$$
$$= PK^2 \sin B \sin C + 4R^2 \cos A \cos B \cos C \sin B \sin C \ldots \text{(ii)}.$$

Now multiplying (i) by $\cot A$, dividing (ii) by $\sin B \sin C$, and adding, the result follows.

7. Since $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

$$\therefore \frac{x^3}{3!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \ldots = \tfrac{1}{2} (\sinh x - \sin x).$$

Hence the given series is the real part of

$$\tfrac{1}{2} \{\sinh (\cos \alpha . e^{i\alpha}) - \sin (\cos \alpha . e^{i\alpha})\}$$

and this is

$$\tfrac{1}{2} \{\sinh (\cos^2 \alpha) \cos (\cos \alpha \sin \alpha) - \sin (\cos^2 \alpha) \cosh (\cos \alpha \sin \alpha)\}.$$

8. Since the chord of curvature at (x', y') makes the same angle with the major axis as the tangent, its equation is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = \frac{x'^2}{a^2} - \frac{y'^2}{b^2}.$$

If this passes through (X, Y) we have

$$\frac{Xx'}{a^2} - \frac{Yy'}{b^2} = \frac{x'^2}{a^2} - \frac{y'^2}{b^2};$$

$$\therefore (a^2 - b^2)\left(\frac{Xx'}{a^2} - \frac{Yy'}{b^2}\right) = x'^2 - \frac{b^2x'^2}{a^2} - \frac{a^2y'^2}{b^2} + y'^2$$

$$= 2(x'^2 + y'^2) - (a^2 + b^2)\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)$$

$$= 2(x'^2 + y'^2) - (a^2 + b^2).$$

Shewing that (x', y') lies on the given circle.

9. Since (x_0, y_0, z_0) is the centre, we have

$$lx_0 = my_0 = nz_0,$$

and the conic can be written in the form

$$\frac{x^2}{x_0} + \frac{y^2}{y_0} + \frac{z^2}{z_0} = 0.$$

To find where the line

$$\frac{x - x_0}{\lambda} = \frac{y - y_0}{\mu} = \frac{z - z_0}{\nu} = \rho$$

meets this we have $$\Sigma \frac{(\lambda\rho + x_0)^2}{x_0} = 0,$$

$$\text{i.e. } \rho^2 . \Sigma \frac{\lambda^2}{x_0} + 1 = 0 \quad \text{.....................(i)},$$

since $$\Sigma\lambda = 0 \text{..............................(ii)}.$$

Now the square of the distance between (x, y, z) and (x_0, y_0, z_0) is $$-\Sigma a^2 (y - y_0)(z - z_0).$$

Hence λ, μ, ν must be connected by the equation

$$\Sigma a^2\mu\nu + 1 = 0 \quad \text{........................(iii)}.$$

We have therefore to find the maximum and minimum values of $\Sigma \frac{\lambda^2}{x_0}$ $\left(\text{since this is } -\frac{1}{\rho^2}\right)$ subject to (ii) and (iii).

Using the multipliers k and k', the conditions are

$$\frac{\lambda}{x_0} + k(b^2\nu + c^2\mu) + k' = 0,$$

and two similar equations. Multiplying these by λ, μ, ν and adding, we get

$$-\frac{1}{\rho^2} - 2k = 0.$$

We therefore have

$$-\frac{2\rho^2}{x_0}\lambda + c^2\mu + b^2\nu - 2\rho^2 k' = 0$$

and two similar equations. Eliminating λ, μ, ν, k' between these and $\Sigma\lambda = 0$, we get the determinant given.

10. Let R be the pressure between the stick and the hemisphere, R' that between the hemisphere and the ground (acting at some point whose position is not required). The only equation for the stick which does not introduce unknown forces is that obtained by resolving vertically. This is

$$\mu R \sin\alpha + R\cos\alpha = W_1,$$

$$\text{i.e.}\quad R\,.\,\frac{\cos(\alpha-\lambda)}{\cos\lambda} = W_1 \quad \text{..................(i)}.$$

Also, resolving horizontally for the hemisphere,

$$\mu R' + \mu R\cos\alpha = R\sin\alpha,$$

$$\text{i.e.}\quad \mu R' = R\,.\,\frac{\sin(\alpha-\lambda)}{\cos\lambda} \quad \text{..................(ii)}.$$

But resolving vertically for the system, $R' = W + W_1$. Hence from (i) and (ii),

$$(W + W_1)\tan\lambda = W_1\tan(\alpha-\lambda).$$

11. At any given instant let v be the velocity of A, v' that of B relative to A (perpendicular to AB), ϕ the angle between OA produced and AB. Then the actual velocity of B is

$$(v^2 + v'^2 + 2vv'\cos\phi)^{\frac{1}{2}}.$$

Hence, if V be the velocity of projection, the energy equation is

$$mv^2 + \tfrac{1}{2}m(v^2 + v'^2 + 2vv'\cos\phi) = \tfrac{1}{2}mV^2,$$

$$\text{i.e. } 3v^2 + v'^2 + 2vv'\cos\phi = V^2 \qquad\text{(i).}$$

Also the equation of angular momentum is

$$2mv \cdot a + mv(a + a\cos\phi) + mv'(a + a\cos\phi) = mV \cdot 2a,$$

where $OA = AB = a$,

$$\text{i.e. } 3v + v' + (v + v')\cos\phi = 2V \qquad\text{(ii).}$$

From (i) and (ii) we have

$$4(3v^2 + v'^2 + 2vv'\cos\phi) = \{3v + v' + (v + v')\cos\phi\}^2 \qquad\text{(iii),}$$

which may be written

$$3(v - v')^2 = 2(3v^2 + v'^2)\cos\phi + (v + v')^2\cos^2\phi.$$

This shews that $\cos\phi$ can never be negative, for if it were, the right-hand side would be negative, since $\cos\phi$ is numerically greater than $\cos^2\phi$. Hence ϕ is always an acute angle, i.e. OAB is always obtuse.

Again, when the string is straight, $\phi = 0$. Hence from (iii)

$$3v^2 + v'^2 + 2vv' = (2v + v')^2,$$

whence $v' = -\tfrac{1}{2}v$, and the velocity of B is now

$$v + v' = \tfrac{1}{2}v.$$

12. Taking P as origin and the tangent at P as axis of x, the equation to the cubic is of the form

$$u_3 + u_2 + y = 0 \qquad\text{(i),}$$

where u_n is the aggregate of terms of degree n.

If the tangent at (x', y') passes through the origin, we have $u_2' + 2y' = 0$, as we see by making (i) homogeneous, u_2' being the result of substituting x', y' for x, y in u_2. Thus the points of contact of tangents through P lie on the conic $u_2 + 2y = 0$, which touches the cubic at the origin. This conic will intersect the cubic in six points, two of which must coincide with P, leaving four other intersections. Hence there are four other tangents passing through P. Further, if $u_2 = ax^2 + bxy + cy^2$, then since the radius of curvature is in either case the limit of $\dfrac{x^2}{2y}$, it is $-\dfrac{1}{2a}$ for the cubic and $-\dfrac{1}{a}$ for the conic, i.e. the curvature of the conic at P is half that of the cubic.

LXXXV.

1. Let T be the point of intersection of the fixed tangents, and let A, T on one tangent correspond to T, B on the other. Then clearly, by symmetry, $TA = TB$. Hence, if perpendiculars AC, BC be drawn to TA, TB respectively, C will be the centre of the circle. Draw ICJ perpendicular to TC meeting TA, TB in I, J. Then I and J are the vanishing points required.

Let P, Q be any pair of corresponding points on TA, TB. Then, since PQ is a tangent to the circle, $I\hat{P}C = C\hat{P}Q$ and similarly $J\hat{Q}C = C\hat{Q}P$. Hence

$$P\hat{C}Q = 180° - (C\hat{P}Q + C\hat{Q}P)$$
$$= 180° - \tfrac{1}{2}(I\hat{P}Q + J\hat{Q}P) = C\hat{I}A \text{ or } C\hat{J}B.$$

Hence $I\hat{P}C = J\hat{C}Q$ and $J\hat{Q}C = I\hat{C}P$, so that the triangles IPC, JCQ are similar,

$$\therefore \frac{IP}{IC} = \frac{JC}{JQ};$$

$$\therefore IP \,.\, JQ = IC^2 = IA \,.\, JT.$$

Hence I and J are the vanishing points, considering these points as defined by the property $IP \,.\, JQ =$ constant.

2. Let S be the given focus, S' the other focus of any one of the conics, SY, $S'Y'$ perpendiculars to the fixed tangent, P the point of contact, and let $S'P$ produced meet SY produced in H. Then if a and b are the semi-axes,

$$SY \,.\, S'Y' = b^2, \quad S'H = 2a.$$

But $\dfrac{b^2}{a}$ is given, as also is SY; therefore $S'Y' : S'H$ is a fixed ratio, i.e. the locus of S' is a conic of which H is a focus, and the given tangent the corresponding directrix. Also, since S' and H are on opposite sides of the tangent, the conic is a hyperbola.

3. If $S = x + 3x^2 + 5x^3 + \dots + (2n-1)x^n$,

then $xS = \qquad x^2 + 3x^3 + \dots + (2n-3)x^n + (2n-1)x^{n+1}$,

and $(1-x)S = x + 2x^2 + 2x^3 + \dots + 2x^n - (2n-1)x^{n+1}$

$$= x + \frac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1};$$

$$\therefore\ S = \frac{x + x^2 - (2n+1)x^{n+1} + (2n-1)x^{n+2}}{(1-x)^2},$$

and if $x = \dfrac{2n+1}{2n-1}$, this is

$$\frac{x(1+x)}{(1-x)^2} = n(2n+1).$$

4. We have

$$N \equiv 2^{37\times 73} - 2 = (2^{37\times 73} - 2^{73}) + (2^{73} - 2).$$

Now $2^{37\times 73} - 2^{73} = 2^{73}(2^{36\times 73} - 1)$, and this is divisible by $2^{36}-1$, and therefore by $2^9 - 1 = 7 \times 73$, i.e. $N = M(73)$, since by Fermat's Theorem $2^{73} - 2 = M(73)$.

Again $\qquad N = (2^{37\times 73} - 2^{37}) + (2^{37} - 2)$

and $2^{37\times 73} - 2^{37} = 2^{37}(2^{36\times 73} - 1)$, and this is divisible by $2^{36}-1$, which by Fermat is $M(37)$.

Also $\qquad 2^{37} - 2 = M(37), \quad \therefore\ N = M(37)$.

5. We have

$$(\alpha - \alpha^4)^2 = \alpha^2 + \alpha^3 - 2, \text{ since } \alpha^5 = 1,$$

$$(\alpha^2 - \alpha^3)^2 = \alpha^4 + \alpha - 2.$$

The sum of these is

$$(\alpha + \alpha^2 + \alpha^3 + \alpha^4) - 4 = -5$$

(since $\alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1$), and the product is, on reduction,

$$-(\alpha + \alpha^2 + \alpha^3 + \alpha^4) + 4 = 5.$$

Hence $\qquad \{x^2 - (\alpha - \alpha^4)^2\}\{x^2 - (\alpha^2 - \alpha^3)^2\} \equiv x^4 + 5x^2 + 5$,

so that the roots of the given equation are $\pm(\alpha - \alpha^4)$, $\pm(\alpha^2 - \alpha^3)$.

6. From a figure,

$$y^2 = \rho^2 + \left(c - x\cos\frac{A}{2}\right)^2, \quad z^2 = \rho^2 + \left(b - x\cos\frac{A}{2}\right)^2;$$

$$\therefore\ by^2 + cz^2 = (b+c)\,\rho^2 + bc\,(b+c) - 4bc\,x\cos\frac{A}{2} + (b+c)\,x^2\cos^2\frac{A}{2}.$$

Now $\rho = x\sin\dfrac{A}{2}$, $\qquad \therefore\ x^2\cos^2\dfrac{A}{2} = x^2 - \rho^2$;

$$\therefore\ ax^2 + by^2 + cz^2 - abc = x^2\,.\,2s + bc\,(b+c-a) - 4bc\rho\cot\frac{A}{2}.$$

Also

$$\cot\frac{A}{2} = \frac{s-a}{r},\quad x^2 = \rho^2\operatorname{cosec}^2\frac{A}{2},\quad r^2\operatorname{cosec}^2\frac{A}{2} = \frac{bc\,(s-a)}{s};$$

$$\begin{aligned}\therefore\ ax^2 + by^2 + cz^2 - abc &= 2bc\,(s-a)\,.\,\frac{\rho^2}{r^2} + bc\,(b+c-a) \\ &\qquad - 4bc\,(s-a)\,.\,\frac{\rho}{r} \\ &= 2bc\,(s-a)\left(\frac{\rho^2}{r^2} + 1 - \frac{2\rho}{r}\right) \\ &= bc\,(b+c-a)\left(\frac{\rho}{r} - 1\right)^2.\end{aligned}$$

7. We have

$$(\cos a + \cos 3a + \cos 9a)^2 = \cos^2 a + \cos^2 3a + \cos^2 5a,$$

for the cross-terms are

$$\begin{aligned}&2\cos a\cos 3a + 2\cos a\cos 9a + 2\cos 3a\cos 9a \\ &\qquad = \cos 2a + \cos 4a + \cos 8a + \cos 10a + \cos 6a + \cos 12a,\end{aligned}$$

which has a factor $\cos 7a$ (using the formula for angles in A.P.).

Now if in the equation $\dfrac{\cos 7\theta}{\cos\theta} = 0$ we put $\cos^2\theta = x$, the roots of the cubic in x will be $\cos^2 a$, $\cos^2 3a$, $\cos^2 5a$. The cubic in question is

$$x^3 - 21x^2(1-x) + 35x(1-x)^2 - 7(1-x)^3,$$

or

$$64x^3 - 112x^2 + 56x - 7 = 0;$$

$$\therefore\ \cos^2 a + \cos^2 3a + \cos^2 5a = \tfrac{112}{64} = \tfrac{7}{4}.$$

Now $\cos a + \cos 3a - \cos 5a$ is evidently positive, since the angles are all acute and $\cos a$, $\cos 3a$ are both $> \cos 5a$. Hence

$$\cos a + \cos 3a + \cos 9a = \frac{\sqrt{7}}{2}.$$

Further $\cos 2\alpha + \cos 6\alpha + \cos 18\alpha$

$$= 2(\cos^2\alpha + \cos^2 3\alpha + \cos^2 9\alpha) - 3$$
$$= \tfrac{7}{2} - 3 = \tfrac{1}{2}.$$

8. Taking the parabolas in the form

$$y^2 = 4a(x-h), \quad x^2 = 4a'(y-k),$$

any normal to the first is

$$m(x-h) + y = am^3 + 2am \quad \text{..................(i)},$$

and to the second

$$m'(y-k) + x = a'm'^3 + 2a'm'.$$

If these coincide we have

$$m = \frac{1}{m'} = \frac{am^3 + (2a+h)m}{a'm'^3 + (2a'+k)m'},$$

whence m is determined by the equation

$$am^6 + (2a+h)m^4 - (2a'+k)m^3 - a'm = 0 \quad \text{.........(ii)},$$

which gives five values of m other than $m = 0$.

If the line (i) touches the general conic, we have

$$Am^2 + B + C[am^3 + (2a+h)m]^2 - 2F[am^3 + (2a+h)m]$$
$$- 2G[am^4 + (2a+h)m^2] + 2Hm = 0 \quad \text{...(iii)}.$$

If (ii) and (iii) give the same values of m, we must have

$$B = 0, \quad A + C(2a+h)^2 - 2G(2a+h) = 0,$$
$$\frac{Ca^2}{a} = \frac{2Ca(2a+h) - 2Ga}{2a+h}.$$

From the third of these equations $C(2a+h) - 2G = 0$, whence, from the second, $A = 0$. Hence since A and B are both zero, the conic touches the axes of co-ordinates, i.e. the axes of the parabolas.

9. If a circle has double contact with an ellipse, its centre must be on an axis of the ellipse. Hence, in the case in question, since the centre of the circle is the origin, and the ellipse passes through the points $(\pm a, 0)$, we have an oblique chord of the ellipse (viz. the axis of x) bisected at a point on an axis of the ellipse. But this is impossible unless the point is the centre. Hence the ellipse and circle are concentric, and in this case the chord of contact must be one of the axes of the

ellipse. Hence one of the semi-axes of the ellipse is c. If c is the semi-major axis, and r the distance of the focus from the centre, the semi-minor axis is $\sqrt{c^2-r^2}$, and the first of the given equations follows from the fact that the semi-diameter making an angle θ with the major axis is of length a. Similarly, if c is the semi-minor axis, we get the second equation.

10. Taking any vertical section, let O, O' be the lower and upper centres,

A the point of contact of lower cylinder and ground,
B ,, ,, ,, ,, two cylinders,
C ,, ,, ,, ,, upper cylinder and wall,
D the highest point of the upper cylinder,

and let DO' meet the ground in N.

Then ABD is a straight line, and the total friction at B must act along it, since the other two forces on the lower cylinder both act through A.

Suppose each of the four angles OAB, OBA, $O'BD$, $O'DB$ is θ. Then $\tan\theta = \dfrac{AN}{DN} = \dfrac{a\cot\alpha - b}{b\cot\beta + b}$, and for equilibrium the angle of friction at B is $\not< \theta$. Hence the first result.

Further the friction at C must act along CD, since the other two forces on the upper cylinder pass through D. But

$$D\hat{C}O' = 45^\circ;$$

therefore the coefficient of friction at C is $\not< 1$.

Now let S be the total friction at B, F and R the friction (horizontal) and reaction at A. Then for the lower cylinder

$$F = S\sin\theta, \quad R = W + S\cos\theta.$$

But from the upper cylinder

$$\frac{S}{\sin 45^\circ} = \frac{W'}{\sin(45^\circ+\theta)}; \quad \therefore\ S = \frac{W'}{\cos\theta + \sin\theta};$$

$$\therefore\ \frac{F}{R} = \frac{W'\sin\theta}{W(\cos\theta+\sin\theta) + W'\cos\theta} = \frac{W'\tan\theta}{W(1+\tan\theta) + W'},$$

and substituting the value of $\tan\theta$ found above, we get the third result.

11. Let S be the total friction in the lower limiting position of equilibrium, l the distance of the bead from the point where the wire meets the vertical axis. Then

$$m\omega^2 . l \cos \alpha = S \cos (90^\circ - \alpha + \lambda) = S \sin (\alpha - \lambda),$$

$$mg = S \cos (\alpha - \lambda);$$

$$\therefore \tan (\alpha - \lambda) = \frac{\omega^2 l \cos \alpha}{g},$$

$$\text{i.e. } l = \frac{g}{\omega^2} . \frac{\tan (\alpha - \lambda)}{\cos \alpha}.$$

Changing the sign of λ for the upper limiting position, we get

$$l' = \frac{g}{\omega^2} . \frac{\tan (\alpha + \lambda)}{\cos \alpha},$$

and the bead must lie in the portion $l' - l$ of the wire.

12. (i) Since $\cos^4 \theta + \sin^4 \theta = \cos^2 2\theta + \frac{1}{2} \sin^2 2\theta$, the indefinite integral takes the form

$$\int \frac{d\theta}{\cos^2 2\theta + \frac{1}{2} \sin^2 2\theta} = \int \frac{\sec^2 2\theta \, d\theta}{1 + \frac{1}{2} \tan^2 2\theta}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \tan 2\theta \right),$$

and this takes the value $\frac{1}{\sqrt{2}} \tan^{-1} 0$ at both limits, the value of the integral thus depending on the interpretation given to the inverse tangent. But, if we put $\theta = \frac{\pi}{2} - \phi$, then

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\theta}{\cos^4 \theta + \sin^4 \theta} = - \int_{\frac{\pi}{4}}^{0} \frac{d\phi}{\sin^4 \phi + \cos^4 \phi} = \int_0^{\frac{\pi}{4}} \frac{d\theta}{\cos^4 \theta + \sin^4 \theta}.$$

Hence the given integral is twice the same indefinite integral taken between the limits $\frac{\pi}{4}$ and 0, so that its value is

$$2 \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \tan 2\theta \right) \right]_0^{\frac{\pi}{4}},$$

and regarding the inverse function as changing continuously with

θ throughout the range of integration, if we take it as zero at the lower limit, it will be $\frac{\pi}{2}$ at the upper limit. Hence the value of the given integral is $\frac{\pi}{\sqrt{2}}$.

(ii) Since $\int_0^{\frac{\pi}{2}} \log \sin \theta \, d\theta = \int_0^{\frac{\pi}{2}} \log \cos \theta \, d\theta$, we have, denoting either integral by I,

$$2I = \int_0^{\frac{\pi}{2}} \log (\sin \theta \cos \theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2\theta \, d\theta + \frac{\pi}{2} \log \tfrac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots(\text{i}).$$

Now putting $2\theta = \phi$, we have

$$\int_0^{\frac{\pi}{2}} \log \sin 2\theta \, d\theta = \tfrac{1}{2} \int_0^{\pi} \log \sin \phi \, d\phi$$

$$= \tfrac{1}{2} \int_0^{\frac{\pi}{2}} \log \sin \phi \, d\phi + \tfrac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \log \sin \phi \, d\phi \quad \ldots(\text{ii}).$$

If in the latter integral we put $\phi = \pi - \psi$, the integral becomes

$$\tfrac{1}{2} \int_0^{\frac{\pi}{2}} \log \sin \psi \, d\psi, \text{ i.e. } \tfrac{1}{2} I.$$

Hence (ii) becomes $\quad \int_0^{\frac{\pi}{2}} \log \sin 2\theta \, d\theta = I,$

whence, from (i), $\quad I = \frac{\pi}{2} \log \tfrac{1}{2}.$

LXXXVI.

1. Let $ABCD$ be the quadrilateral. Draw CE equal and parallel to BD, and AM, CN perpendicular to BD, and let AM produced meet CE in K. Then if H is the middle point of BD, we have

$$AB^2 - AD^2 = BM^2 - MD^2 = 2BD \,.\, HM.$$

Similarly $$CD^2 - BC^2 = 2BD \,.\, HN;$$

$$\therefore\; AB^2 - AD^2 + CD^2 - BC^2 = 2BD \,.\, MN,$$

i.e. $CE \,.\, CK$ is given. Now angle ACK is greatest when the ratio $CK : AC$ is least, i.e. when $AC \,.\, CE$ is greatest. But the rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by pairs of opposite sides, except when the quadrilateral is cyclic, in which case the two magnitudes are equal. Hence the angle ACK is greatest when the quadrilateral is cyclic.

2. Draw SK parallel to the asymptote. Then since TP and the asymptote are tangents,

$$T\hat{S}P = T\hat{S}K = S\hat{T}Q;$$

$$\therefore\; SQ = QT.$$

3. Denoting the expression on the left by s_n, we have, provided t be properly chosen,

$$\Sigma \log \{1 + x(a+\lambda)t\} - \Sigma \log \{1 + x(a-\lambda)t\} = s_1 t - \frac{s_2 t^2}{2} + \frac{s_3 t^3}{3} - \ldots.$$

If $s_n = 0$ for all values of n, then the right side is zero, and we must therefore have

$$\frac{\Pi\{1 + x(a+\lambda)t\}}{\Pi\{1 + x(a-\lambda)t\}} = 1,$$

independently of t, i.e.

$$\lambda \,.\, \Sigma xt(1 + byt)(1 + czt) + \lambda^3 \,.\, xyz \,.\, t^3 = 0.$$

By virtue of the given equations, this reduces to

$$(\Sigma bc)\, xyz \,.\, t^2 + \lambda^2 \,.\, xyz \,.\, t^2 = 0,$$

which will be satisfied if $\lambda^2 = -\Sigma bc$.

4. Evidently n must be even, and a multiple of 3, since $n \pm 1$ are both prime; therefore $n = M(6)$.

Further n cannot be of the form $5p \pm 1$, and it is therefore of the form $5p$ or $5p \pm 2$. In the first case, then, $n = M(30)$. In the second case $5p \pm 2 = 6q$, and since the primes $n-1$ and $n+1$ are of the forms $5p+1$, $5p+3$ or $5p-3$, $5p-1$, therefore p must be even, say $2r$; therefore $5r \pm 1 = 3q$,

$$\text{i.e. } 5r - 3q = \mp (3 \,.\, 2 - 5 \,.\, 1),$$

$$\text{i.e. } 5(r \mp 1) = 3(q \mp 2);$$

$$\therefore \frac{r \mp 1}{3} = \frac{q \mp 2}{5} = t, \text{ an integer};$$

$$\therefore r = 3t \pm 1, \text{ and } n = 10r \pm 2 = 30t \pm 12.$$

Also, if $n = 30m$, then

$$n^2(n^2+16) = 900m^2(900m^2+16)$$
$$= M(180) \times M(4) = M(720).$$

If $n = 30m \pm 12$, then

$$n^2(n^2+16) = (30m \pm 12)^2(900m^2 \pm 720m + 160)$$
$$= M(36) \times M(20) = M(720).$$

5. Let the roots be $a_1, a_2, a_3 \dots a_n$, and consider the value of the function $\Sigma(a_1 - a_2)^2$, the sum being taken for all possible pairs of roots. In this sum $a_1{}^2$ will occur $(n-1)$ times, and so for each of the other squares. Hence the sum is

$$(n-1) \,.\, \Sigma a_1{}^2 - 2\Sigma a_1 a_2 = (n-1)(a_1{}^2 - 2a_2) - 2a_2$$
$$= (n-1)\, a_1{}^2 - 2na_2.$$

Hence if this expression is negative, $\Sigma(a_1 - a_2)^2$ is negative, and therefore the roots cannot be all real.

6. Here a, b, c are the roots of

$$\cot^{-1}(t+x) + \cot^{-1}(t+y) + \cot^{-1}(t+z) = \cot^{-1} t,$$

$$\text{i.e. } \frac{(t+x)(t+y)(t+z) - (3t+x+y+z)}{\Sigma(t+x)(t+y) - 1} = t,$$

or $t^3 + t^2 \,.\, \Sigma x + t(\Sigma yz - 3) + xyz - \Sigma x = t(3t^2 + 2t \,.\, \Sigma x + \Sigma yz - 1)$,

$$\text{i.e. } 2t^3 + t^2 \,.\, \Sigma x + 2t + \Sigma x - xyz = 0,$$

and we must therefore have $\Sigma bc = 1$.

7. Since

$$\cot n\theta \equiv \frac{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n} \cdot i,$$

if $\cot n\theta = \infty$, we have, putting $\cot\theta = x$, the equation

$$(x+i)^n - (x-i)^n = 0,$$

and the roots of this are $\cot\frac{\pi}{n}$, $\cot\frac{2\pi}{n} \dots \cot\frac{(n-1)\pi}{n}$. Hence since $\cot\frac{r\pi}{n} = -\cot\frac{(n-r)\pi}{n}$, we have

$$(x+i)^n - (x-i)^n \equiv 2ni\left(x^2 - \cot^2\frac{\pi}{n}\right)\dots\left\{x^2 - \cot^2\frac{(n-1)\pi}{2n}\right\},$$

and putting $x = 2i$ in this identity, we get

$$(3^n - 1)\, i^n = 2ni\,(-1)^{\frac{n-1}{2}} \,.\, P,$$

where P is the given product.

Now $\qquad i^n = i\,(-1)^{\frac{n-1}{2}}; \quad \therefore\ P = \frac{3^n - 1}{2n}.$

8. The pole of $x\cos\alpha + y\sin\alpha = p$ for the confocal is

$$\left(\frac{a'^2\cos\alpha}{p},\ \frac{b'^2\sin\alpha}{p}\right),$$

and the condition of tangency $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$, so that $a'^2\cos^2\alpha + b'^2\sin^2\alpha = p^2 + \lambda$. Now the line

$$\frac{x - \dfrac{a'^2\cos\alpha}{p}}{\cos\theta} = \frac{y - \dfrac{b'^2\sin\alpha}{p}}{\sin\theta} = r$$

meets the confocal where

$$\frac{1}{a'^2}\left(r\cos\theta + \frac{a'^2\cos\alpha}{p}\right)^2 + \frac{1}{b'^2}\left(r\sin\theta + \frac{b'^2\sin\alpha}{p}\right)^2 = 1,$$

$$\text{i.e. } r^2\left(\frac{\cos^2\theta}{a'^2} + \frac{\sin^2\theta}{b'^2}\right) + \frac{2r\cos(\theta - \alpha)}{p} + \frac{\lambda}{p^2} = 0,$$

and is therefore a tangent if

$$\cos^2(\theta - \alpha) = \lambda\left(\frac{\cos^2\theta}{a'^2} + \frac{\sin^2\theta}{b'^2}\right).$$

Now the angle at which this line cuts the curve is

$$\phi = \frac{\pi}{2} + \alpha - \theta.$$

Hence the above equation becomes

$$\sin^2\phi = \lambda\left[\frac{\sin^2(\phi-\alpha)}{a'^2} + \frac{\cos^2(\phi-\alpha)}{b'^2}\right],$$

and putting $\tan\phi = t$, this is

$$t^2 = \lambda\left[\frac{(\sin\alpha - t\cos\alpha)^2}{a'^2} + \frac{(\cos\alpha + t\sin\alpha)^2}{b'^2}\right],$$

i.e. $(a^2+\lambda)(b^2+\lambda)\,t^2$

$$= \lambda\left[(p^2+\lambda) + t(a^2-b^2)\sin 2\alpha + t^2(a^2+b^2+\lambda-p^2)\right],$$

or $$(a^2b^2+p^2\lambda)\,t^2 - (a^2-b^2)\,\lambda\sin 2\alpha\,.\,t - \lambda\,(p^2+\lambda) = 0.$$

If $\lambda = -p^2$, one value of t is zero, and

$$a'^2\cos^2\alpha + b'^2\sin^2\alpha = 0\,;$$

so the confocal must be a hyperbola, and the line parallel to one of its asymptotes.

If $\lambda = \pm ab$, the product of the values of t is -1, i.e. the two tangents are at right angles, i.e. the chord of contact of two perpendicular tangents to a conic always touches a confocal.

9. Suppose $lx + my = 1$ is the tangent at the vertex. The axis of the parabola is evidently parallel to the line $\frac{x}{a} - \frac{y}{b} = 1$. Hence these two lines are perpendicular, and we have

$$\frac{l}{a} - \frac{m}{b} - \left(-\frac{l}{b} + \frac{m}{a}\right)\cos\omega = 0,$$

$$\text{i.e. } l\,(b + a\cos\omega) - m\,(a + b\cos\omega) = 0 \;\ldots\ldots\ldots\ldots\text{(i)}.$$

Further, the condition of tangency is

$$\frac{l}{b} + \frac{m}{a} - lm = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

(This is easily obtained by forming the equation to the lines joining the origin to the intersections of $lx + my = 1$ with the parabola, and making these lines coincide.)

From (i) and (ii) we obtain

$$\frac{l^{-1}}{b + a \cos \omega} = \frac{m^{-1}}{a + b \cos \omega} = \frac{1}{\dfrac{b + a \cos \omega}{a} + \dfrac{a + b \cos \omega}{b}},$$

and substituting these values, the equation can easily be thrown into the form given.

10. Let S be the pressure on each support, and let the point of support of the rth rod $A_{r-1}A_r$ be at distance x_r from A_{r-1}. Also let Y be the vertical reaction at the point A_{r-1}, $2a$ the length, and W the weight of each rod. Then, taking moments about A_{r-2} for the $(r-1)$th rod, we have

$$x_{r-1} S + 2aY = aW,$$

and about A_r for the rth rod,

$$(2a - x_r) S = 2aY + aW,$$

and from these

$$x_r - x_{r-1} = 2a - \frac{2aW}{S};$$

$$\therefore \quad x_r = (r-1)\, d,$$

where $d = 2a - \dfrac{2aW}{S}$, remembering that $x_1 = 0$, since there is no support under the first rod.

Again, taking moments about the fixed end for the system, we have

$$(2a + x_2) S + (4a + x_3) S + \ldots + (\overline{2n-1}\, a + x_n) S$$

$$= a \,.\, W + 3a \,.\, W + \ldots + (2n-1)\, a \,.\, W,$$

$$\text{i.e. } n(n-1)\, aS + \frac{n(n-1)}{2} dS = n^2 aW,$$

and substituting for d, this becomes

$$2(n-1) S = (2n-1) W;$$

$$\therefore \quad d = 2a \left\{1 - \frac{2(n-1)}{2n-1}\right\} = \frac{2a}{2n-1};$$

$$\therefore \quad \frac{x_r}{2a} = \frac{r-1}{2n-1}, \text{ i.e. } \frac{x_r}{2a - x_r} = \frac{r-1}{2n-r}.$$

11. If θ is the angle of projection, and $AB = a$, we have

$$a = \frac{2u^2}{g} \cdot \frac{\cos\theta \sin(\theta - \overline{90^\circ - a})}{\cos^2(90^\circ - a)} = -\frac{2u^2}{g} \cdot \frac{\cos\theta\cos(\theta + a)}{\sin^2 a};$$

$$\therefore\ u^2 = -ga \cdot \frac{\sin^2 a}{\cos(2\theta + a) + \cos a}.$$

Hence u^2 will be least when

$$\cos(2\theta + a) = -1,$$

$$\text{i.e. } 2\theta + a = 180^\circ \text{ or } \theta = 90^\circ - \tfrac{1}{2}a.$$

In this case

$$u^2 = ga \cdot \frac{\sin^2 a}{1 - \cos a} = ga(1 + \cos a) = 2ga\cos^2 \tfrac{1}{2}a,$$

and the greatest height is

$$\frac{u^2 \sin^2\theta}{2g} = \frac{u^2}{2g}\cos^2\tfrac{1}{2}a = a\cos^4\tfrac{1}{2}a.$$

12. If $u = e^{e^x}$, then $\log u = e^x$;

$$\therefore\ \frac{1}{u}\frac{du}{dx} = e^x, \text{ i.e. } \frac{du}{dx} = u \cdot e^x;$$

$$\therefore\ \Sigma(n+1)a_{n+1}x^n = (\Sigma a_n x^n) \cdot \left(\Sigma \frac{x^n}{n!}\right),$$

and equating coefficients of x^n, we obtain the first result.

Using successive cases of this equation, and remembering that $a_0 = e$ (as is seen by putting $x = 0$), we find

$$a_1 = e, \quad a_2 = e, \quad a_3 = \frac{5e}{6},$$

$$a_4 = \frac{5e}{8}, \quad a_5 = \frac{13e}{30}.$$

LXXXVII.

1. Let ABC be the triangle, L and L' the points of contact of the inscribed circle and escribed circle with BC, D the middle point of BC (and therefore also of LL'), AN the perpendicular from A, $A\alpha$ the bisector of the angle A. Draw CGH perpendicular to $A\alpha$, meeting AB in H. Then $BH = c - b$ $(c > b)$;

$$\therefore\ DG = \tfrac{1}{2}(c - b) = DL = DL',$$

and is therefore given and $D\hat{G}\alpha = \frac{1}{2}A$. But the quadrilateral $AGNC$ is cyclic, therefore $G\hat{N}\alpha = G\hat{A}C = \frac{1}{2}A$. Hence DG touches the circle $GN\alpha$; therefore $D\alpha \,.\, DN = DG^2$. Hence α is a given point and therefore G is given, since it is the intersection with $A\alpha$ of a circle, centre D and radius DL. If then we draw through G a line perpendicular to AG cutting LL' in C, it follows that C will be an angular point.

2. Let the tangent at P cut the asymptotes in L and L', the angle $CL'L$ being a right angle, and let the normal at P cut the axes in G and g and CL in O.

Then PG is parallel to CL';

$$\therefore\ P\hat{G}C = L'\hat{C}G = L\hat{C}G;$$

therefore $CO = OG$. Hence, since GCg is a right angle, O must be the middle point of Gg.

3. This depends on the identity

$$(x + y + z)^5 - \Sigma x^5 = 5\,(x + y)\,(y + z)\,(z + x)\,(\Sigma x^2 + \Sigma xy).$$

From this

$$\frac{5\,(x + y + z)^4 - 5z^4}{(x + y + z)^5 - \Sigma x^5} = \frac{(x + y + 2z)\,\{(x + y + z)^2 + z^2\}}{(y + z)\,(z + x)\,(\Sigma x^2 + \Sigma xy)}$$

$$= \frac{1}{y + z} + \frac{1}{z + x} + \frac{x + y + 2z}{\Sigma x^2 + \Sigma xy}.$$

If now we put $z = -1$, and use the given condition, we get

$$\frac{5\,(x + y - 1)^4 - 5}{(x + y - 1)^5} = \frac{1}{y - 1} + \frac{1}{x - 1} + \frac{x + y - 2}{x^2 + y^2 + 1 - x - y + xy},$$

i.e. $$5-\frac{5}{(x+y-1)^4}=\frac{x+y-1}{y-1}+\frac{x+y-1}{x-1}+\frac{(x+y-1)(x+y-2)}{x^2+y^2+1-x-y+xy}$$

$$=1+\frac{x}{y-1}+1+\frac{y}{x-1}+2-\frac{x^2+y^2+x+y}{x^2+y^2+1-x-y+xy},$$

whence the identity follows.

4. Writing $x+h$ for x, we have

$$\{1+c_1x+c_2x^2+\ldots+h(c_1+2c_2x+3c_3x^2+\ldots)+\ldots\}^n$$
$$=1+k_1x+k_2x^2+\ldots+h(k_1+2k_2x+\ldots)+\ldots,$$

and equating coefficients of h, this gives

$$n(1+c_1x+c_2x^2+\ldots)^{n-1}(c_1+2c_2x+3c_3x^2+\ldots)$$
$$=k_1+2k_2x+3k_3x^2+\ldots,$$

and multiplying both sides by $1+c_1x+c_2x^2+\ldots$, we have

$$n(1+k_1x+k_2x^2+\ldots)(c_1+2c_2x+3c_3x^2+\ldots)$$
$$=(1+c_1x+c_2x^2+\ldots)(k_1+2k_2x+\ldots).$$

Equating coefficients of x^r on each side, we obtain

$$n(c_1k_r+2c_2k_{r-1}+\ldots+rc_rk_1+\overline{r+1}\,c_{r+1})$$
$$=c_rk_1+2c_{r-1}k_2+\ldots+rc_1k_r+(r+1)k_{r+1},$$

i.e. $$(r+1)k_{r+1}+(r-n)c_1k_r+(r-1-2n)c_2k_{r-1}+\ldots=0.$$

5. We have

$$a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\equiv ax^4-bx^3+cx^2-dx+e$$

for all values of x.

In this identity put

$$x=\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)=\frac{1}{2}\cdot\frac{b}{a},$$

and we obtain

$$a\,.\,\Pi\,\tfrac{1}{2}(\beta+\gamma+\delta-\alpha)=\frac{b^4}{16a^3}-\frac{b^4}{8a^3}+\frac{b^2c}{4a^2}-\frac{bd}{2a}+e,$$

and, clearing of fractions, this gives

$$a^4\,.\,\Pi(\beta+\gamma+\delta-\alpha)=-b^4+4ab^2c-8a^2bd+16a^3e.$$

6. If AL is the perpendicular from A on BC, then

$$I\hat{A}L = \tfrac{1}{2}(B - C),$$

and we therefore have

$$\tfrac{1}{4}x^2 = r^2 - AI^2 \sin^2 \frac{B-C}{2}$$

$$= r^2 \operatorname{cosec}^2 \frac{A}{2}\left(\sin^2 \frac{A}{2} - \sin^2 \frac{B-C}{2}\right)$$

$$= r^2 \operatorname{cosec}^2 \frac{A}{2} \,.\, \cos B \cos C\,;$$

$$\therefore \quad \frac{1}{x^2} = \frac{1}{8r^2} \cdot \frac{1 - \cos A}{\cos B \cos C};$$

$$\therefore \quad \Sigma \frac{1}{x^2} - \frac{1}{4r^2} = \frac{1}{8r^2} \cdot \frac{\Sigma \cos A - \Sigma \cos^2 A - 2 \cos A \cos B \cos C}{\cos A \cos B \cos C}$$

$$= \frac{1}{8r^2} \cdot \frac{\Sigma \cos A - 1}{\cos A \cos B \cos C}.$$

But $$\Sigma \cos A - 1 = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{R};$$

$$\therefore \quad \Sigma \frac{1}{x^2} - \frac{1}{4r^2} = \frac{1}{8rR \cos A \cos B \cos C}.$$

7. From the identity

$$1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \ldots \equiv \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right) \ldots$$

we have $$\frac{4^2}{\pi^4} \,.\, \Sigma \frac{1}{a^2b^2} = \frac{1}{4\,!},$$

$$\frac{4^4}{\pi^8} \,.\, \Sigma \frac{1}{a^2b^2c^2d^2} = \frac{1}{8\,!},$$

where a, b, c, d are any unequal odd numbers.

But $$\Sigma \frac{1}{a^4b^4} \equiv \left(\Sigma \frac{1}{a^2b^2}\right)^2 - 6 \,.\, \Sigma \frac{1}{a^2b^2c^2d^2} - 2 \,.\, \Sigma \frac{1}{a^4b^2c^2},$$

and this is

$$\frac{\pi^8}{4^4}\left\{\frac{1}{(4\,!)^2} - \frac{6}{8\,!} - 2\left(\frac{1}{2\,!} \cdot \frac{1}{6\,!} - \frac{4}{8\,!}\right)\right\} = \frac{\pi^8}{4^4} \cdot \frac{16}{8\,!} = \frac{\pi^8}{2^4 \,.\, 8\,!}.$$

8. Suppose that the circle through the three points is

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

If $(am^2, 2am)$ is on this, we have

$$a^2m^4 + 4a^2m^2 + 2gam^2 + 4fam + c = 0,$$

$$\text{i.e. } a^2m^4 + 2a(2a + g)m^2 + 4fam + c = 0 \quad \text{...........(i)}.$$

This equation gives the parameters of the four points in which the circle cuts the parabola. But three of these are the roots of the equation given, and their sum is $-p$. Also, by (i) the sum of the four m's is zero. Hence the remaining m is p, and we must therefore have the equation

$$(m - p)(m^3 + pm^2 + qm + r) = 0$$

identical with (i). Equating coefficients, we find

$$\frac{2(2a + g)}{a} = q - p^2, \quad \frac{4f}{a} = r - pq, \quad \frac{c}{a^2} = -pr,$$

whence we obtain the values of g, f and c required.

Next consider the triangle formed by the three tangents. If the parameters of the points of contact are m_1, m_2, m_3, then one of the angular points of the triangle is

$$x = am_1m_2 = a \cdot \frac{m_1m_2m_3}{m_3} = -\frac{ar}{m_3},$$

$$y = a(m_1 + m_2) = -a(p + m_3).$$

We may therefore take any angular point to be $-\dfrac{ar}{m}$, $-a(p + m)$.

If this lies on the circle given by the general equation, we find, on substituting and re-arranging,

$$a^2m^4 + 2am^3(ap - f) + m^2(a^2p^2 - 2fap + c) - 2mgar + a^2r^2 = 0.$$

As before, the roots of this equation are m_1, m_2, m_3 and one other, which is

$$-\frac{2(ap - f)}{a} + p = \frac{2f}{a} - p.$$

Hence the biquadratic above must be identical with

$$\left(m + p - \frac{2f}{a}\right)(m^3 + pm^2 + qm + r) = 0.$$

Comparing the two equations, we find

$$a^2q = c, \quad ar + apq - 2qf = -2gr, \quad ar = ap - 2f,$$

whence $\qquad 2g = -a(1+q), \quad 2f = a(p-r), \quad c = a^2q.$

9. The polars of (x_1, y_1) for the two conics are

$$axx_1 + byy_1 = 1, \quad x(y_1+g) + y(x_1+f) + gx_1 + fy_1 = 0.$$

If these coincide,

$$\frac{y_1+g}{ax_1} = \frac{x_1+f}{by_1} = -(gx_1+fy_1).$$

Eliminating y_1, we obtain for x_1 the equation

$$-b \cdot \frac{ax^2g+g}{axf+1} \cdot \left(g - \frac{ax^2g+g}{axf+1}\right) = ax(x+f),$$

and the roots of this, other than $x=0$, must be x_1, x_2, x_3.

On simplifying, the equation is

$$(a^2f^2 - abg^2)x^3 + \ldots + \ldots + (f + bfg^2) = 0;$$

$$\therefore \; x_1x_2x_3 = -\frac{f(1+bg^2)}{a(af^2-bg^2)},$$

i.e. $\dfrac{a}{f} \cdot x_1x_2x_3 = -\dfrac{1+bg^2}{af^2-bg^2}$, and similarly $\dfrac{b}{g} \cdot y_1y_2y_3 = -\dfrac{1+af^2}{bg^2-af^2}$.

10. Let T be the tension of the string, R and R' the normal pressures for w_2 and w_1 respectively. Then for w_2, resolving along and perpendicular to the line of greatest slope, we have

$$w_2 \sin\alpha = \mu_2 R \sin\theta + T\cos\theta,$$

$$\mu_2 R\cos\theta = T\sin\theta$$

(since $\mu_2 R$ acts perpendicular to the string).

Also resolving perpendicular to the plane

$$w_2 \cos\alpha = R.$$

From these we get

$$\tan\alpha = \frac{\mu_2}{\sin\theta}, \quad T = w_2 \sin\alpha\cos\theta.$$

Again, let F and F' be the frictions at w_1 along and perpendicular to the line of greatest slope, R' the normal pressure. Then, for w_1,

$$F = T\cos\theta + w_1 \sin\alpha = \sin\alpha\,(w_2\cos^2\theta + w_1),$$

$$F' = T\sin\theta = \sin\alpha\,(w_2\sin\theta\cos\theta),$$

whence $$F^2 + F'^2 = \sin^2\alpha\left[(w_1+w_2)^2\cos^2\theta + w_1{}^2\sin^2\theta\right].$$

Also $R' = w_1\cos\alpha$, and we must have $\mu_1 > \dfrac{\sqrt{F^2+F'^2}}{R'}$.

11. Let u be the striking velocity, u' the velocity of the striking ball after impact, v that of each of the others. The equation of momentum is

$$2v\cos 30^\circ + u' = u \quad \text{.....................(i)},$$

and by Newton's Law

$$v - u'\cos 30^\circ = -e(-u\cos 30^\circ)\text{..............(ii)}.$$

From (i) and (ii) we find

$$u' = \frac{2-3e}{5}\,.\,u, \quad v = \frac{\sqrt{3}\,(1+e)}{5}\,.\,u,$$

and the loss of kinetic energy is

$$\tfrac{1}{2}mu^2 - \tfrac{1}{2}mu'^2 - mv^2 = \tfrac{1}{2}mu^2\left[1 - \frac{(2-3e)^2}{25} - \frac{6\,(1+e)^2}{25}\right]$$

$$= \tfrac{1}{2}mu^2\,.\,\frac{3\,(1-e^2)}{5}.$$

12. We have

$$\frac{\partial V}{\partial\rho} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial\rho} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial\rho}$$

$$= -\frac{\partial V}{\partial x}\cdot\frac{\sec\theta}{\rho^2} - \frac{\partial V}{\partial y}\cdot\frac{\tan\theta}{\rho^2};$$

$$\therefore\ \rho\frac{\partial V}{\partial\rho} = -x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} \quad \text{..........................(i)},$$

i.e. the operators $\rho\dfrac{\partial}{\partial\rho}$ and $-\left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)$ are equivalent. Operating with these on the two sides of (i) we get

$$\rho \frac{\partial V}{\partial \rho} + \rho^2 \frac{\partial^2 V}{\partial \rho^2} = x^2 \frac{\partial^2 V}{\partial x^2} + 2xy \frac{\partial^2 V}{\partial x \partial y} + y^2 \frac{\partial^2 V}{\partial y^2} + x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y};$$

$$\therefore\ 2\rho \frac{\partial V}{\partial \rho} + \rho^2 \frac{\partial^2 V}{\partial \rho^2} = x^2 \frac{\partial^2 V}{\partial x^2} + 2xy \frac{\partial^2 V}{\partial x \partial y} + y^2 \frac{\partial^2 V}{\partial y^2} \quad \text{......(ii)}.$$

Again, operating on this, we get

$$-2\rho \frac{\partial V}{\partial \rho} - 4\rho^2 \frac{\partial^2 V}{\partial \rho^2} - \rho^3 \frac{\partial^3 V}{\partial \rho^3} = x^3 \frac{\partial^3 V}{\partial x^3} + 3x^2 y \frac{\partial^3 V}{\partial x^2 \partial y} + 3xy^2 \frac{\partial^3 V}{\partial x \partial y^2}$$
$$+ y^3 \frac{\partial^3 V}{\partial y^3} + 2x^2 \frac{\partial^2 V}{\partial x^2} + 4xy \frac{\partial^2 V}{\partial x \partial y} + 2y^2 \frac{\partial^2 V}{\partial y^2},$$

whence, using (ii), the result follows.

LXXXVIII.

1. Let X, Y, Z, α, β, γ be the middle points of BC, CA, AB, OA, OB, OC. Then αZ is parallel to OB and therefore perpendicular to AC, and therefore to ZX, i.e. αZX is a right angle. Similarly αYX is a right angle. Also $\alpha A'X$ is a right angle; therefore A' lies on the sphere αXYZ. But αX, βY, γZ are the lines joining the middle points of opposite edges of the tetrahedron $OABC$; therefore they meet in a point and bisect each other, i.e. α, β, γ, X, Y, Z lie on a sphere and this sphere passes through A'. Similarly it passes through B' and C', and since X, Y, Z lie on it, its section by the plane ABC must be the nine-point circle of the triangle ABC.

2. Since the six points O, O', P, Q, R, S lie on a conic,

$$\therefore\ O\,(PQRS) = O'\,(PQRS),$$

$$\text{i.e. } O\,(ABCD) = O'\,(BCDA).$$

But

$$O\,(ABCD) = O'\,(ABCD);$$

$$\therefore\ O'\,(BCDA) = O'\,(ABCD),$$

$$\text{i.e. } O'\,(AC,\ BD) = O'\,(BD,\ CA) = O'\,(CA,\ BD).$$

Hence $O'\,(ABCD)$ is harmonic, and therefore so also is $O\,(ABCD)$.

3. Expanding the first determinant in terms of the minors formed from the first two columns, it is equal to

$$A(b'c'+a'd')+B(c'a'+b'd')+C(a'b'+c'd'),$$

where $A=(b-c)(a-d)$, $B=(c-a)(b-d)$, $C=(a-b)(c-d)$, and using the relation $A+B+C\equiv 0$, this becomes

$$BC'-B'C.$$

But in the second determinant if we subtract the first row from each of the others, it becomes

$$\begin{vmatrix} 1, & bc+ad, & b'c'+a'd' \\ 0, & C, & C' \\ 0, & -B, & -B' \end{vmatrix},$$

i.e. $BC'-B'C$.

4. Denote the nth convergent of the first fraction by $\dfrac{p_n}{q_n}$. We then have

$$p_{3n}=cp_{3n-1}+p_{3n-2},$$
$$p_{3n-1}=bp_{3n-2}+p_{3n-3},$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$p_{3n-4}=bp_{3n-5}+p_{3n-6}.$$

Multiplying these by 1, c, $1+bc$, $-b$, 1 and adding, we find

$$p_{3n}=(a+b+c+abc)\,p_{3n-3}+p_{3n-6}\quad\dots\dots\dots\dots\text{(i)},$$

and similarly for q_{3n}.

Also $\dfrac{p_3}{q_3}=\dfrac{1}{a+}\,\dfrac{1}{b+}\,\dfrac{1}{c}=\dfrac{bc+1}{abc+a+c}$. Hence, evidently from (i), the $3n$th convergent of the first fraction will be equal to the nth convergent of the second.

5. If we take

$$ax^3+3bx^2+3cx+d\equiv l(x-\alpha)^3+m(x-\beta)^3,$$

we have

$$l+m=a,\quad l\alpha+m\beta=-b,\quad l\alpha^2+m\beta^2=c,\quad l\alpha^3+m\beta^3=-d.$$

If then we assume α, β as the roots of $\lambda x^2+\mu x+\nu=0$, we easily get

$$\lambda c-\mu b+\nu a=0, \quad -\lambda d+\mu c-\nu b=0,$$

so that the quadratic is

$$\begin{vmatrix} x^2, & x, & 1 \\ c, & -b, & a \\ -d, & c, & -b \end{vmatrix}=0,$$

$$\text{i.e. } (ac-b^2)\,x^2+(ad-bc)\,x+(bd-c^2)=0.$$

If therefore the left-hand side be factorised in the form $(px+q)(p'x+q')$, it follows that the cubic is of the form

$$A\,(px+q)^3+D\,(p'x+q')^3=0,$$

and therefore the substitution

$$y=\frac{px+q}{p'x+q'}, \text{ or } x=\frac{q'y-q}{p-p'y},$$

will reduce the cubic to the form $Ay^3+D=0$.

6. Let Q be a point on the circumcircle, and suppose the pedal line of Q makes an acute angle θ with AB, Q and C being on opposite sides of AB. Then if QL, QM, QN are the perpendiculars, we have $R^2-x^2=BL\,.\,LC$.

Now if B', C' are the middle points of QB, QC, then

$$\frac{BL}{\sin 2\theta}=\frac{B'L}{\sin QBC}=\frac{1}{2}\frac{QB}{\sin QBC}=R\,.\,\frac{\sin QCB}{\sin QBC}.$$

Similarly $$\frac{CL}{\sin 2\,(A-\theta)}=R\,.\,\frac{\sin QBC}{\sin QCB};$$

$$\therefore\ BL\,.\,CL=R^2\sin 2\theta\sin 2\,(A-\theta)=R^2-x^2.$$

Similarly $$R^2-y^2=-R^2\sin 2\theta\sin 2\,(B+\theta),$$

$$R^2-z^2=R^2\sin 2\,(A-\theta)\sin 2\,(B+\theta).$$

Hence putting $2\theta=\phi$, $2A=\alpha$, etc. we have

$$\Sigma\,(R^2-x^2)\sin 2A=R^2\left[\sin\alpha\sin\phi\sin(\alpha-\phi)-\sin\beta\sin\phi\sin(\beta+\phi)\right.$$
$$\left.-\sin(\alpha+\beta)\sin(\alpha-\phi)\sin(\beta+\phi)\right],$$

and the expression in the bracket reduces to

$$-\sin\alpha\sin\beta\sin(\alpha+\beta).$$

Hence $\qquad \Sigma(R^2-x^2)\sin 2A = R^2 \sin 2A \sin 2B \sin 2C,$

$$\text{i.e. } \Sigma x^2 \sin 2A = \sin A \sin B \sin C\,(4R^2 - 8R^2 \cos A \cos B \cos C)$$
$$= (3R^2 + OP^2) \sin A \sin B \sin C.$$

7. We have, since the quantities are in H.P.,

$$1-y=\frac{2x(1+x)}{1+2x}, \quad x = 2y(1-y).$$

Putting $x+y=\lambda$, we find from the first equation $\quad \dfrac{1+x}{1+2x}=\lambda$;

$$\therefore\ x=\frac{1-\lambda}{2\lambda-1}, \quad y=\frac{2\lambda^2-1}{2\lambda-1}.$$

Hence from the second equation we find

$$2\lambda-1=4\lambda(2\lambda^2-1),$$
$$\text{i.e. } 4\lambda^3-3\lambda=-\tfrac{1}{2},$$

the roots of which (comparing with $4\cos^3\theta - 3\cos\theta = \cos 3\theta$) are $\cos\dfrac{2\pi}{9}$, $\cos\dfrac{4\pi}{9}$, $\cos\dfrac{8\pi}{9}$, and the only negative root is

$$\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}.$$

8. Let the eccentric angles of A, B, C be α, β, γ: those of A', B', C', α', β', γ'. Then the normal at α' is the chord α, α'. Hence

$$\frac{a\sin\alpha'}{b\cos\dfrac{\alpha+\alpha'}{2}} = \frac{-b\cos\alpha'}{a\sin\dfrac{\alpha+\alpha'}{2}},$$

$$\text{i.e. } \tan\alpha' \tan\frac{\alpha+\alpha'}{2} = -\frac{b^2}{a^2},$$

$$\text{or } \tan\frac{\alpha+\alpha'}{2} = -\frac{b^2}{a^2}\cdot\frac{1}{t_1},$$

with two similar equations.

Again the tangent at α' is parallel to the chord β, γ, and therefore $\alpha' - \dfrac{\beta+\gamma}{2}$ is a multiple of π, and so also are $\beta' - \dfrac{\gamma+\alpha}{2}$

and $\gamma' - \frac{\alpha+\beta}{2}$. Hence $\alpha + \alpha'$ and $\beta' + \gamma'$ differ by a multiple of π;

$$\therefore \tan(\alpha + \alpha') = \tan(\beta' + \gamma'),$$

$$\text{i.e. } \frac{t_2 + t_3}{1 - t_2 t_3} = \frac{-2 \cdot \frac{b^2}{a^2} \cdot \frac{1}{t_1}}{1 - \frac{b^4}{a^4} \cdot \frac{1}{t_1^2}} = \frac{-2(1-e^2)t_1}{t_1^2 - (1-e^2)^2},$$

whence $\quad t_1^2(t_2 + t_3) - (1 - e^2)^2(t_2 + t_3) = -2(1 - e^2)t_1(1 - t_2 t_3)$.

Similarly $t_2^2(t_3 + t_1) - (1 - e^2)^2(t_3 + t_1) = -2(1 - e^2)t_2(1 - t_3 t_1)$.

Subtracting and dividing by $t_1 - t_2$, we get

$$\Sigma t_1 t_2 + (1 - e^2)^2 = -2(1 - e^2),$$

$$\text{i.e. } \Sigma t_1 t_2 + (3 - e^2)(1 - e^2) = 0.$$

9. Considering the curves

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$$(x - \alpha)^2 + (y - \beta)^2 - r^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

$$(x - \alpha)^2 + (y - \beta)^2 + \tfrac{1}{2}r^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)},$$

the two latter being the circumscribed and polar circles of the equilateral triangle with centre (α, β), we have a triangle circumscribed to (i) and inscribed in (ii), leading to the invariant relation

$$\Theta^2 = 4\Delta\Theta'.$$

Again we have a triangle circumscribed to (i) and self-conjugate for (iii), involving the relation $\Theta = 0$.

Now the invariants for (i) and (ii) are

$$\Delta = -\frac{1}{a^2 b^2}, \qquad \Theta = \frac{1}{a^2 b^2}(\alpha^2 + \beta^2 - a^2 - b^2 - r^2),$$

$$\Theta' = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 - r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

Also, since $\Theta = 0$ for (i) and (iii), we have

$$\alpha^2 + \beta^2 - a^2 - b^2 + \tfrac{1}{2}r^2 = 0.$$

Hence the relation $\Theta^2 = 4\Delta\Theta'$ for (i) and (ii) becomes, on substituting for r^2,

$$9(\alpha^2+\beta^2-a^2-b^2)^2$$
$$= -4a^2b^2\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1+2(\alpha^2+\beta^2-a^2-b^2)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right],$$

reducing to the locus given for (α, β).

10. Let $ABCD\ldots$ be the polygon, and on OA take $OA' = \frac{2}{3}OA$, and find similar points B', C', $D'\ldots$. Then the centroid of the triangle OAB is the middle point of $A'B'$. Also the areas of the triangles OAB, OBC, etc. are proportional to AB, $BC\ldots$ and therefore to $A'B'$, $B'C'\ldots$. Hence the centroid of the area of the polygon $ABCD\ldots$ is the centroid of the perimeter of $A'B'C'D'\ldots$, i.e. K and G are the centroids of the perimeters of these two polygons. But the polygons are similar, the lines joining corresponding vertices all meeting in O, and

$$A'B' = \tfrac{2}{3}AB, \text{ etc.}$$

Hence evidently $\qquad OG = \frac{2}{3}OK.$

11. Let O be the centre of the circle, and suppose the string first becomes slack at P and tight again at Q. If the tension vanishes at height h above the lowest point, we have

$$\frac{u^2-2gh}{a} = g\cdot\frac{h-a}{a},$$
$$\text{i.e. } u^2 = g(3h-a).$$

But $u^2 = ga\left(2+\frac{3\sqrt{3}}{2}\right)$, whence $h = a + a\cdot\frac{\sqrt{3}}{2}$, i.e. OP makes an angle 30° with the upward vertical.

Also since the circular path must be the circle of curvature to the free parabolic path at P, therefore PQ and the tangent at P are equally inclined to the axis of the parabola, i.e. to the vertical, i.e. PQ makes an angle 60° with the vertical. Hence, from the figure, OQ is horizontal.

Again, the velocity at P is given by

$$v^2 = u^2 - 2gh = ga\left(2+\frac{3\sqrt{3}}{2}\right) - 2g\left(a + a\frac{\sqrt{3}}{2}\right) = ag\cdot\frac{\sqrt{3}}{2}.$$

Hence, if V be the velocity (vertical) at Q, we have

$$V^2 = v^2 \cos^2 60^\circ + 2g \,.\, a\frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{8}ag,$$

and the velocity on again reaching the lowest point is

$$u_1^2 = \frac{9\sqrt{3}}{8}ag + 2ga.$$

The tension again vanishes at height h_1 given by $u_1^2 = g(3h_1 - a)$, whence

$$h_1 = a + \frac{3\sqrt{3}}{8}a.$$

12. Since $3ay^2 = x(a-x)^2$, we have

$$6ayy_1 = (a-x)^2 - 2x(a-x) = (a-x)(a-3x);$$

$$\therefore\ y_1^2 = \frac{(a-x)^2(a-3x)^2}{36a^2y^2} = \frac{(a-3x)^2}{12ax};$$

$$\therefore\ 1 + y_1^2 = \frac{(a+3x)^2}{12ax}.$$

Hence the length of the loop is

$$2\int_0^a \sqrt{1 + y_1^2}\,dx = \frac{1}{\sqrt{3a}}\int_0^a \frac{a+3x}{x^{\frac{1}{2}}}\,dx$$

$$= \frac{1}{\sqrt{3a}}\left[2ax^{\frac{1}{2}} + 2x^{\frac{3}{2}}\right]_0^a = \frac{4a}{\sqrt{3}},$$

and the area is

$$2\int_0^a y\,dx = \frac{2}{\sqrt{3a}}\int_0^a x^{\frac{1}{2}}(a-x)\,dx = \frac{2}{\sqrt{3a}}\left[\frac{2}{3}ax^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^a$$

$$= \frac{8a^2}{15\sqrt{3}}.$$

The curve touches the axis of y at the origin, and is entirely on the right-hand side of this axis. It is symmetrical about the axis of x, and the tangent is parallel to the axis of x at the points $\left(\frac{a}{3}, \pm\frac{2a}{9}\right)$. There is a double point at $(a, 0)$, but no points of inflection and no asymptotes.

LXXXIX.

1. The radical centre of the escribed circles cannot be on a side of the triangle, since the tangents from it to the three circles must be equal. Hence the inverses of the sides for the orthogonal circle will be three circles, and the inverse of the nine-point circle will be another circle, and these four circles will touch the inverses of the escribed circles. But inverting for the orthogonal circle, the escribed circles invert into themselves: hence the theorem.

2. Let PCP' be the diameter of one of the circles, O the foot of the radical axis, A and B the points in which the circle cuts the line of centres. Draw OK parallel to PC, and bisect the angle COK. Draw AL, PM perpendicular to the bisector: then MP produced will pass through B. Then since AOL is a constant angle, the ratios

$$OM : OB \text{ and } AL : OA$$

are constant. Hence $PM \,.\, OM : OA \,.\, OB$ is constant.

But, since the circles are coaxal, $OA \,.\, OB$ is constant, being equal to the square of the tangent from O to any circle. Thus $PM \,.\, OM$ is constant, i.e. the locus of P is a rectangular hyperbola, of which O is the centre, and OM one asymptote.

3. Suppose we take p letters out of the first group, q out of the second, and r out of the third, so that $p+q+r=n$. Then p and q may be chosen, for a given value of r, in $(n-r+1)$ ways, for p may have any value from 0 to $n-r$. Hence the number of combinations in this mode of selection is

$$\begin{aligned}\sum_{r=0}^{r=n}(n-r+1)\,{}^nC_r &= (n+1)\,.\sum_0^n {}^nC_r - n\,.\sum_1^n {}^{n-1}C_{r-1} \\ &= (n+1)(1+1)^n - n\,(1+1)^{n-1} \\ &= (n+1)\,2^n - n\,.\,2^{n-1} = (n+2)\,.\,2^{n-1}.\end{aligned}$$

4. A may win (i) three consecutive games exactly, or (ii) four consecutive games, or (iii) all five games.

(i) Here the sequence may begin with the first, second or third game. The respective chances are

$$\left(\frac{4}{7}\right)^3 \cdot \frac{3}{7}, \quad \left(\frac{4}{7}\right)^3 \cdot \left(\frac{3}{7}\right)^2, \quad \left(\frac{4}{7}\right)^3 \cdot \frac{3}{7},$$

and the sum of these is $\dfrac{3264}{7^5}$.

(ii) Here the sequence may begin with either the first or the second game, the chance being in either case

$$\left(\frac{4}{7}\right)^4 \cdot \frac{3}{7}.$$

Thus the chance under these conditions is $2 \cdot \left(\frac{4}{7}\right)^4 \cdot \frac{3}{7} = \frac{1536}{7^5}$.

(iii) Here the chance is $\left(\frac{4}{7}\right)^5 = \frac{1024}{7^5}$.

Hence the total chance that A wins three games in succession is

$$\frac{3264 + 1536 + 1024}{7^5} = \frac{832}{2401},$$

i.e. the odds against it are 1569 : 832.

5. If P is the product of squared differences of the equation $f(x) = 0$, then

$$P = (-1)^{\frac{n(n-1)}{2}} \cdot f'(a_1) f'(a_2) \ldots f'(a_n).$$

Now here $$f'(x) = nx^{n-1} + a;$$

$$\therefore \; xf'(x) = nx^n + ax = -(n-1)ax - nb$$

$$= (n-1)a\left[-\frac{nb}{(n-1)a} - x\right], \text{ if } x \text{ is a root};$$

$$\therefore \; (-1)^n b \cdot f'(a_1) f'(a_2) \ldots f'(a_n)$$

$$= (n-1)^n \cdot a^n \cdot \prod_1^n \left\{-\frac{nb}{(n-1)a} - a_r\right\}.$$

Now $$\Pi(x - a_r) \equiv x^n + ax + b;$$

$$\therefore \; \Pi\left\{-\frac{nb}{(n-1)a} - a_r\right\} = (-1)^n \frac{n^n b^n}{(n-1)^n a^n} - \frac{nb}{n-1} + b$$

$$= (-1)^n \frac{n^n b^n}{(n-1)^n a^n} - \frac{b}{n-1};$$

$$\therefore\ f'(a_1) f'(a_2) \dots f'(a_n) = n^n b^{n-1} - (-1)^n . (n-1)^{n-1} a^n,$$

giving the required value of P.

6. If $x = 2R \cos A$, we have

$$2R + x = 4R \cos^2 \frac{A}{2}, \quad 2R - x = 4R \sin^2 \frac{A}{2}.$$

Also since

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}, \quad r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2};$$

$$\therefore\ r_1 - r = 4R \sin \frac{A}{2} \cos \frac{B+C}{2} = 4R \sin^2 \frac{A}{2} = 2R - x;$$

$$\therefore\ r_1 = r + 2R - x.$$

But $r_1 = s \tan \dfrac{A}{2}$; $\quad \therefore\ (r + 2R - x)^2 = s^2 . \dfrac{2R - x}{2R + x}$,

giving the required equation, which is similarly satisfied by $2R \cos B$ and $2R \cos C$.

7. We have

$$\frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \dots,$$

and putting $x = e^{i\alpha} \tan \beta$, we see that the given series is the coefficient of i in

$$\frac{1}{2} \log \frac{1 + e^{i\alpha} \tan \beta}{1 - e^{i\alpha} \tan \beta} = \frac{1}{2} \log \frac{1 + \cos \alpha \tan \beta + i \sin \alpha \tan \beta}{1 - \cos \alpha \tan \beta - i \sin \alpha \tan \beta},$$

and this coefficient is

$$\frac{1}{2}\left(\tan^{-1} \frac{\sin \alpha \tan \beta}{1 + \cos \alpha \tan \beta} + \tan^{-1} \frac{\sin \alpha \tan \beta}{1 - \cos \alpha \tan \beta}\right)$$

$$= \frac{1}{2} \tan^{-1} \frac{2 \sin \alpha \tan \beta}{1 - \tan^2 \beta}$$

$$= \tfrac{1}{2} \tan^{-1} (\sin \alpha \tan 2\beta).$$

8. The normal at $\left(\dfrac{a}{2m}, \dfrac{am}{2}\right)$ is

$$x - m^2 y = \frac{a(1 - m^4)}{2m},$$

the tangent being $m^2 x + y = am$.

If the normal passes through (ξ, η) we have

$$am^4 - 2\eta m^3 + 2\xi m - a = 0,$$

and putting $\dfrac{1}{m^2} = t$, this gives $m = \dfrac{a(t^2-1)}{2t(t\xi - \eta)}$.

So that the equation to the tangent is

$$x \cdot \frac{1}{t} + y - \frac{a^2(t^2-1)}{2t(t\xi-\eta)} = 0,$$

or $$2x(t\xi - \eta) + 2ty(t\xi - \eta) - a^2(t^2 - 1) = 0.$$

Thus the four tangents belong to the system of lines

$$\lambda^2(2y\xi - a^2) + 2\lambda(x\xi - y\eta) - (2x\eta - a^2) = 0,$$

where λ is a variable parameter, and the envelope of this system is

$$(x\xi - y\eta)^2 + (2y\xi - a^2)(2x\eta - a^2) = 0,$$

i.e. $$(x\xi + y\eta)^2 - 2a^2(x\eta + y\xi) + a^4 = 0.$$

9. Taking P as origin, and the tangent at P as axis of x, the equation to the conic is of the form

$$ax^2 + 2hxy + by^2 - 2y = 0 \quad \text{.................(i).}$$

To find the polar reciprocal, we want the locus of a point whose polar with respect to $x^2 + y^2 = c^2$ shall touch the conic (i), whose tangential equation is

$$-\lambda^2 + (ab - h^2)\nu^2 + 2a\mu\nu - 2h\nu\lambda = 0.$$

Substituting x', y', $-c^2$ for λ, μ, ν we get

$$-x'^2 + (ab - h^2)c^4 - 2ac^2y' + 2hc^2x' = 0,$$

and therefore the polar reciprocal is

$$(x - hc^2)^2 = -2ac^2\left(y - \frac{bc^2}{2}\right)$$

a parabola of latus-rectum $2ac^2$.

But the radius of curvature at P to (i) is Lt $\dfrac{x^2}{2y}$ at the origin, i.e. $\dfrac{1}{a}$. Hence the latus-rectum of the reciprocal parabola is $\dfrac{2c^2}{\rho}$.

10. Supposing the weight W at P about to slip down, then the friction at P makes an angle λ with the outward normal on the upward side. Let θ be the acute angle between the focal

distance SP and the tangent at P. Since the length of the string is equal to the major axis, the rings are evidently at the ends of a diameter. If T is the tension of the string, we have, for the equilibrium of W,

$$\frac{T}{\sin(\alpha+\lambda)} = \frac{W}{\sin(90^\circ - \lambda + \theta)} = \frac{W}{\cos(\lambda - \theta)},$$

where α is the acute angle the normal at P makes with the major axis.

Similarly for the other ring,

$$\frac{T}{\sin(\alpha+\lambda)} = \frac{W'}{\cos(\lambda+\theta)};$$

$$\therefore \quad \frac{W}{W'} = \frac{\cos(\lambda-\theta)}{\cos(\lambda+\theta)}, \text{ i.e. } \frac{W-W'}{W+W'} = \mu\tan\theta \quad \ldots\ldots(\text{i}).$$

Now if p, p' be the perpendiculars from the foci on the tangent at P, then

$$\sin\theta = \frac{p}{SP} = \frac{p'}{S'P};$$

$$\therefore \quad \sin^2\theta = \frac{p^2}{SP \,.\, S'P} = \frac{b^2}{CD^2},$$

where CD is the semi-diameter conjugate to CP.

Hence $\quad \sin^2\theta = \dfrac{b^2}{a^2\sin^2\phi + b^2\cos^2\phi} = \dfrac{1-e^2}{1-e^2+e^2\sin^2\phi},$

$$\therefore \quad \tan^2\theta = \frac{1-e^2}{e^2\sin^2\phi},$$

so that from (i) we have

$$e^2\sin^2\phi = \mu^2(1-e^2)\,.\left(\frac{W+W'}{W-W'}\right)^2.$$

11. Suppose the buffer compressed a length x, and let the final velocities of the carriages be V and V'. The common velocity immediately after impact is $\dfrac{M}{M+M'}v$. Hence the equations of energy and momentum are

$$\tfrac{1}{2}MV^2 + \tfrac{1}{2}M'V'^2 = \tfrac{1}{2}(M+M')\left(\frac{M}{M+M'}v\right)^2 + mgx \ldots(\text{i}),$$

$$MV + M'V' = Mv.$$

The latter will be satisfied, if we put

$$V = \frac{Mv + M'k}{M + M'}, \quad V' = \frac{Mv - Mk}{M + M'} \quad \text{............(ii)},$$

where k is to be found. Substituting these in (i) we find

$$\frac{MM'k^2}{M + M'} = 2mgx,$$

$$\text{i.e. } k^2 = 2mgx\left(\frac{1}{M} + \frac{1}{M'}\right).$$

Now since the backing is inelastic, V' must be positive, i.e. $v > k$, so that

$$v^2 > 2mgx\left(\frac{1}{M} + \frac{1}{M'}\right),$$

and if the buffer is completely compressed, this must hold when $x = l$.

In this case $k = \left\{2mgl\left(\frac{1}{M} + \frac{1}{M'}\right)\right\}^{\frac{1}{2}}$, and substituting this value in the expressions (ii), we get the second result.

12. The determinant on the left-hand side is equal to

$$\begin{vmatrix} y_1 & sy_1 & ty_1 \\ y_1' & sy_1' + s'y_1 & ty_1' + t'y_1 \\ y_1'' & sy_1'' + 2s'y_1' + s''y_1 & ty_1'' + 2t'y_1' + t''y_1 \end{vmatrix}.$$

Multiplying the first column by s, and subtracting from the second, and multiplying the first column by t, and subtracting from the third, this determinant becomes

$$\begin{vmatrix} y_1 & 0 & 0 \\ y_1' & s'y_1 & t'y_1 \\ y_1'' & 2s'y_1' + s''y_1 & 2t'y_1' + t''y_1 \end{vmatrix}$$

$$= y_1^2 \begin{vmatrix} s' & t' \\ 2s'y_1' + s''y_1 & 2t'y_1' + t''y_1 \end{vmatrix}.$$

Multiplying the first row of this last determinant by $2y_1'$, and subtracting from the second, it becomes

$$\begin{vmatrix} s' & t' \\ s''y_1 & t''y_1 \end{vmatrix},$$

and the result given follows.

XC.

1. If we invert from A, the circle becomes a straight line on which are B', C', D' the inverses of B, C, D.

Now since ACB, $AB'C'$ are similar triangles,

$$\therefore \frac{BC}{B'C'} = \frac{AC}{AB'} = \frac{AB \, . \, AC}{r^2},$$

where r is the constant of inversion.

Similarly $$\frac{CD}{C'D'} = \frac{AC \, . \, AD}{r^2},$$

and hence $$\frac{B'C'}{C'D'} = \frac{AD \, . \, BC}{AB \, . \, CD},$$

so that $$\frac{AD \, . \, BC}{B'C'} = \frac{AB \, . \, CD}{C'D'} = \frac{AC \, . \, BD}{B'D'},$$

and we have merely to find a point D', such that

$$B'C'^2 = C'D' \, . \, B'D'.$$

To do this divide any line $\beta\delta$ in 'extreme and mean ratio' in γ, and take $C'D'$ a fourth proportional to $\beta\gamma$, $\gamma\delta$, $B'C'$.

2. The locus of the centres of conics touching the four sides of a quadrilateral is the line joining the middle points of the diagonals. Now let the tangent at P meet the other common tangent in T. Then in this case, the conic and circle are inscribed in a quadrilateral, three of whose sides coincide with the tangent at P, and the remaining side is the other common tangent. But since TP is now a diagonal of the quadrilateral, the line joining the centres must pass through its middle point. Hence the line joining Q and R, the other extremities of the diameters through P, must pass through T.

3. If x be so chosen that the series

$$u_1x + u_2x^2 + \dots + u_rx^r + \dots$$

is convergent, then evidently v_r is the coefficient of x^r in the product

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots\right)(u_1x + \dots + u_rx^r + \dots),$$

$$\text{i.e. } \sum_{r=1} v_rx^r = e^x(u_1x + \dots + u_rx^r + \dots);$$

$$\therefore \sum_{r=1} u_rx^r = e^{-x} \cdot \sum_{r=1} v_rx^r$$

$$= \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots\right)(v_1x + \dots + v_rx^r + \dots),$$

and equating coefficients of x^r, the result follows.

4. Denote the given series by $f(n)$. Then since

$$\frac{n(n+1)\dots(n+r-1)}{r!} - \frac{(n-1)n\dots(n+r-2)}{r!} \equiv \frac{n(n+1)\dots(n+r-2)}{(r-1)!},$$

we have

$$f(n) - f(n-1) = \frac{1}{2} + n \cdot \frac{1}{2^2} + \frac{n(n+1)}{2!} \cdot \frac{1}{2^3} + \dots$$

$$+ \frac{n(n+1)\dots(2n-4)}{(n-3)!} \cdot \frac{1}{2^{n-2}} + \frac{n(n+1)\dots(2n-2)}{(n-1)!} \cdot \frac{1}{2^{n-1}}$$

$$= \frac{1}{2}\left[f(n) - \frac{n(n+1)\dots(2n-3)}{(n-2)!} \cdot \frac{1}{2^{n-2}} - \frac{n(n+1)\dots(2n-2)}{(n-1)!} \cdot \frac{1}{2^{n-1}}\right]$$

$$+ \frac{n(n+1)\dots(2n-2)}{(n-1)!} \cdot \frac{1}{2^{n-1}}$$

$$= \frac{1}{2} f(n) + \frac{n(n+1)\dots(2n-2)}{(n-1)!}\left(-\frac{1}{2^n} - \frac{1}{2^n} + \frac{1}{2^{n-1}}\right) = \frac{1}{2} f(n);$$

$$\therefore f(n) = 2f(n-1).$$

But $\qquad f(1) = 1; \quad \therefore f(n) = 2^{n-1}.$

5. If $f(x) \equiv 2x^5 - 5px^2 + 3q$, then $f'(x) = 10(x^4 - px)$ and the only real roots of $f'(x) = 0$ are 0 and $p^{\frac{1}{3}}$ (real value). Now substitute $-\infty$, 0, $p^{\frac{1}{3}}$, ∞ for x in $f(x)$. The values obtained are

$$-\infty, \ 3q, \ 3(q - p^{\frac{5}{3}}), \ \infty,$$

and every change of sign in this series gives one real root, and only one, of $f(x)=0$.

If $q>p^{\frac{5}{3}}$ the signs are $-$, $+$, $+$, $+$ giving one real root,

„ $q<p^{\frac{5}{3}}$ „ „ „ $-$, $+$, $-$, $+$ „ three real roots.

If $q=p^{\frac{5}{3}}$ the equation has equal roots ($p^{\frac{1}{3}}$) and therefore must have three real roots.

6. Since

$$1-\tan^2\theta\tan^2\phi=\frac{\cos(\theta+\phi)\cos(\theta-\phi)}{\cos^2\theta\cos^2\phi},$$

the left side is equal to

$$\left\{\frac{\cos(x+y+2a)\cos(x-y)}{\cos^2(x+a)\cos^2(y+a)}\div\frac{\cos(x+y-2a)\cos(x-y)}{\cos^2(x-a)\cos^2(y-a)}\right\}$$

$$\times\left\{\frac{\cos(x+y)\cos(x+y-2a)}{\cos^2 a\cos^2(x+y-a)}\div\frac{\cos(x+y+2a)\cos(x+y)}{\cos^2 a\cos^2(x+y+a)}\right\}$$

$$=\left\{\frac{\cos(x-a)\cos(y-a)\cos(x+y+a)}{\cos(x+a)\cos(y+a)\cos(x+y-a)}\right\}^2.$$

Now putting $\tan x=X$, $\tan y=Y$, $\tan(x+y)=Z$, $\tan a=\alpha$, the expression in the bracket is

$$\frac{(1+\alpha X)(1+\alpha Y)(1-\alpha Z)}{(1-\alpha X)(1-\alpha Y)(1+\alpha Z)},$$

and since $Z(1-XY)=X+Y$, this may be written

$$\frac{1-\alpha XY\,.\,\dfrac{Z-\alpha}{1+\alpha Z}}{1+\alpha XY\,.\,\dfrac{Z+\alpha}{1-\alpha Z}}=\frac{1-\tan a\tan x\tan y\tan(x+y-a)}{1+\tan a\tan x\tan y\tan(x+y+a)}.$$

7. If ω, ω^2 are the imaginary cube roots of unity, then

$$\log(1-\omega x)-\log(1-\omega^2 x)=(\omega^2-\omega)(x-\tfrac{1}{2}x^2+\tfrac{1}{4}x^4-\tfrac{1}{5}x^5+\dots).$$

Hence putting $x=1$, and taking

$$\omega=-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3},\quad \omega^2=-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3},$$

we get

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\ldots=\frac{1}{i\sqrt{3}}\log\frac{1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}{1+\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}}$$

$$=\frac{1}{i\sqrt{3}}\log\left(\frac{\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}\right)$$

$$=\frac{1}{i\sqrt{3}}\log\left(e^{\frac{i\pi}{3}}\right)=\frac{\pi}{3\sqrt{3}}.$$

8. Let A be the common vertex, P the other point of intersection. Let PA make angles θ and θ' with the axes of the two parabolas. Then it is easily seen from a figure that $\theta+\theta'=\alpha$, and that the tangents to the parabolas at P make angles $\tan^{-1}(\frac{1}{2}\tan\theta)$ and $\tan^{-1}(\frac{1}{2}\tan\theta')$ with the respective axes. Hence, since the angle between the tangents is $\alpha-\epsilon$, and that between the axes α, we must have

$$\tan^{-1}(\tfrac{1}{2}\tan\theta)+\tan^{-1}(\tfrac{1}{2}\tan\theta')=\epsilon.$$

Now put $\tan\theta=t$, $\tan\theta'=t'$, $\tan\alpha=a$, $\tan\epsilon=e$. Then

$$\frac{t+t'}{1-tt'}=a,\quad \frac{\frac{1}{2}(t+t')}{1-\frac{1}{4}tt'}=e,$$

from which $$tt'=\frac{2(a-2e)}{2a-e},\quad t+t'=\frac{3ae}{2a-e}\quad\ldots\ldots\ldots\ldots\ldots(\text{i}).$$

Also $$\frac{l}{l'}=\frac{\sin^2\theta\cos\theta'}{\sin^2\theta'\cos\theta},$$

$$\therefore\ \frac{l}{l'}+\frac{l'}{l}=\frac{\sin^4\theta\cos^2\theta'+\sin^4\theta'\cos^2\theta}{\sin^2\theta\sin^2\theta'\cos\theta\cos\theta'}$$

$$=\cos\theta\cos\theta'.\,\frac{t^4(1+t'^2)+t'^4(1+t^2)}{t^2t'^2};$$

$$\therefore \frac{l}{l'}+\frac{l'}{l}-2\cos\alpha=\cos\theta\cos\theta'\left[\frac{t^4(1+t'^2)+t'^4(1+t^2)}{t^2t'^2}-2+2tt'\right]$$

$$=\cos\theta\cos\theta'\,.\,\frac{(t^2-t'^2)^2+t^2t'^2(t+t')^2}{t^2t'^2}$$

$$=\cos\theta\cos\theta'\,.\,(t+t')^2\,.\,\frac{(t-t')^2+t^2t'^2}{t^2t'^2}\quad\ldots\ldots\text{(ii)}.$$

Now $$(t-t')^2+t^2t'^2=\left(\frac{3ae}{2a-e}\right)^2-\frac{8(a-2e)}{2a-e}+\frac{4(a-2e)^2}{(2a-e)^2}$$

$$=\frac{3a\{3ae^2-4(a-2e)\}}{(2a-e)^2}.$$

Also from (i) $$\cos\theta\cos\theta'=\frac{(2a-e)\cos\alpha}{3e}.$$

Hence substituting in (ii), the result follows.

9. To find the lines through A parallel to the asymptotes, put $\beta=k\gamma$ in the equation of the conic, and we then get $\gamma=0$, or

$$lk\gamma+(m+nk)\alpha=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

This equation together with $\beta=k\gamma$ determines a point which is to lie on $a\alpha+b\beta+c\gamma=0$ and therefore satisfies

$$a\alpha+(bk+c)\gamma=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) and (ii) we get

$$lka=(m+nk)(bk+c)=bnk^2+(bm+cn)k+cm.$$

But since the centre lies on BC, we have $la-mb-nc=0$, and this equation becomes $bnk^2+cm=0$, giving equal and opposite values of k.

But if θ is the angle between the lines $\beta=\pm k\gamma$, we have

$$\tan\theta=\frac{2k\sin A}{1-k^2}.$$

Hence θ being now the angle between the asymptotes, we have

$$\tan^2\theta=\frac{-4\left(\frac{cm}{bn}\right)\sin^2 A}{\left(1+\frac{cm}{bn}\right)^2}=\frac{-4bcmn\sin^2 A}{(bn+cm)^2}.$$

Also $$e = \sec\frac{\theta}{2}; \quad \therefore\ \tan\theta = \frac{2\sqrt{e^2-1}}{2-e^2}.$$

Equating these values of $\tan\theta$, we get the result as given.

10. The length of each string is $\sqrt{3}a\sin\theta$. Hence if T be the tension of a string, the equation of virtual work is

$$3W\,.\,\delta\left(\frac{a\cos\theta}{2}\right) + W\,.\,\delta\,(b\operatorname{cosec}\theta) + 3T\,.\,\delta\,(\sqrt{3}a\sin\theta) = 0,$$

$$\text{i.e.}\ -\frac{3W}{2}\,a\sin\theta - Wb\operatorname{cosec}\theta\cot\theta + 3T\,.\,\sqrt{3}a\cos\theta = 0.$$

Now since the modulus is W, we have $T = W\,.\,\dfrac{\sqrt{3}a\sin\theta - a}{a}$.

Hence

$$-\tfrac{3}{2}\,a\sin\theta - b\operatorname{cosec}\theta\cot\theta + 3\sqrt{3}a\,(\sqrt{3}\sin\theta - 1)\cos\theta = 0,$$

$$\text{i.e.}\ 3a\sec\theta + 2b\operatorname{cosec}^3\theta = (18 - 6\sqrt{3}\operatorname{cosec}\theta)\,a.$$

11. Let v be the velocity, θ the angle of projection, α the angle between the plane of projection and the second wall. Then the initial horizontal velocities parallel to the walls are $v\cos\theta\cos\alpha$ and $v\cos\theta\sin\alpha$, while the initial vertical velocity is $v\sin\theta$. Also at any impact the velocities in these directions are not altered in magnitude. Hence the time of flight is equal to both

$$\frac{a+b}{v\cos\theta\cos\alpha} \text{ and } \frac{a+b}{v\cos\theta\sin\alpha},$$

so that $\alpha = 45°$, and the time is $\dfrac{\sqrt{2}\,(a+b)}{v\cos\theta}$.

Again since the vertical distance described in this time is zero, we have

$$v\sin\theta - \tfrac{1}{2}g\,.\,\frac{\sqrt{2}\,(a+b)}{v\cos\theta} = 0,$$

$$\text{i.e.}\ \frac{v^2\sin 2\theta}{2g} = \frac{a+b}{\sqrt{2}} \quad \text{..................(i).}$$

Hence the least possible value of $\frac{v^2}{2g}$ is $\frac{a+b}{\sqrt{2}}$, occurring when $\theta = 45^\circ$, provided the particle does not hit the ceiling, i.e. provided $\left(\frac{v}{\sqrt{2}}\right)^2 < 2gc$, i.e. $\frac{a+b}{\sqrt{2}} < 2c$

If this condition be not satisfied, the least velocity will be that which causes the particle just to graze the ceiling, i.e. for which $v^2 \sin^2\theta = 2gc$. Combining this with (i) we get

$$\tan\theta = \frac{2\sqrt{2c}}{a+b},$$

whence
$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{4\sqrt{2c}\,(a+b)}{(a+b)^2+8c^2},$$

and by (i) the height required is

$$\frac{v^2}{2g} = \frac{a+b}{\sqrt{2}\sin 2\theta} = \frac{(a+b)^2+8c^2}{8c}.$$

12. Taking the co ordinates of any point on the curve in the form $x = am$, $y = am^3$, we have

$$ds^2 = dx^2 + dy^2 = (a^2 + 9a^2m^4)\,dm^2,$$

so that the surface required is

$$\int 2\pi y\,ds = \int_0^1 2\pi a m^3 \,.\, a\sqrt{1+9m^4}\,dm,$$

and putting $1 + 9m^4 = z^2$, this becomes

$$2\pi a^2 \int_1^{\sqrt{10}} z \,.\, \frac{z}{18}\,dz = \frac{\pi a^2}{9}\left[\frac{1}{3}z^3\right]_1^{\sqrt{10}}$$

$$= \frac{\pi a^2}{27}(10\sqrt{10}-1).$$

XCI.

1. Inverting the system from A, two of the circles become straight lines at right angles, and the other a circle cutting these lines orthogonally, i.e. a circle of which these lines are diameters, while the inverses of the circles ABC, $AB'C'$ are a pair of lines joining the extremities of these diameters. But these lines are obviously parallel, therefore their inverses are circles touching at A.

2. Let AY be the perpendicular from A on BC, O the centre of the circle. Make $O\hat{A}O' = B\hat{A}Y$, so that for a given angle ABC, AO' is a fixed line. Draw OO' perpendicular to AO', and join $O'Y$. Then, since $AY : AB = AO' : AO$, and the angles BAO, YAO' are equal, the triangles BAO, YAO' are similar, therefore $AO' : O'Y = AO : OB$, i.e. $O'Y$ is constant. Hence the locus of Y is a circle, and the envelope of BC is a conic of which A is a focus, and this circle the auxiliary circle. Moreover, the eccentricity is $AO' : O'Y$, i.e. $AO : OB$, and is therefore the same for all the conics, i.e. the conics are similar.

Again, let X be the foot of a directrix and draw XP perpendicular to AO', meeting AO in P. Then

$$AP : AO = AX : AO' = e^2 : 1 - e^2 ;$$

therefore P is a fixed point, and there is another similar point on the other directrix.

3. We want the number of solutions of the equation

$$x + y + z = r,$$

where $x, y, z \not> n$.

This is the coefficient of t^r in $(1 + t + t^2 + \dots + t^n)^3$

$$= (1 - t^{n+1})^3 (1 - t)^{-3}$$

$$= (1 - 3t^{n+1} + 3t^{2n+2} - \dots) \left\{ 1 + \sum_{r=1} \frac{(r+1)(r+2)}{2} \cdot t^r \right\}.$$

Now since $r < 2n + 1$, we need only retain the first two terms in the first bracket, and the coefficient is

$$\frac{(r+1)(r+2)}{2} - 3 \cdot \frac{(r-n)(r-n+1)}{2},$$

which is equivalent to the expression given.

4. If $a-b$ is even, then $a+b$ must be even. Let

$$a+b=2m, \quad a-b=2n.$$

Then $ab=m^2-n^2$, and this is a perfect square, say p^2, so that $m^2-n^2=p^2$, while $(a^2-b^2)\sqrt{ab}=4mnp$, so that we have to shew that $mnp=M(60)$.

Now every square number is of one of the forms $5r$ or $5r\pm 1$. Hence since $m^2=n^2+p^2$, n and p cannot both be of the form $5r\pm 1$, for then m^2 would be of the form $5r+2$. Hence unless either n or p is a multiple of 5, one of them must be of the form $5r\pm 1$, and the other $5r\pm 2$, and in this case m^2, and therefore m, is a multiple of 5. Hence in any case mnp is a multiple of 5.

Similarly, from the fact that every square number is of the form $4r$ or $4r+1$, and also of the form $3r$ or $3r+1$, we can shew that mnp is a multiple of 4 and 3.

5. If α and β be the two roots such that $\alpha^2+\beta^2=\alpha\beta$, they must be the roots of a quadratic of the form

$$x^2-\lambda x+\tfrac{1}{3}\lambda^2=0.$$

Hence putting

$$x^3+p_1x^2+p_2x+p_3\equiv(x^2-\lambda x+\tfrac{1}{3}\lambda^2)(x+p_1+\lambda),$$

we find $\qquad -\tfrac{2}{3}\lambda^2-\lambda p_1=p_2, \quad \tfrac{1}{3}\lambda^2(p_1+\lambda)=p_3,$

and hence the equations

$$2\lambda^2+3\lambda p_1+3p_2=0,$$

$$\lambda^2p_1+3\lambda p_2+6p_3=0,$$

whence, eliminating λ, we get

$$(18p_1p_3-9p_2^2)(6p_2-3p_1^2)=(3p_1p_2-12p_3)^2,$$

which reduces to the condition given.

The equation in question satisfies the condition, and writing it in the form

$$(x^2-\lambda x+\tfrac{1}{3}\lambda^2)(x-6a+\lambda)=0\ldots\ldots\ldots\ldots\text{(i)},$$

we have $\tfrac{1}{3}\lambda^2+\lambda(6a-\lambda)=0$, i.e. $\lambda=9a$, since $\lambda\neq 0$. Hence by (i) the roots are $-3a$ and $\dfrac{9\pm 3\sqrt{-3}}{2}a$.

6. Since the nine-point circle touches the inscribed and escribed circles, we have $\alpha = \frac{1}{2}R - r$, $\beta = \frac{1}{2}R + r_1$, etc. Also

$$r = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$

$$r_1 = 4R \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C, \text{ etc.};$$

$$\therefore\ r_1 - r = 4R \sin^2 \tfrac{1}{2}A, \quad r_2 + r_3 = 4R \cos^2 \tfrac{1}{2}A,$$

$$\text{i.e. } r_1 + r_2 + r_3 - r = 4R \quad \text{.................(i).}$$

But $$\beta + \gamma + \delta - 11\alpha = -4R + (r_1 + r_2 + r_3 - r) + 12r$$
$$= 12r \text{ from (i),}$$

and $$\gamma + \delta + \alpha - 11\beta = -4R + (r_1 + r_2 + r_3 - r) - 12r_1$$
$$= -12r_1, \text{ etc.}$$

Hence since $\Sigma \dfrac{1}{r_1} = \dfrac{1}{r}$, the first result follows.

Further

$$\Sigma\alpha^2 = R^2 + R(r_1 + r_2 + r_3 - r) + r^2 + \Sigma r_1^2$$
$$= 5R^2 + 2R^2(1 - \cos A)(1 - \cos B)(1 - \cos C)$$
$$+ 2R^2 . \Sigma(1 - \cos A)(1 + \cos B)(1 + \cos C)$$
$$= 5R^2 + 2R^2(1 - \Sigma \cos A + \Sigma \cos B \cos C - \cos A \cos B \cos C)$$
$$+ 2R^2(3 + \Sigma \cos A - \Sigma \cos B \cos C - 3 \cos A \cos B \cos C)$$
$$= R^2(13 - 8 \cos A \cos B \cos C).$$

7. All the angles $\dfrac{(4r \pm 1)\pi}{16}$ satisfy $\tan^2 4\theta = 1$. Putting $\tan\theta = t$, this equation is

$$(4t - 4t^3)^2 = (1 - 6t^2 + t^4)^2,$$

and the roots of this are therefore $\tan \dfrac{r\pi}{16}$, $(r = 1, 3, 5 \dots 15)$. Hence putting $t^2 = x$, the roots of

$$16x(1 - x)^2 = (1 - 6x + x^2)^2$$

are $\tan^2 \dfrac{r\pi}{16}$, $(r = 1, 3, 5, 7)$, and this is the equation in question.

8. Any tangent to the parabola is

$$x - my + am^2 = 0.$$

Hence if the tangents at m_1, m_2 intersect in (X, Y), then m_1 and m_2 are the roots of

$$X - mY + am^2 = 0 \quad \text{......................(i)},$$

so that if m_1, m_2, m_3 be the roots of the cubic

$$\alpha - \beta m + \gamma m^2 - \delta m^3 = 0 \quad \text{..................(ii)},$$

we must have an identity of the form

$$a(\alpha - \beta m + \gamma m^2 - \delta m^3) = (X - mY + am^2)(k - \delta m),$$

so that

$$a\alpha = kX, \quad a\beta = kY + \delta X, \quad a\gamma = ka + \delta Y \quad \text{......(iii)}.$$

Now suppose we also have another triangle determined by the cubic

$$\alpha' - \beta' m + \gamma' m^2 - \delta' m^3 = 0 \quad \text{...............(iv)}.$$

Consider the conic

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \\ x, & y, & a, & 0 \\ 0, & x, & y, & a \end{vmatrix} = 0.$$

Multiplying the first, third and fourth rows by $-a$, k, δ respectively, and adding, we see from the equations (iii) that (X, Y) lies on this conic; and since the first two rows may be interchanged it follows that this is the conic circumscribing the two circumscribing triangles, and if it passes through the origin,

$$\begin{vmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{vmatrix} = 0,$$

i.e. $\dfrac{\beta}{\alpha} = \dfrac{\beta'}{\alpha'}$ or from (ii) and (iv) $\Sigma \dfrac{1}{m_1} = \Sigma \dfrac{1}{m_1'}$.

9. The tangent at B to the circle is

$$a\gamma + c\alpha = kb\alpha, \quad \text{i.e. } a\gamma = (kb - c)\alpha.$$

Hence if this makes an angle θ with BC, we have

$$\frac{\sin\theta}{\sin(B - \theta)} = \frac{a}{kb - c},$$

$$\text{i.e. } \sin B \cot\theta - \cos B = \frac{kb - c}{a},$$

whence
$$\cot\theta = \frac{k - \cos A}{\sin A}.$$

Now if O is the centre, N the middle point of BC, and R' the radius, then $BON = \theta$; therefore $\frac{a}{2R'} = \sin\theta$;

$$\therefore \quad R' = \frac{R\sin A}{\sin\theta} \quad \text{and} \quad \sin\theta = \frac{\sin A}{\sqrt{1 - 2k\cos A + k^2}}.$$

10. Let W be the weight of each rod, and unity its length. Let FG, GA, AB make angles θ, ϕ, ψ with the horizontal.

The reaction X at F is horizontal by symmetry. Hence taking moments about A for the rods AG, GF together, we have

$$X(\sin\theta + \sin\phi) = \tfrac{1}{2}W\cos\phi + W(\cos\phi + \tfrac{1}{2}\cos\theta)$$
$$= \tfrac{3}{2}W\cos\phi + \tfrac{1}{2}W\cos\theta \quad \ldots\ldots\ldots\ldots\text{(i)}.$$

Also taking moments about G for FG, we get

$$X\sin\theta = W \cdot \tfrac{1}{2}\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) and (ii)

$$\frac{\sin\theta + \sin\phi}{\sin\theta} = \frac{3\cos\phi + \cos\theta}{\cos\theta},$$

whence
$$\tan\theta = \tfrac{1}{3}\tan\phi \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Again, resolving horizontally for AG, GF, the horizontal reaction at A is X, and the vertical reaction due to the three lower rods is $\frac{3}{2}W$. Hence taking moments about B for AB, we have

$$X\sin\psi = \tfrac{3}{2}W\cos\psi - \tfrac{1}{2}W\cos\psi = W\cos\psi.$$

Hence, from (ii), $\tan\psi = 2\tan\theta \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$

But $2\cos\phi + 2\cos\theta = AD = 2\cos\psi + 1$;

$$\therefore \quad \cos\phi - \cos\psi + \cos\theta = \tfrac{1}{2}.$$

But if $\tan\theta = x$, then $\cos\theta = (1 + x^2)^{-\frac{1}{2}}$, and from (iii) and (iv)

$$\cos\phi = (1 + 9x^2)^{-\frac{1}{2}}, \quad \cos\psi = (1 + 4x^2)^{-\frac{1}{2}}.$$

11. Let β be the angle the direction of projection makes with the horizontal, α the inclination of the plane of projection to the vertical plane in direction east, V the velocity of projection. Then the initial vertical velocity of the second shot is $V\sin\beta$, while its horizontal velocities are $V\cos\beta\cos\alpha$ east and $V\cos\beta\sin\alpha$ north. Hence if the shots meet after time t, we must have

$$(V\sin\beta \,.\, t - \tfrac{1}{2}gt^2) + \tfrac{1}{2}gt^2 = c,$$
$$V\cos\beta\sin\alpha \,.\, t = b,$$
$$Vt - V\cos\beta\cos\alpha \,.\, t = a.$$

From these we easily get

$$\cot\beta = \frac{b}{c}\operatorname{cosec}\alpha, \quad \operatorname{cosec}\beta = \frac{a}{c} + \frac{b}{c}\cot\alpha;$$

$$\therefore\quad 1 = \frac{a^2}{c^2} + \frac{2ab}{c^2}\cot\alpha - \frac{b^2}{c^2},$$

$$\text{i.e.}\quad \cot\alpha = \frac{b^2+c^2-a^2}{2ab};$$

$$\therefore\quad \operatorname{cosec}\beta = \frac{a}{c} + \frac{b^2+c^2-a^2}{2ac} = \frac{a^2+b^2+c^2}{2ac}.$$

Again if θ be the inclination of the direction of projection to the east, we easily see from the figure that

$$\cos\theta = \cos\alpha\cos\beta,$$

$$\text{i.e.}\quad \cos\theta = \frac{b^2+c^2-a^2}{\sqrt{(b^2+c^2-a^2)^2+4a^2b^2}} \cdot \frac{\sqrt{(b^2+c^2+a^2)^2-4a^2c^2}}{a^2+b^2+c^2}$$
$$= \frac{b^2+c^2-a^2}{a^2+b^2+c^2}.$$

12. Taking the co-ordinates of any point as (m^2, m^3), the tangent at this point is

$$\begin{vmatrix} x, & y, & 1 \\ m^2, & m^3, & 1 \\ 2m, & 3m^2, & 0 \end{vmatrix} = 0, \quad \text{i.e.}\quad 3mx - 2y - m^3 = 0,$$

and the corresponding normal is

$$2x + 3my = 2m^2 + 3m^4 \text{.....................(i).}$$

Hence, if the normal at m' coincides with the tangent at m, we must have

$$\frac{2}{3m} = \frac{3m'}{-2} = \frac{2m'^2 + 3m'^4}{m^3},$$

whence $m = -\frac{4}{9m'}$, and we get $32 = 243m'^2(2m'^2 + 3m'^4)$, which, putting $9m'^2 = t$, becomes $t^3 + 6t^2 - 32 = 0$,

$$\text{i.e. } (t-2)(t^2 + 8t + 16) = 0,$$

giving $t = 2$ or -4. This gives only one real value of m'^2, viz. $\frac{2}{9}$, and the corresponding value of m^2 is $\frac{8}{9}$, and these are the abscissae of the two points, the lines being given by (i), when

$$m = \pm\frac{2\sqrt{2}}{3}.$$

XCII.

1. Let PQ and AB meet in F, and AQ, BP in E. Then EF is the polar of O and is therefore fixed. Also OE is the polar of F, i.e. the pole of OE is on PQ; therefore the pole of PQ is on OE; let it be T.

Now since AB is a diameter, OPE and OQE are right angles, and therefore, since $TP = TQ$, T must be the middle point of OE, i.e. the locus of T is a fixed line parallel to EF.

2. If ∞ and ∞' are the circular points, and S the given focus, then $S\infty$ and $S\infty'$ are tangents to the conic, so that all the conics are touched by the same four lines. Hence the director circles have a common radical axis, i.e. they all pass through the same two points.

3. The expanded form of the determinant is

$$\{(b+c)(c+a)(a+b)\}^3 - 2a^3b^3c^3 - \Sigma a^6(b+c)^3,$$

and if a, b, c are the roots of $x^3 - px^2 + qx - r = 0$, the latter sum is $\Sigma(aq-r)^3$, i.e. $q^3 . \Sigma a^3 - 3q^2r . \Sigma a^2 + 3qr^2 . \Sigma a - 3r^3$, and this is

$$\begin{aligned} E &\equiv (p^3 - 3pq + 3r)q^3 - 3(p^2 - 2q)q^2r + 3pqr^2 - 3r^3 \\ &= p^3q^3 - 3pq^4 + 9q^3r - 3p^2q^2r + 3pqr^2 - 3r^3. \end{aligned}$$

Hence the value of the determinant is

$$(pq-r)^3 - 2r^3 - E = 3q^3(pq-3r)$$

as given, since $$pq - 3r = \Sigma a^2 b.$$

4. Since $N \equiv 16^{99} - 1 = (2^{18})^{22} - 1$, and 2^{18} is prime to 23; therefore by Fermat's Theorem $N = M(23)$. So also

$$N = (2^{22})^{18} - 1, \text{ so that } N = M(19).$$

$$\therefore \quad N = M(19 \times 23) = M(437).$$

Again, by Wilson's Theorem

$$1 + 18! = M(19) \text{ and } 1 + 22! = M(23).$$

But

$$\begin{aligned}(1+22!) - (1+18!) &= 18!\,(22 \,.\, 21 \,.\, 20 \,.\, 19 - 1)\\ &= 18!\,\{(23-1)(23-2)(23-3)(23-4) - 1\}\\ &= 18!\,\{M(23) + 24 - 1\} = 18!\,.\,M(23).\end{aligned}$$

Hence, since $1 + 22!$ is $M(23)$, so also is $1 + 18!$, so that

$$1 + 18! = M(19 \times 23) = M(437).$$

5. If

$$f(x) \equiv x^5 - 5ax - 1 = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon),$$

then $$f'(x) = 5x^4 - 5a = \Sigma\,(x-\beta)(x-\gamma)(x-\delta)(x-\epsilon).$$

Hence, putting $x = \alpha$, we have

$$\begin{aligned}(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon) &= 5\alpha^4 - 5a\\ &= \frac{5\,(5a\alpha+1)}{\alpha} - 5a = 20a + \frac{5}{\alpha}.\end{aligned}$$

Hence the given function is

$$\begin{aligned}25\,.\,\Sigma\left(4a + \frac{1}{\alpha}\right)^2 &= 25\,.\,\Sigma\left(16a^2 + 8a\,.\,\frac{1}{\alpha} + \frac{1}{\alpha^2}\right)\\ &= 25\left(80a^2 + 8a\,.\,\Sigma\frac{1}{\alpha} + \Sigma\frac{1}{\alpha^2}\right).\end{aligned}$$

Now $$\Sigma\frac{1}{\alpha} = -5a, \quad \Sigma\frac{1}{\beta\gamma} = 0; \quad \therefore \quad \Sigma\frac{1}{\alpha^2} = 25a^2.$$

Hence the given function is

$$25\,(80a^2 - 40a^2 + 25a^2) = 1625a^2.$$

6. Let $\dfrac{2\pi}{n} = \alpha$, and call the given series S, so that

$$S = \cos\alpha + 2\cos 2\alpha + 3\cos 3\alpha + \ldots + (n-1)\cos(n-1)\alpha.$$

If we multiply by $2(1-\cos\alpha)$, the coefficient of $\cos r\alpha$ becomes

$$2r - (r+1) - (r-1) = 0.$$

This applies to all values of r from 1 to $n-2$. The coefficient of $\cos(n-1)\alpha$ is $2(n-1)-(n-2) = n$, and that of $\cos n\alpha$ is $-(n-1)$. We thus have

$$2(1-\cos\alpha)S = -1 + n\cos(n-1)\alpha - (n-1)\cos n\alpha.$$

But $\cos n\alpha = 1$, and $\cos(n-1)\alpha = \cos\alpha$;

$$\therefore\ 2(1-\cos\alpha)S = -n + n\cos\alpha,$$

$$\text{i.e. } S = -\tfrac{1}{2}n.$$

7. If x be a positive proper fraction, we have

$$\log(1+x) > x - \tfrac{1}{2}x^2 \text{ and } < x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3.$$

Putting $x = \dfrac{1}{n}$, we get

$$\frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} + \frac{1}{3}\cdot\frac{1}{n^3} > \log(n+1) - \log n > \frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2}.$$

Writing $n-1$, $n-2$, etc. for n and adding, this gives

$$\Sigma\frac{1}{n} - \frac{1}{2}\cdot\Sigma\frac{1}{n^2} + \frac{1}{3}\cdot\Sigma\frac{1}{n^3} > \log(n+1) > \Sigma\frac{1}{n} - \frac{1}{2}\cdot\Sigma\frac{1}{n^2},$$

i.e. $\Sigma\dfrac{1}{n} - \log(n+1)$ lies between

$$\frac{1}{2}\cdot\Sigma\frac{1}{n^2} \text{ and } \frac{1}{2}\cdot\Sigma\frac{1}{n^2} - \frac{1}{3}\cdot\Sigma\frac{1}{n^3}.$$

Now, when n is infinite $\Sigma\dfrac{1}{n^2}$ becomes $\dfrac{\pi^2}{6}$, and since

$$\log(n+1) - \log n = \log\left(1 + \frac{1}{n}\right),$$

which ultimately vanishes, we have $\Sigma\dfrac{1}{n} - \log n$, when n is infinite, lying between

$$\frac{\pi^2}{12} \text{ and } \frac{\pi^2}{12} - \frac{1}{3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots\right).$$

8. Any conic having double contact at P and Q is of the form

$$b^2x^2 + a^2y^2 - a^2b^2 + k\,(ax\sin\alpha - by\cos\alpha - \overline{a^2-b^2}\sin\alpha\cos\alpha)^2 = 0.$$

If this is a parabola,

$$(b^2 + ka^2\sin^2\alpha)(a^2 + kb^2\cos^2\alpha) = k^2a^2b^2\sin^2\alpha\cos^2\alpha,$$

whence $k = -\dfrac{a^2b^2}{a^4\sin^2\alpha + b^4\cos^2\alpha}$, and the equation becomes

$$(b^3\cos\alpha\,.\,x + a^3\sin\alpha\,.\,y)^2 + 2a^3b^2(a^2-b^2)\sin^2\alpha\cos\alpha\,.\,x$$
$$- 2a^2b^3(a^2-b^2)\sin\alpha\cos^2\alpha\,.\,y + \ldots = 0,$$

and the axis is $\quad b^3\cos\alpha\,.\,x + a^3\sin\alpha\,.\,y + \lambda = 0,$

where $\quad \lambda = \dfrac{a^3b^5(a^2-b^2)\sin^2\alpha\cos^2\alpha - a^5b^3(a^2-b^2)\sin^2\alpha\cos^2\alpha}{b^6\cos^2\alpha + a^6\sin^2\alpha}$

$$= -\frac{a^3b^3(a^2-b^2)^2\sin^2\alpha\cos^2\alpha}{b^6\cos^2\alpha + a^6\sin^2\alpha}.$$

9. The pole of (x', y') for the circle $(x-\xi)^2 + (y-\eta)^2 = r^2$ is

$$(x-\xi)(x'-\xi) + (y-\eta)(y'-\eta) = r^2.$$

If this coincides with $\dfrac{x}{a}\cos\alpha + \dfrac{y}{b}\sin\alpha = 1$, we have

$$\frac{a(x'-\xi)}{\cos\alpha} = \frac{b(y'-\eta)}{\sin\alpha} = r^2 + \xi(x'-\xi) + \eta(y'-\eta),$$

whence, eliminating α, the locus of (x', y') is

$$a^2(x-\xi)^2 + b^2(y-\eta)^2 = (x\xi + y\eta - \xi^2 - \eta^2 + r^2)^2.$$

The terms of the second degree in this equation are

$$(a^2-\xi^2)x^2 - 2\xi\eta xy + (b^2-\eta^2)y^2.$$

Hence the locus is similar to the original conic if

$$\frac{(a^2+b^2-\xi^2-\eta^2)^2}{(a^2-\xi^2)(b^2-\eta^2)-\xi^2\eta^2} = \frac{(a^2+b^2)^2}{a^2b^2},$$

giving, on reduction, the required locus of (ξ, η).

10. Let A and B be the lower and upper ends of the rod, C its point of contact with the table. Then since the rod begins to move at A, and not at C, the friction at A is perpendicular to CA. Hence, if R be the vertical pressure at A, we have, taking moments about an axis through C perpendicular to the rod and in the vertical plane containing it,

$$\mu R \,.\, AC = P \cos \beta \,.\, CB,$$

$$\text{i.e. } \mu R = P(2n-1)\cos\beta \quad \text{.................(i)}.$$

Now draw AD perpendicular to the edge to meet it in D, and DM vertical to the ground. Then taking moments about the edge, we have

$$\mu R \sin\beta \,.\, DM + R \,.\, AM = W \,.\, (1-n) AM,$$

and since $\dfrac{DM}{AM} = \tan\alpha$, this is

$$R(\mu \sin\beta \tan\alpha + 1) = W(1-n) \quad \text{..............(ii)}.$$

Eliminating R from (i) and (ii), we get the given value for P.

11. At the point P where the particle leaves the curve the pressure vanishes and we have $\dfrac{v^2}{\rho} = g\cos\theta$, where θ is the angle which the normal at P makes with the vertical. But if the eccentric angle of P is ϕ,

$$v^2 = 2g(a - a\cos\phi);$$

$$\therefore \quad \rho\cos\theta = 2a(1-\cos\phi) \quad \text{..............(i)}.$$

Also if b' is the semi-diameter conjugate to CP,

$$\cos\theta = \frac{b\cos\phi}{b'}, \quad \rho = \frac{b'^3}{ab},$$

and
$$b'^2 = a^2(1 - e^2\cos^2\phi).$$

Substituting in (i), we find that ϕ satisfies the equation

$$(1 - e^2\cos^2\phi)\cos\phi = 2(1-\cos\phi).$$

But

$$\sin^2\theta = \frac{a^2\sin^2\phi}{b'^2} = \frac{\sin^2\phi}{1-e^2\cos^2\phi} = \frac{\sin^2\phi\cos\phi}{2(1-\cos\phi)} = \frac{1}{2}\cos\phi(1+\cos\phi);$$

$$\therefore \quad \cos^2\theta = \frac{2-\cos\phi-\cos^2\phi}{2} = \frac{1}{2}(1-\cos\phi)(2+\cos\phi),$$

and the latus-rectum of the parabola is

$$\frac{2v^2\cos^2\theta}{g} = 4a(1-\cos\phi)\cos^2\theta$$

$$= 2a(1-\cos\phi)^2(2+\cos\phi).$$

12. Since $$\frac{1+\sin x}{1+\cos x} = \frac{1}{1+\cos x} + \frac{\sin x}{1+\cos x}$$

$$= \frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2},$$

the value of the first integral is evidently $e^x \tan\dfrac{x}{2}$.

Again, $$\frac{1}{\sin\beta\sin(x-\alpha)} - \frac{1}{\sin\alpha\sin(x-\beta)}$$

$$= \frac{\sin(\alpha-\beta)\sin x}{\sin\alpha\sin\beta\sin(x-\alpha)\sin(x-\beta)}$$

$$= (\cot\beta-\cot\alpha)\cdot\frac{\sin x}{\sin(x-\alpha)\sin(x-\beta)}.$$

Hence the second integral is

$$\frac{1}{\cot\beta-\cot\alpha}\left[\frac{1}{\sin\beta}\int\frac{dx}{\sin(x-\alpha)} - \frac{1}{\sin\alpha}\int\frac{dx}{\sin(x-\beta)}\right]$$

$$= \frac{1}{\cot\beta-\cot\alpha}\left(\frac{1}{\sin\beta}\log\tan\frac{x-\alpha}{2} - \frac{1}{\sin\alpha}\log\tan\frac{x-\beta}{2}\right).$$

XCIII.

1. Let O, O' be the centres of the circles, a and b their radii. Let P be any point, Q its inverse for the first circle, Q' the inverse of Q for the second. Then, if the operations are commutative, the inverse of P for the second circle must lie on OQ', and must therefore be R, the intersection of OQ', $O'P$. We have then $PRQ'Q$ a cyclic quadrilateral and $OP \,.\, OQ = a^2$,

$O'Q \,.\, O'Q' = b^2$. But the sum of the squares of the tangents from O and O' to the circle round $PRQ'Q$ is equal to OO'^2. Hence

$$OO'^2 = a^2 + b^2,$$

i.e. the two circles cut orthogonally.

2. Let QQ' meet the asymptotes in R and R', and let PQ, PQ' meet the polar of P in V and V'. Then since PV is divided harmonically by the curve, and the second intersection is at infinity, $PQ = QV$.

So also $PQ' = Q'V'$, and therefore QQ' is parallel to the polar of P. It is therefore also parallel to the tangent at the extremity of the given diameter, i.e. QQ' is fixed in direction. But $PQ : PQ' = CR : CR'$, and this is a constant ratio.

3. The equations are equivalent to $ax^2 + bx + c = 0$, $x^7 = 1$, where $x \neq 1$. Now

$$(\alpha + \beta)^7 - \alpha^7 - \beta^7 \equiv 7\alpha\beta\,(\alpha + \beta)\,(\alpha^2 + \alpha\beta + \beta^2)^2.$$

Putting $\alpha = ax$, $\beta = b$, we get

$$(ax + b)^7 - a^7x^7 - b^7 = 7abx\,(ax + b)\,(a^2x^2 + abx + b^2)^2.$$

Now $\qquad (ax + b)^7 = x^7\,(ax + b)^7 = -c^7$, since $ax^2 + bx = -c$.

Also $\qquad a^2x^2 + abx + b^2 = a\,(ax^2 + bx) + b^2 = b^2 - ac.$

Hence the above becomes

$$-c^7 - a^7 - b^7 = 7ab\,(-c)\,(b^2 - ac)^2,$$

$$\text{i.e. } \Sigma a^7 - 7abc\,(b^2 - ac)^2 = 0.$$

This includes the result of the elimination when $x = 1$, viz.

$$a + b + c = 0.$$

4. If u_n denotes either p_n or q_n, we have

$$u_n = u_{n-1} + (n+1)\,(n-1)^3\,u_{n-2},$$

$$\text{i.e. } u_n - n^2u_{n-1} = -(n^2 - 1)\left[u_{n-1} - (n-1)^2u_{n-2}\right],$$

whence $\quad u_n - n^2u_{n-1} = (-1)^{n-2}\,(n^2 - 1)\,\ldots\,(3^2 - 1)\,(u_2 - 2^2u_1).$

For the p's we have $u_2 - 2^2 u_1 = -6$, and for the q's $= 0$. Hence, we find

$$\frac{p_n}{(n!)^2} - \frac{p_{n-1}}{\{(n-1)!\}^2} = (-1)^{n-1}\left(1 + \frac{1}{n}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{p_2}{(2!)^2} - p_1 = (-1)\left(1 + \frac{1}{2}\right).$$

Adding these, if n is even, we have

$$\frac{p_n}{(n!)^2} - 2 = -1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{n},$$

$$\text{i.e. } \frac{p_n}{q_n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{n},$$

since $q_n = (n!)^2$. Hence $\underset{n=\infty}{\mathrm{Lt}} \dfrac{p_n}{q_n} = \log 2$.

5. Taking

$$x^4 + px^3 + qx^2 + rx + s \equiv (x^2 - ax + b)(x^2 - a'x + b'),$$

let α, δ be the roots of the first quadratic, β, γ those of the second. Then $\lambda = \dfrac{ab' - a'b}{a' - a}$, whence

$$\frac{a}{\lambda + b} = \frac{a'}{\lambda + b'} = \frac{-p}{2\lambda + b + b'} = \frac{-r}{\lambda(b + b') + 2bb'}$$

$$= \frac{-r}{\lambda(b + b') + 2s} = \frac{p\lambda - r}{2s - 2\lambda^2} \quad\cdots\cdots\cdots\cdots\text{(i)},$$

since $a + a' = -p$, $b + b' + aa' = q$, $ab' + a'b = -r$, $bb' = s$.

From (i) we get $\qquad b + b' = \dfrac{2(\lambda r - ps)}{p\lambda - r}$.

Also $\qquad aa' = (\lambda + b)(\lambda + b')\left(\dfrac{p\lambda - r}{2s - 2\lambda^2}\right)^2$

$$= \left\{\lambda^2 + \frac{2\lambda(\lambda r - ps)}{p\lambda - r} + s\right\}\left(\frac{p\lambda - r}{2s - 2\lambda^2}\right)^2$$

$$= \frac{1}{4} \cdot \frac{p^2\lambda^2 - r^2}{\lambda^2 - s}.$$

Hence the equation is

$$\frac{2(\lambda r - ps)}{p\lambda - r} + \frac{1}{4} \cdot \frac{p^2\lambda^2 - r^2}{\lambda^2 - s} = q,$$

which reduces to the form given.

6. Putting $\cos^2\alpha = x$, etc. the determinant is

$$\Sigma x^3 \left[(1-y)^3 - (1-z)^3\right] = -3\Sigma x^3 (y-z) + 3\Sigma x^3 (y^2 - z^2)$$
$$= 3(y-z)(z-x)(x-y)(\Sigma x - \Sigma yz).$$

Now since x, y, z are positive proper fractions, it is evident that $\Sigma x > \Sigma yz$, i.e. $\Sigma x - \Sigma yz$ cannot vanish. Hence one of the differences $y - z$, etc. must vanish. But

$$y - z = \cos^2\beta - \cos^2\gamma = \tfrac{1}{2}(\cos 2\beta - \cos 2\gamma).$$

Hence one of the differences $\cos 2\beta - \cos 2\gamma$, etc. must vanish.

7. If $\qquad (1 + ia)(1 + ib)(1 + ic) \ldots = A + iB,$

then $\Sigma \log(1 + ia) = \log(A + iB)$, and equating the imaginary parts, we get

$$\Sigma \tan^{-1} a = \tan^{-1}\frac{B}{A}.$$

Now

$$\left(1 + i\frac{p^2}{1^2}\right)\left(1 + i\frac{p^2}{2^2}\right) \ldots = \frac{\sinh(\sqrt{i}\,.\,p\pi)}{\sqrt{i}\,.\,p\pi}.$$

Also $\sqrt{i} = \dfrac{1+i}{\sqrt{2}}$, and therefore the given product is

$$\frac{\sinh(1+i)\theta}{(1+i)\theta}.$$

But $\qquad \sinh(\theta + i\theta) = \sinh\theta\cos\theta + i\cosh\theta\sin\theta;$

$$\therefore \quad \frac{\sinh(\theta + i\theta)}{(1+i)\theta} = \frac{1}{2\theta}(1 - i)(\sinh\theta\cos\theta + i\cosh\theta\sin\theta).$$

Hence the given series is equal to

$$n\pi + \tan^{-1}\left(\frac{\cosh\theta\sin\theta - \sinh\theta\cos\theta}{\sinh\theta\cos\theta + \cosh\theta\sin\theta}\right)$$
$$= n\pi + \tan^{-1}\left(\frac{\tan\theta - \tanh\theta}{\tan\theta + \tanh\theta}\right),$$

the inverse tangent having its principal value, and n being some integer depending on the value of p.

8. Suppose the normals at the extremities of the lines

$$\frac{lx}{a}+\frac{my}{b}=1 \;\ldots\ldots\ldots\text{(i)}, \qquad \frac{l'x}{a}+\frac{m'y}{b}=1 \;\ldots\ldots\ldots\text{(ii)}$$

intersect at the point α on the curve, then for some value of λ

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{lx}{a}+\frac{my}{b}-1\right)\left(\frac{l'x}{a}+\frac{m'y}{b}-1\right)=0$$

must coincide with

$$(a^2-b^2)\,xy+b^3x\sin\alpha-a^3y\cos\alpha=0.$$

Comparing these we get

$$1+\lambda ll'=0, \quad 1+\lambda mm'=0, \quad \lambda=1,$$

$$\frac{lm'+l'm}{a^2-b^2}=-\frac{l+l'}{b^2\sin\alpha}=\frac{m+m'}{a^2\cos\alpha},$$

leading to

$$\left.\begin{aligned}\sin\alpha &= \frac{a^2-b^2}{b^2}\cdot\frac{(l^2-1)\,m}{l^2+m^2}\\ \cos\alpha &= -\frac{a^2-b^2}{a^2}\cdot\frac{(m^2-1)\,l}{l^2+m^2}\end{aligned}\right\}\;\ldots\ldots\ldots\ldots\text{(iii)}.$$

Now we want the envelope of (i) when (ii) passes through α. The latter fact yields the condition

$$\frac{\cos\alpha}{l}+\frac{\sin\alpha}{m}+1=0.$$

Hence from (iii), substituting and reducing, we find

$$a^4l^2+b^4m^2=(a^2-b^2)^2,$$

which is the condition that (i) should touch

$$\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{1}{(a^2-b^2)^2}.$$

9. The equation to a conic touching the four lines is

$$L\alpha^2+M\beta^2+N\gamma^2=0,$$

with the single condition

$$\Sigma\frac{l^2}{L}=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The tangents from $(\alpha', \beta', \gamma')$ are

$$(L\alpha^2 + \ldots + \ldots)(L\alpha'^2 + \ldots + \ldots) = (L\alpha\alpha' + \ldots + \ldots)^2 \ \ldots\text{(ii)}.$$

Now, if the triangle of reference is equilateral, the conditions that the general conic should represent a circle are

$$v + w - 2u' = w + u - 2v' = u + v - 2w'.$$

Applying these to the locus (ii), we obtain on reduction

$$L\alpha'^2(M+N) + MN(\beta' + \gamma')^2$$
$$= \text{two similar expressions},$$

$$\text{i.e. } \frac{(\beta' + \gamma')^2}{L} + \frac{\alpha'^2}{M} + \frac{\alpha'^2}{N} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

whence, using (i), the locus of the foci is

$$\begin{vmatrix} l^2, & m^2, & n^2, & 0 \\ (\beta+\gamma)^2, & \alpha^2, & \alpha^2, & 1 \\ \beta^2, & (\gamma+\alpha)^2, & \beta^2, & 1 \\ \gamma^2, & \gamma^2, & (\alpha+\beta)^2, & 1 \end{vmatrix} = 0,$$

in which the coefficient of l^2 is

$$(\alpha + \beta + \gamma)\left[-2\alpha^3 + \alpha^2(\alpha + \beta + \gamma) + 2\alpha\beta\gamma\right].$$

Hence, excluding the line infinity, this gives the locus stated.

10. Let x be the distance of the points of attachment of the wire from the apex, and suppose each rod makes an angle θ with the vertical. Then $l = \sqrt{3}x \sin\theta$.

For a symmetrical displacement the equation of virtual work is

$$3W \,.\, \delta(c\cos\theta) + 3W \,.\, \delta\left(\frac{c}{2}\cos\theta\right) + 3T \,.\, \delta(\sqrt{3}x\sin\theta) = 0,$$

$$\text{i.e. } -\frac{9}{2}W \,.\, c\sin\theta \,.\, \delta\theta + 3\sqrt{3}\,T \,.\, x\cos\theta \,.\, \delta\theta = 0.$$

$$\therefore \quad T = \frac{\sqrt{3}}{2} \,.\, \frac{c}{x}\tan\theta \,.\, W$$
$$= \frac{3}{2}c \,.\, \frac{\sin^2\theta}{l\cos\theta} \,.\, W.$$

But $\cos\theta = \dfrac{h}{c}$; $\qquad \therefore \quad T = \dfrac{3}{2} \,.\, \dfrac{c^2 - h^2}{lh} \,.\, W.$

11. Suppose the plane of projection makes an angle θ with the lines of greatest slope, and that its intersections with the inclined plane and the ground are inclined at an angle ϕ. Then, from a figure,

$$\sin\phi = \cos\theta \sin\alpha.$$

The maximum range in this direction is

$$\frac{u^2}{g(1+\sin\phi)} = \frac{2h}{1+\cos\theta\sin\alpha}.$$

Hence the extreme points which can be reached lie on the curve whose polar equation is

$$\frac{2h}{r} = 1 + \cos\theta \sin\alpha,$$

and this curve is an ellipse of eccentricity $\sin\alpha$ and semi-latus rectum $2h$. Its semi-axes are therefore

$$\frac{2h}{\cos^2\alpha} \text{ and } \frac{2h}{\cos\alpha},$$

so that its area is $$4\pi h^2 \sec^3\alpha.$$

12. The tangent at the point 2α is

$$u = U\cos(\theta - 2\alpha) + U'\sin(\theta - 2\alpha),$$

where U, U' are the values of u and $\frac{du}{d\theta}$ at the point.

But $$\frac{1}{u^2}\frac{du}{d\theta} = -\frac{dr}{d\theta} = a\sin\theta;$$

$$\therefore \quad \frac{1}{u}\frac{du}{d\theta} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2};$$

$$\therefore \quad \frac{U'}{U} = \tan\alpha, \quad \text{and} \quad U = \frac{1}{2a\cos^2\alpha}.$$

Hence the tangent is

$$u = \frac{1}{2a\cos^2\alpha}\left[\cos(\theta - 2\alpha) + \tan\alpha \sin(\theta - 2\alpha)\right]$$

$$= \frac{1}{2a\cos^2\alpha}\cdot\frac{\cos(\theta - 3\alpha)}{\cos\alpha},$$

$$\text{i.e.} \quad r\cos(\theta - 3\alpha) = 2a\cos^3\alpha.$$

To find where this meets the curve, we have

$$(1+\cos\theta)\cos(\theta-3\alpha)=2\cos^3\alpha,$$

i.e. $$2\cos(\theta-3\alpha)+\cos(2\theta-3\alpha)+\cos 3\alpha=4\cos^3\alpha\,;$$

$$\therefore\quad 2\cos(\theta-3\alpha)+\cos(2\theta-3\alpha)=3\cos\alpha.$$

Writing this in the form

$$-\tfrac{1}{2}\cos(2\theta-3\alpha)-\tfrac{1}{2}\cos\alpha+2\cos\alpha-\cos(\theta-3\alpha)=0,$$

we see that it is equivalent to

$$[1-\cos(\theta-2\alpha)][\cos(\theta-\alpha)+2\cos\alpha]=0.$$

The first factor is $2\sin^2\dfrac{\theta-2\alpha}{2}$, and corresponds to the repeated root 2α. Hence 2β must satisfy

$$\cos(\theta-\alpha)+2\cos\alpha=0.$$

XCIV.

1. Through C draw any line cutting the given lines in a and b. Join Qb and draw Pa' parallel to Qb meeting the other line in a'. Then evidently a' and b will form homographic ranges on the two lines, and so also will a and b. Hence the points a and a' form homographic ranges on the same lines, and the double points of these ranges will give two solutions of the problem.

2. Let PR, QS meet in O. Then evidently the line joining the middle points of PQ, RS will pass through O, and it also passes through the centre C.

Now if PQ meets the asymptotes in P' and Q', then $PP'=QQ'$, so that V, the middle point of PQ, is also the middle point of $P'Q'$. Hence, since $P'CQ'$ is a right angle, it follows that $CV=VP'$;

$$\therefore\quad V\hat{C}P'=V\hat{P}'C=P\hat{P}'L$$

if the tangent at P meets the asymptote CP' in L.

But $CP = PL$; $\therefore\ P\hat{C}L = P\hat{L}C.$

$$\therefore\ V\hat{C}P = P\hat{P}L + P\hat{L}C$$

$$= \text{a right angle, since } PQ \text{ is normal at } P.$$

Hence CO is at right angles to CP.

3. Let I be the integral part of $(\sqrt{5}+2)^n$, and f the fractional part. Since $\sqrt{5}-2$ is a proper fraction, so also is $(\sqrt{5}-2)^n$, say f'. We have then

$$I + f = (\sqrt{5}+2)^n, \quad f' = (\sqrt{5}-2)^n;$$

$$\therefore\ I + f + f' = 2\{(\sqrt{5})^n + {}^nC_2 . (\sqrt{5})^{n-2} . 2^2 + {}^nC_4 . (\sqrt{5})^{n-4} . 2^4 + \ldots\},$$

$$I + f - f' = 2\{{}^nC_1 . (\sqrt{5})^{n-1} . 2 + {}^nC_3 . (\sqrt{5})^{n-3} . 2^3 + \ldots\}.$$

If n is even, the first series is rational and of the form $M(4)+2$, since every term after the first is $M(4)$. Since then $I+f+f'$ is an integer, and f and f' are proper fractions, $f+f'$ must be unity. Therefore I is of the form $M(4)+1$.

If n is odd, the second series is rational and of the form $M(4)$. Hence since $I+f-f'$ is an integer, $f-f'$ must be zero (it cannot be 1 or -1). Therefore I is of the form $M(4)$.

4. The highest power of a contained in $n!$ is

$$I\left(\frac{n}{a}\right) + I\left(\frac{n}{a^2}\right) + \ldots,$$

where $I(x)$ is the greatest integer in x, and this is less than

$$\frac{n}{a} + \frac{n}{a^2} + \frac{n}{a^3} + \ldots, \quad < \frac{\frac{n}{a}}{1-\frac{1}{a}} < \frac{n}{a-1},$$

since the series only extends to a finite number of terms. Similarly for $\beta, \gamma \ldots$. But denoting these highest powers by $p, q, r \ldots$, we must have $n! = \alpha^p \beta^q \gamma^r \ldots$;

$$\therefore\ n! < \alpha^{\frac{n}{\alpha-1}} \beta^{\frac{n}{\beta-1}} \ldots.$$

5. The sums of successive powers of the roots of the equation

$$x^5 - 5px^3 + 5p^2x - q = 0$$

are easily calculated from the equation

$$s_{r+5} - 5ps_{r+3} + 5p^2s_{r+1} - qs_r = 0.$$

We find

$$s_1 = 0, \quad s_2 = 10p, \quad s_3 = 0, \quad s_4 = 30p^2, \quad s_5 = 5q,$$

$$s_6 = 100p^3, \quad s_8 = 350p^4, \quad s_{10} = 1250p^5 + 5q^2.$$

Now

$$\Sigma\alpha_1{}^4\alpha_2{}^3\alpha_3{}^2 . \Sigma\alpha_1 \equiv S + \Sigma\alpha_1{}^5\alpha_2{}^3\alpha_3{}^2 + \Sigma\alpha_1{}^4\alpha_2{}^4\alpha_3{}^2 + \Sigma\alpha_1{}^4\alpha_2{}^3\alpha_3{}^3 \ldots \text{(i)},$$

where S is the given sum; and when two indices are alike, each permutation of the corresponding suffixes is to count as a separate term.

Now applying the formula

$$\Sigma\alpha_1{}^p\alpha_2{}^q\alpha_3{}^r = s_p s_q s_r - \Sigma s_{q+r}s_p + 2s_{p+q+r}$$

to the sums on the right of (i), (which vanishes, since $\Sigma\alpha_1 = 0$), they are respectively

$$- s_5{}^2 - s_8 s_2 + 2s_{10}, \quad s_4{}^2 s_2 - 2s_6 s_4 - s_8 s_2 + 2s_{10}, \quad - s_6 s_4 + 2s_{10},$$

whence $\qquad S = - 6s_{10} + 2s_8 s_2 + s_5{}^2 - s_2 s_4{}^2 + 3s_6 s_4,$

and substituting the values for the quantities s previously found, we obtain, on reduction,

$$S = - 500p^5 - 5q^2.$$

6. Let E and F be the middle points of AB, CD and let $EF = p$. Then we have

$$ED^2 + EC^2 = 2p^2 + \frac{c^2}{2}, \quad d^2 + y^2 = 2DE^2 + \frac{a^2}{2}, \quad b^2 + x^2 = 2EC^2 + \frac{a^2}{2}$$

(where $AC = x$, $BD = y$), whence

$$b^2 + d^2 + x^2 + y^2 = 4p^2 + a^2 + c^2,$$

$$\text{i.e.} \quad 4p^2 = b^2 + d^2 - a^2 - c^2 + x^2 + y^2.$$

Similarly, for the other line,

$$4q^2 = a^2 + c^2 - b^2 - d^2 + x^2 + y^2,$$

whence

$$4(p^2+q^2)=2(x^2+y^2)=4(\sigma^2+\tau^2); \quad \therefore\ p^2+q^2=\sigma^2+\tau^2,$$

and
$$\begin{aligned}16p^2q^2 &= (x^2+y^2)^2-(b^2+d^2-a^2-c^2)^2\\ &= (x^2-y^2)^2+\{4x^2y^2-(b^2+d^2-a^2-c^2)^2\}\\ &= 16\sigma^2\tau^2+16\Delta^2\end{aligned}$$

(since $\Delta=\frac{1}{2}xy\sin\phi$, and $2xy\cos\phi=b^2+d^2-a^2-c^2$, where ϕ is the angle between the diagonals).

Hence p^2 and q^2 are the roots of

$$z^2-(\sigma^2+\tau^2)z+\sigma^2\tau^2+\Delta^2=0.$$

7. From the logarithmic series, we find

$$x+\tfrac{1}{2}x^2-\tfrac{1}{4}x^4-\tfrac{1}{5}x^5+\ldots=\frac{1}{\omega-\omega^2}\log\frac{1+\omega x}{1+\omega^2 x},$$

where ω and ω^2 are the imaginary cube roots of unity, so that we may take

$$\omega=e^{\frac{2i\pi}{3}}, \quad \omega^2=e^{-\frac{2i\pi}{3}}.$$

Hence the given series is the coefficient of i in

$$\frac{1}{\sqrt{3}i}\log\frac{1+e^{i\left(\theta+\frac{2\pi}{3}\right)}}{1+e^{i\left(\theta-\frac{2\pi}{3}\right)}}$$

$$=\frac{1}{\sqrt{3}i}\log\frac{1+\cos\left(\theta+\frac{2\pi}{3}\right)+i\sin\left(\theta+\frac{2\pi}{3}\right)}{1+\cos\left(\theta-\frac{2\pi}{3}\right)+i\sin\left(\theta-\frac{2\pi}{3}\right)}.$$

Now the real part of

$$\log\left\{1+\cos\left(\theta+\frac{2\pi}{3}\right)+i\sin\left(\theta+\frac{2\pi}{3}\right)\right\}$$

is
$$\begin{aligned}&\tfrac{1}{2}\log\left\{\left[1+\cos\left(\theta+\frac{2\pi}{3}\right)\right]^2+\sin^2\left(\theta+\frac{2\pi}{3}\right)\right\}\\ &=\tfrac{1}{2}\log\left\{2+2\cos\left(\theta+\frac{2\pi}{3}\right)\right\}\\ &=\tfrac{1}{2}\log\left\{4\cos^2\left(\frac{\theta}{2}+\frac{\pi}{3}\right)\right\}=\log\left\{2\cos\left(\frac{\theta}{2}+\frac{\pi}{3}\right)\right\},\end{aligned}$$

and similarly for the denominator. Hence the value of the given series is

$$\frac{1}{\sqrt{3}}\log\frac{\cos\left(\frac{\theta}{2}-\frac{\pi}{3}\right)}{\cos\left(\frac{\theta}{2}+\frac{\pi}{3}\right)}=\frac{1}{\sqrt{3}}\log\frac{1+\sqrt{3}\tan\frac{\theta}{2}}{1-\sqrt{3}\tan\frac{\theta}{2}}.$$

8. The pole of $l'x+m'y+n'=0$ for the second circle is given by

$$\frac{x'+b}{l'}=\frac{y'}{m'}=\frac{bx'+c^2}{n'}=\frac{c^2-b^2}{n'-bl'},$$

and if this lies on $lx+my+n=0$, we find

$$c^2ll'+(c^2-b^2)\,mm'-b\,(ln'+l'n)+nn'=0.$$

Hence if $Ax^2+2Hxy+\ldots=0$ represents a pair of lines, they are conjugate for the circle if

$$Ac^2+B\,(c^2-b^2)-2Gb+C=0 \quad\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Now the tangents from (ξ,η) to the first circle are

$$(x^2+y^2+2ax+c^2)\,(\xi^2+\eta^2+2a\xi+c^2)=\{x\,(\xi+a)+y\eta+a\xi+c^2\}^2,$$

for which

$$A=\eta^2+c^2-a^2,\qquad B=\xi^2+2a\xi+c^2,$$

$$G=a\eta^2+(a^2-c^2)\,\xi,\qquad C=(c^2-a^2)\,\xi^2+c^2\eta^2.$$

Substituting these in (i), we get the locus given for (ξ,η).

9. The tangential equation to $\Sigma\beta\gamma=0$ is

$$\Sigma l^2-2\Sigma mn=0.$$

Hence that of any confocal is

$$\Sigma l^2-2\Sigma mn+k\,(\Sigma l^2-2\Sigma mn\cos A)=0,$$

$$\text{i.e. } (1+k)\,.\,\Sigma l^2-2\Sigma\,(1+k\cos A)\,mn=0.$$

If the confocal touches $\alpha=0$, this must be satisfied when $m=n=0$ and $l\neq0$, whence $k=-1$, and when this is the case, the conic evidently touches $\beta=0$ and $\gamma=0$ also.

The tangential equation is then $\Sigma mn\sin^2\frac{A}{2}=0$. But the tangential equation of $\Sigma\sqrt{\lambda\alpha}=0$ is $\Sigma\lambda mn=0$, and therefore in this case we must have

$$\frac{\lambda}{\sin^2\frac{A}{2}}=\ldots=\ldots,$$

and the confocal is $$\Sigma\sqrt{\alpha}\sin\frac{A}{2}=0.$$

10. Consider first the spherical segment cut off by a plane at distance z from the centre. To find its centre of gravity we consider it as the difference of the spherical sector and the cone having the same plane base and its vertex at O, the centre of the sphere (radius a).

The centre of gravity of the sector is at a distance $\frac{3}{8}(a+z)$ from O, that of the cone at distance $\frac{3}{4}z$, while their respective volumes are $\frac{2}{3}\pi a^2(a-z)$ and $\frac{1}{3}\pi(a^2-z^2)z$.

Hence the distance of the centre of gravity of the segment from O is

$$\frac{\frac{2}{3}\pi a^2(a-z)\cdot\frac{3}{8}(a+z)-\frac{1}{3}\pi(a^2-z^2)z\cdot\frac{3}{4}z}{\frac{2}{3}\pi a^2(a-z)-\frac{1}{3}\pi(a^2-z^2)z}=\frac{3}{4}\cdot\frac{(a+z)^2}{2a+z},$$

and if $z=\dfrac{a}{2}$, this is $\dfrac{27}{40}a$.

Now calling the centre of gravity G, let N be the point of contact of the solid with the inclined plane. Then NG is vertical, and the angle the plane face makes with the horizontal is the angle OG makes with the vertical, i.e. $180^\circ-O\hat{G}N$.

But $$\frac{\sin O\hat{G}N}{\sin O\hat{N}G}=\frac{ON}{OG}=\frac{40}{27},$$

and $O\hat{N}G=\alpha$.

11. The acceleration of the wedge horizontally is

$$\frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}\text{ (as in VII. 11, putting }\mu=0\text{)}.$$

Hence acceleration of the particle relative to the wedge, parallel to the lines of greatest slope, is

$$f=\frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}\cdot\cos\alpha+g\sin\alpha=\frac{(M+m)\,g\sin\alpha}{M+m\sin^2\alpha}.$$

Since this acceleration is constant in magnitude and direction, the particle describes a parabola on the face of the wedge.

Also the velocity perpendicular to the direction of this acceleration is $V\cos\beta$. Hence the latus-rectum of the path is $2V^2\cos^2\beta/f$.

12. If P is the point m, then since PQ makes the same angle with the axis as the tangent, its equation is

$$x + my = 3am^2,$$

and it meets the parabola again at the point $-3m$.

Hence $\qquad PQ^2 = (9am^2 - am^2)^2 + (2am + 6am)^2,$

whence $\qquad PQ = 8am \, . \sqrt{1+m^2}.$

Also if CY is the perpendicular from C on PQ, we have, since the co-ordinates of C are $(2a + 3am^2, -2am^3)$,

$$CY = \frac{2a - 2am^4}{\sqrt{1+m^2}};$$

$$\therefore \quad \tfrac{1}{2}PQ \, . \, CY = 8a^2m(1-m^4).$$

This is a maximum when $m - m^5$ is a maximum, i.e. when

$$1 - 5m^4 = 0 \text{ or } m = \frac{1}{\sqrt[4]{5}},$$

and CP, being the normal, makes an angle $\tan^{-1}m$ with the axis.

XCV.

1. Suppose the lines meet EF, FD, DE in X, Y, Z respectively, and let OA, OB, OC meet the opposite sides in L, M, N. Then since the sides of DEF are parallel to those of ABC, and DX is parallel to AL, it is evident that $EX : XF = BL : LC$, and similarly

$$FY : YD = CM : MA, \quad DZ : ZE = AN : NB.$$

But since AL, BM, CN meet in a point, therefore the ratio compounded of $BL : LC$, $CM : MA$ and $AN : NB$ is unity. Hence the ratio compounded of $EX : XF$, etc. is unity, and therefore DX, EY, FZ meet in a point.

2. Let P and P' be the two points of contact on one side of the axis, and PG, $P'G'$ the normals, so that G and G' are the centres

of the circles. Let L and M be the internal and external centres of similitude, and bisect LM in O. Then

$$\frac{GL}{G'L}=\frac{GM}{G'M}=\frac{GL+GM}{G'L+G'M}=\frac{GM-GL}{G'M-G'L},$$

$$\text{i.e.}\quad \frac{GL}{G'L}=\frac{OL}{OG'}=\frac{OG}{OL},$$

$$\therefore\quad \frac{OG}{OG'}=\frac{GL^2}{G'L^2}=\frac{PG^2}{P'G'^2}.$$

But, if PN be the ordinate of P,

$$PG^2=PN^2+4AS^2=4AS\,(AN+AS)=4AS\,.\,SP=4AS\,.\,SG.$$

Hence
$$\frac{SG}{SG'}=\frac{PG^2}{P'G'^2},$$

so that S must coincide with O.

3. If n is odd, the sum of the products is

$$\frac{1}{2}\sum_{s=1}^{s=n-1} s\,(n-s)=\frac{1}{2}\left\{\frac{n^2(n-1)}{2}-\frac{n\,(n-1)\,(2n-1)}{6}\right\}=\frac{n\,(n^2-1)}{12},$$

and their number is $\dfrac{n-1}{2}$.

Hence the A.M. is $\dfrac{n\,(n+1)}{6}$, and the ratio of n^2 to this is

$$\frac{6n^2}{n\,(n+1)}=\frac{6}{1+\dfrac{1}{n}},$$

tending to the limit 6 as n increases.

If n is even, the sum of the products is

$$\frac{n\,(n^2-1)}{12}-\frac{n^2}{8}=\frac{n\,(n-2)\,(2n+1)}{24},$$

and their number is $\dfrac{n-2}{2}$, leading to a similar result.

4. Using x flags, they may be selected in ${}^nC_x$ ways, and they have then to be divided into r different groups. This can be done by associating with them $r-1$ points of partition, and then arranging these $x+r-1$ things in all possible ways. Since $r-1$ of them are alike, the number of arrangements is

$$\frac{(x+r-1)\,!}{(r-1)\,!}=r\,(r+1)\,(r+2)\ldots(r+x-1).$$

Thus the number of signals which can be made with x flags is

$$^nC_x \,.\, r(r+1)(r+2)\ldots(r+x-1).$$

Including the case in which no flag is used, x may have all values from 0 to n, and thus the total number of signals is

$$n!\left\{\frac{1}{n!}+\frac{r}{1!\,(n-1)!}+\frac{r(r+1)}{2!\,(n-2)!}+\ldots \text{ to } (n+1) \text{ terms}\right\},$$

which is the coefficient of x^n in the product of the two series

$$n!\left\{1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right\}$$

and $$1+\frac{r}{1!}x+\frac{r(r+1)}{2!}x^2+\frac{r(r+1)(r+2)}{3!}x^3+\ldots,$$

and these are respectively the expansions of $n!\,e^x$ and $(1-x)^{-r}$.

Since this includes the case when no flag is used, and therefore no signal made, the actual number of possible signals is one less than this coefficient.

5. We have

$$\Sigma(\alpha-\beta)^4 \equiv (n-1)\,\Sigma\alpha^4 - 4\Sigma\alpha^3\beta + 6\Sigma\alpha^2\beta^2 \ldots\ldots\ldots\text{(i)}.$$

Now $$\Sigma\alpha^3\beta = \Sigma\alpha^2 \,.\, \Sigma\alpha\beta - \Sigma\alpha^2\beta\gamma,$$

and $$\Sigma\alpha^2\beta\gamma = \Sigma\alpha \,.\, \Sigma\alpha\beta\gamma - 4\Sigma\alpha\beta\gamma\delta = p_1p_3 - 4p_4;$$

$$\therefore\ \Sigma\alpha^3\beta = (p_1^2-2p_2)\,p_2 - (p_1p_3-4p_4).$$

Also $$\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha^2\beta\gamma - 6\Sigma\alpha\beta\gamma\delta$$
$$= p_2^2 - 2p_1p_3 + 2p_4,$$

and $$\Sigma\alpha^4 = (p_1^2-2p_2)^2 - 2\Sigma\alpha^2\beta^2.$$

Substituting these in (i) its value is

$$(n-1)(p_1^4-4p_1^2p_2+4p_2^2) - 2(n-4)(p_2^2-2p_1p_3+2p_4)$$
$$-4(p_1^2p_2-2p_2^2-p_1p_3+4p_4)$$
$$=(n-1)\,p_1^4 - 4np_1^2p_2 + 4(n-3)\,p_1p_3 + 2(n+6)\,p_2^2 - 4np_4.$$

6. Putting $\cos\theta = x$, the equation becomes

$$4x^3 - 3x + a(2x^2-1) + bx + c = 0,$$

or $$4x^3 + 2ax^2 + (b-3)\,x + c - a = 0,$$

and the roots of this are $\cos A$, $\cos B$ and $\cos C$.

But $$1 - \Sigma\cos^2 A - 2\cos A\cos B\cos C \equiv 0;$$

$$\therefore\ 1 - \left(\frac{a^2}{4} - \frac{b-3}{2}\right) - \frac{a-c}{2} = 0,$$

i.e. $a^2 + 2a - 2b - 2c + 2 = 0.$

We also know that $\Sigma \cos A > 1$ and $\not> \frac{3}{2}$, i.e. $-\frac{a}{2} > 1$ and $\not> \frac{3}{2}$; therefore a must lie between -2 and -3. Further

$$\cos A \cos B \cos C \not> \tfrac{1}{8}; \quad \therefore \quad a - c \not> \tfrac{1}{2}.$$

7. Since

$$\tan(\phi + a) = \frac{3\tan a + \tan^3 a}{1 + 3\tan^2 a} = \frac{3\sin a - 2\sin^3 a}{3\cos a - 2\cos^3 a} = \frac{3\sin a + \sin 3a}{3\cos a - \cos 3a},$$

we have $$\frac{e^{i(\phi+a)}}{e^{-i(\phi+a)}} = \frac{3e^{ia} - e^{-3ia}}{3e^{-ia} - e^{3ia}};$$

whence $$e^{2i\phi} = \frac{3 - e^{-4ia}}{3 - e^{4ia}};$$

$$\therefore \quad 2i\phi = \log\left(1 - \frac{e^{-4ia}}{3}\right) - \log\left(1 - \frac{e^{4ia}}{3}\right)$$

$$= \frac{e^{4ia} - e^{-4ia}}{3} + \frac{1}{2} \cdot \frac{e^{8ia} - e^{-8ia}}{3^2} + \ldots,$$

whence $$\phi = \frac{1}{3}\sin 4a + \frac{1}{2 \cdot 3^2}\sin 8a + \ldots .$$

8. Any conic osculating the parabola at m is of the form

$$\lambda(y^2 - 4ax) + (x - my + am^2)^2 = 0 \quad \text{.............(i)}.$$

In this the coefficients of x^2, $2xy$, y^2 are 1, $-m$, $m^2 + \lambda$ and therefore the eccentricity is given by

$$\frac{(2 - e^2)^2}{1 - e^2} = \frac{(1 + m^2 + \lambda)^2}{\lambda} \quad \text{..................(ii)}.$$

Also the centre of (i) is given by

$$-2a\lambda + x - my + am^2 = 0,$$

$$\lambda y - m(x - my + am^2) = 0.$$

From these, $$y = 2am, \quad x = am^2 + 2a\lambda,$$

$$\text{i.e. } m = \frac{y}{2a}, \quad \lambda = \frac{4ax - y^2}{8a^2},$$

and substituting these in (ii), we get the locus.

9. Since similar conics only intersect in two finite points, therefore any conic similar to the general conic is of the form

$$u\alpha^2 + \ldots + \ldots + 2u'\beta\gamma + \ldots + \ldots + (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0.$$

If this circumscribes the triangle of reference, the coefficients of α^2, β^2, γ^2 must vanish, so that

$$l = -\frac{u}{a}, \quad m = -\frac{v}{b}, \quad n = -\frac{w}{c},$$

and the equation then becomes

$$\Sigma\left(2u' - b\,.\,\frac{w}{c} - c\,.\,\frac{v}{b}\right)\beta\gamma = 0,$$

$$\text{i.e. } \Sigma a\,(2u'bc - vc^2 - wb^2)\,\beta\gamma = 0.$$

If the original conic is a circle, then, since all circles are similar, this equation must be that of the circumcircle, viz. $\Sigma a\beta\gamma = 0$. Hence the required conditions are

$$2u'bc - vc^2 - wb^2 = \text{two similar expressions.}$$

10. Let P and P' be the lower and upper extremities of the rod, α the angle the normal at P makes with the major axis. Draw lines through P and P' making angles λ with the normals upwards, and let these lines meet in O. Then O is vertically above G, the centre of the rod.

Thus $\dfrac{PG}{OG} = \dfrac{P'G}{OG}$, whence

$$\frac{\cos(\theta + \alpha - \lambda)}{\sin(\alpha - \lambda)} = \frac{\cos(\alpha - \theta + \lambda)}{\sin(\alpha + \lambda)},$$

$$\text{i.e. } \frac{\cos(\theta + \alpha - \lambda)}{\cos(\alpha - \theta + \lambda)} = \frac{\sin(\alpha - \lambda)}{\sin(\alpha + \lambda)};$$

$$\therefore\ \tan(\theta - \lambda)\tan\alpha = \tan\lambda\cot\alpha.$$

Now, since P is the extremity of a latus-rectum, we have

$$\tan\alpha = \frac{SX}{SP} = \frac{1}{e};$$

$$\therefore\ \tan(\theta - \lambda) = e^2\tan\lambda,$$

$$\text{i.e. } \frac{\tan\theta - \mu}{1 + \mu\tan\theta} = e^2\mu;$$

$$\therefore\ \tan\theta\,(1 - e^2\mu^2) = \mu\,(1 + e^2).$$

11. Let P be the point of projection, S the focus of the path, PG the normal. Then, since PG lies along the radius of the hoop at P, $P\hat{G}S = a$, therefore $PG = 2SP \cos a$.

Now if the particle is to clear the hoop, the axis of the parabola must be on the side of the centre remote from P,

$$\text{i.e. } PG > a, \quad \text{i.e. } 2SP > a \sec a.$$

But the velocity of projection is $\sqrt{2g \, . \, SP}$, which must therefore be $> \sqrt{ag \sec a}$.

12. The origin is a conjugate point, and the asymptotes are $x = a$, $y = a$ and $x + y + 2a = 0$.

Transferring to A (a, a) as origin, the equation becomes

$$a^2 (x + y) + xy (x + y) + 4axy = 0.$$

Near A the form is $a^2 (x + y) = 4ax^2$ or $y = -x + \dfrac{4x^2}{a}$, shewing that the curve is above the tangent.

Hence the curve consists of two quasi-hyperbolic branches in the second and fourth quadrants (both on the far side of the asymptote $x + y + 2a = 0$), and a single infinite branch in the first quadrant, with two points of inflexion, the whole figure being symmetrical about OA.

Also the joins of the origin to the intersections of the curve and the given circle are given by

$$(x^2 + y^2) + \frac{2xy}{x^2 + y^2} (1 + \sqrt{2}) (x + y)^2 - 2 (1 + \sqrt{2})^2 (x + y)^2 \frac{x^2 y^2}{(x^2 + y^2)^2} = 0,$$

and if $x = ky$, this reduces to

$$(k - 1)^2 [(k + 1)^2 + k \sqrt{2}]^2 = 0,$$

shewing that the circle has treble contact with the curve at the intersections of the latter with

$$x = y \quad \text{and} \quad (x + y)^2 + xy \sqrt{2} = 0.$$

XCVI.

1. On the three circles as diametral planes describe spheres tersecting in A and A'; then by symmetry AA' is bisected perpendicularly by the given plane.

Now invert the figure with respect to A; then the given plane inverts into a sphere passing through A, the three spheres into three planes passing through the centre of this sphere, and the three circles into the intersections of these planes and the sphere, i.e. into three great circles.

2. Reciprocating from the common focus S, the reciprocals of the two conics are equal circles, the ends of the latus-rectum becoming their common tangents. A common tangent to the conic becomes a point of intersection of the circles, and its points of contact (P, Q) become the tangents to the circles at this point. Hence the angles between the latus-rectum and SP, SQ are equal to the angles which the tangents to the circles at a point of intersection make with the common tangent, and these are evidently equal.

3. The A.M. of the quantities

$$\frac{n+1}{n},\ \frac{n}{n-1},\ \frac{n-1}{n-2} \dots,\ \frac{2}{1}$$

is greater than their G.M., i.e.

$$\frac{n+\Sigma\frac{1}{n}}{n} > (n+1)^{\frac{1}{n}},$$

$$\text{i.e. } \Sigma\frac{1}{n} > n\left\{(n+1)^{\frac{1}{n}}-1\right\}.$$

Again, using the same theorem for the quantities

$$\frac{n}{n+1},\ \frac{n-1}{n},\ \frac{n-2}{n-1} \dots,\ \frac{1}{2},$$

we get
$$\frac{n - \sum\limits_{n-1} \dfrac{1}{n+1}}{n} > \left(\frac{1}{n+1}\right)^{\frac{1}{n}} > (n+1)^{-\frac{1}{n}},$$

$$\text{i.e. } \Sigma \frac{1}{n} + \frac{1}{n+1} - 1 < n - n(n+1)^{-\frac{1}{n}},$$

or
$$\Sigma \frac{1}{n} < n + \frac{n}{n+1} - n(n+1)^{-\frac{1}{n}}.$$

4. For each position of the two queens there are three symmetrical positions obtained by successive rotations of the board through a right angle. We need therefore only consider the cases where one queen is on one of the 16 squares at one corner. Placed on one of the 7 side squares, she commands 21 squares, giving 7×21 cases in which taking is possible.

Placed on one of the five squares one square in, she commands 23 squares, giving 5×23 possible cases. Similarly the other positions give 3×25 and 1×27 cases.

Hence the number of possible cases of taking is

$$147 + 115 + 75 + 27 = 364.$$

But the total number of possible positions is $16 \times 63 = 1008$.

Hence the chance of not taking is

$$\frac{1008 - 364}{1008} = \frac{23}{36}.$$

5. Writing the equation in the form

$$(ax^2 + 2bx + c + 2a\theta)^2 - (2Mx + N)^2 = 0,$$

and comparing with the original, we easily find for θ the cubic

$$4a^3\theta^3 - Ia\theta + J = 0 \quad \text{.....................(i)},$$

and calling the roots of this θ_1, θ_2, θ_3, we find from the two quadratics,

$$\beta\gamma + \alpha\delta = 4\theta_1 + \frac{2c}{a}, \text{ etc.}$$

If we take $x = 4y + \dfrac{2c}{a}$, the equation in the question becomes

$$\sqrt[3]{\theta_1 - y} + \sqrt[3]{\theta_2 - y} + \sqrt[3]{\theta_3 - y} = 0,$$

of which the rational form is

$$(\Sigma\theta_1 - 3y)^3 = 27\,(\theta_1 - y)\,(\theta_2 - y)\,(\theta_3 - y).$$

But by (i) this is

$$(-3y)^3 = -\frac{27}{4a^3}\,(4a^3y^3 - Iay + J),$$

and this has only one root, viz. $y = \frac{J}{Ia}$, giving

$$x = \frac{4J}{Ia} + \frac{2c}{a}.$$

6. Let $A'B'C'$ be the projections of ABC on the horizontal plane, and let the areas of ABC, $A'B'C'$ be Δ, Δ', so that

$$\cos\theta = \frac{\Delta'}{\Delta}.$$

Now calling the perpendiculars p_1, p_2, p_3, we have

$$a'^2 = a^2 - (p_2 - p_3)^2 = a^2 - \frac{a^2}{4R^2}\,(b-c)^2 \text{ ....(i).}$$

Hence

$$b'^2 + c'^2 - a'^2 = 2bc\cos A - \frac{1}{4R^2}\left[b^2(c-a)^2 + c^2(a-b)^2 - a^2(b-c)^2\right]$$

$$= 2bc\left[\cos A - \frac{1}{4R^2}\,(a-b)\,(a-c)\right] \text{ ....(ii),}$$

whence

$$16\Delta'^2 = 4b'^2c'^2 - (b'^2 + c'^2 - a'^2)^2$$

$$= 4b^2c^2\left\{\sin^2 A - \frac{1}{4R^2}\left[2a^2(1-\cos A) - 2a(b+c)(1-\cos A) + a^2\right]\right\}$$

from (i) and (ii).

Now $$(b+c)(1-\cos A) = a\,\frac{\sin B + \sin C}{\sin A}\,.\,2\sin^2\frac{A}{2}$$

$$= a(\cos B + \cos C);$$

$$\therefore\; 16\Delta'^2 = 4b^2c^2\left\{\sin^2 A - \frac{a^2}{4R^2}\left[2(1-\cos A) - 2(\cos B + \cos C) + 1\right]\right\}$$

$$= 4b^2c^2\sin^2 A\,\{1 - (3 - 2\Sigma\cos A)\}$$

$$= 16\Delta^2(2\Sigma\cos A - 2);$$

$$\therefore\; \cos^2\theta = 2\Sigma\cos A - 2; \qquad \therefore\; \cos 2\theta = 4\Sigma\cos A - 5.$$

7. If in the identity

$$\sin n\theta = 2^{n-1} \cdot \prod_{m=0}^{m=n-1} \sin\left(\theta + \frac{m\pi}{n}\right)$$

we put $\theta = 0$, we have, remembering that $\text{Lt}\,\dfrac{\sin n\theta}{\sin\theta} = n$,

$$\prod_{m=1}^{m=n-1} \sin\frac{m\pi}{n} = \frac{n}{2^{n-1}}.$$

Hence the value of the double product is

$$\prod_{n=2}^{n=s} . n . 2^{-(n-1)} = s\,!\; 2^{-\sum\limits_{n=2}^{n=s}(n-1)}$$

$$= s\,!\; 2^{-\frac{s(s-1)}{2}},$$

i.e. $s\,! = 2^{\frac{1}{2}s(s-1)}$ times the double product.

8. The parallels through the origin to the four normals through (x_0, y_0) are

$$(a^2x^2 + b^2y^2)(xy_0 - x_0y)^2 - c^4x^2y^2 = 0 \quad (\text{XLIII. } 8),$$

i.e. $a^2y_0^2x^4 - 2a^2x_0y_0x^3y + (a^2x_0^2 + b^2y_0^2 - c^4)\,x^2y^2 - 2b^2x_0y_0xy^3$

$$+ b^2x_0^2y^4 = 0 \ldots\text{(i)}.$$

If these four lines form a harmonic pencil, the biquadratic (i) must break up into factors

$$(x^2 + 2pxy + qy^2)(x^2 + 2p'xy + q'y^2) = 0,$$

satisfying the condition

$$q + q' - 2pp' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Now writing the biquadratic (i) in the form

$$a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4 = 0,$$

and comparing the two forms, we get

$$p + p' = \frac{2a_1}{a_0}, \quad q + q' + 4pp' = \frac{6a_2}{a_0}, \quad pq' + p'q = \frac{2a_3}{a_0}, \quad qq' = \frac{a_4}{a_0},$$

whence by (ii) $\qquad pp' = \dfrac{a_2}{a_0}, \quad q + q' = \dfrac{2a_2}{a_0}.$

Also $$(p+p')(q+q') = \frac{4a_1a_2}{a_0^2},$$

$$\therefore\ pq + p'q' = \frac{4a_1a_2 - 2a_0a_3}{a_0^2};$$

and now, using the identity

$$(p^2+p'^2)(q^2+q'^2) \equiv (pq'-p'q)^2 + (pq+p'q')^2,$$

we have

$$\frac{4a_1^2 - 2a_0a_2}{a_0^2} \cdot \frac{4a_2^2 - 2a_0a_4}{a_0^2} = \frac{4a_3^2 - 4a_2a_4}{a_0^2} + \left(\frac{4a_1a_2 - 2a_0a_3}{a_0^2}\right)^2,$$

reducing to

$$a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3 = 0.$$

Applying this to (i), we find

$$a^2b^2x_0^2y_0^2 \,.\, a_2 + \tfrac{1}{2}a^2b^2x_0^2y_0^2 \,.\, a_2 - \tfrac{1}{4}a^2b^4x_0^2y_0^4 - \tfrac{1}{4}a^4b^2x_0^4y_0^2 - a_2^3 = 0,$$

where $$a_2 = \tfrac{1}{6}(a^2x_0^2 + b^2y_0^2 - c^4).$$

From this $$4a_2^3 = a^2b^2x_0^2y_0^2(6a_2 - a^2x_0^2 - b^2y_0^2)$$
$$= -a^2b^2c^4x_0^2y_0^2,$$

giving the required locus.

9. Since the directrices pass through the intersections (imaginary) of the curve with its director-circle, their equation must be of the form

$$\lambda S + K = 0,$$

and if this represents a pair of parallel straight lines, we must have

$$(\lambda a + C)(\lambda b + C) = \lambda^2 h^2,$$

$$\text{i.e.}\quad \lambda^2 + \lambda(a+b) + C = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Now the equation to the conic referred to its centre is

$$ax^2 + 2hxy + by^2 = c',$$

where $c' = -\dfrac{\Delta}{C}$. Hence, if r_1, r_2 are the semi-axes, we have

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{a+b}{c'},$$

$$\frac{1}{r_1^2 r_2^2} = \frac{C}{c'^2},$$

shewing that r_1^2, r_2^2 are the roots of

$$\frac{c'^2}{r^4} - (a+b)\frac{c'}{r^2} + C = 0 \quad \text{..................(ii)},$$

and we see that (i) and (ii) are identical if $\lambda = -\frac{c'}{r^2}$, i.e. for a semi-axis a, the value of λ is

$$-\frac{c'}{a^2} = \frac{\Delta}{a^2 C},$$

and the equation to the directrices is $\Delta S + a^2 CK = 0$.

10. Let PP' cut the axis in N, then $l^2\cos^2\theta = 4a \,.\, AN$, where θ is the inclination of PQ to the horizontal. For a small symmetrical displacement the equation of virtual work is

$$2w \,.\, \delta\left(\frac{l^2\cos^2\theta}{4a}\right) + 2W \,.\, \delta\left(\frac{l^2\cos^2\theta}{4a} + \tfrac{1}{2}l\sin\theta\right) = 0,$$

i.e. $\quad -w \,.\, \frac{l^2}{a}\cos\theta\sin\theta \,.\, \delta\theta - W \,.\, \frac{l^2}{a}\cos\theta\sin\theta \,.\, \delta\theta + W \,.\, l\cos\theta \,.\, \delta\theta = 0,$

whence $$\sin\theta = \frac{Wa}{l(W+w)}.$$

11. Let P be the point of impact, v the velocity of P, and θ the angle the chord makes with the vertical. Then the direction of motion after the impact makes an angle $90° - 3\theta$ with the horizontal, and the equation to the path, referred to horizontal and vertical axes through P, is

$$y = x\cot 3\theta - \tfrac{1}{2}g \,.\, \frac{x^2}{v^2}\operatorname{cosec}^2 3\theta.$$

Also $v^2 = 2ga(1+\cos 2\theta) = 4ga\cos^2\theta$, so that this equation is

$$y = x\cot 3\theta - \tfrac{1}{8}\frac{x^2\operatorname{cosec}^2 3\theta}{a\cos^2\theta}.$$

If this passes through the lowest point, whose co-ordinates are $a\sin 2\theta$, $-a(1-\cos 2\theta)$, we have

$$-2a\sin^2\theta = a\sin 2\theta\cot 3\theta - \tfrac{1}{2}a\frac{\sin^2\theta}{\sin^2 3\theta},$$

$$\text{i.e.} \quad -2 = 2\cot\theta\cot 3\theta - \tfrac{1}{2}\operatorname{cosec}^2 3\theta,$$

or $$\operatorname{cosec}^2 3\theta = 4(\cot\theta\cot 3\theta + 1)$$
$$= \frac{4\cos 2\theta}{\sin\theta\sin 3\theta},$$
$$\text{i.e.}\quad 4\cos 2\theta\sin 3\theta = \sin\theta.$$

But $$\frac{\sin 3\theta}{\sin\theta} = 2\cos 2\theta + 1;$$
$$\therefore\quad 4\cos 2\theta(2\cos 2\theta + 1) = 1,$$
$$\text{i.e.}\quad 8\cos^2 2\theta + 4\cos 2\theta - 1 = 0,$$

whence $$\cos 2\theta = \frac{-1 \pm \sqrt{3}}{4},$$

and evidently, from the figure, 2θ must be an acute angle;
$$\therefore\quad \theta = \tfrac{1}{2}\cos^{-1}\left(\frac{\sqrt{3}-1}{4}\right).$$

12. If $y^2 = \dfrac{x-c}{x-b}$, then $y^2 - 1 = \dfrac{b-c}{x-b}$;
$$\therefore\quad 2y\,dy = \frac{c-b}{(x-b)^2}\,dx.$$

Also $\sqrt{(x-b)(x-c)} = (x-b)\,y$, so that
$$\frac{dx}{\sqrt{(x-b)(x-c)}} = \frac{2(x-b)}{c-b}\,.\,dy = \frac{2dy}{1-y^2}.$$

Also $$x - a = \frac{(c-a)-(b-a)y^2}{1-y^2}.$$

Hence, if we put $b-a=p^2$, $c-a=q^2$ $(a<b<c)$, the integral becomes
$$2\int\frac{1-y^2}{(q^2-p^2y^2)^2}\,dy = \frac{1}{p^2q^2}\int\left\{\frac{p^2+q^2}{q^2-p^2y^2} + \frac{(p^2-q^2)(q^2+p^2y^2)}{(q^2-p^2y^2)^2}\right\}dy$$
$$= \frac{p^2+q^2}{2p^3q^3}\,.\,\log\frac{q+py}{q-py} + \frac{p^2-q^2}{p^2q^2}\,.\,\frac{y}{q^2-p^2y^2},$$

and, replacing the values of p, q, y, we obtain the value of the integral in terms of x.

XCVII.

1. Let the plane of the common section cut the line of centres at distances x and y from the respective centres. Then

$$x^2 - y^2 = a^2 - b^2, \quad x + y = c; \quad \therefore \; x - y = \frac{a^2 - b^2}{c}.$$

Hence
$$x = \frac{c^2 + a^2 - b^2}{2c}, \quad y = \frac{b^2 + c^2 - a^2}{2c}.$$

Now the common volume consists of two spherical caps of heights $a - x$ and $b - y$. Also the volume of a cap of height h on a sphere of radius r is $\pi h^2 (r - \frac{1}{3} h)$.

Hence the volume required is

$$\pi (a - x)^2 . \frac{2a + x}{3} + \pi (b - y)^2 . \frac{2b + y}{3},$$

and since

$$a - x = \frac{b^2 - (c - a)^2}{2c}, \quad b - y = \frac{a^2 - (b - c)^2}{2c},$$

this is

$$\frac{\pi}{3} \cdot \left(\frac{a + b - c}{2c}\right)^2 [(b + c - a)^2 (2a + x) + (a + c - b)^2 (2b + y)].$$

The expression in the square bracket is

$$[c^2 + (a - b)^2] (2a + 2b + x + y) - 2c (a - b) (2a - 2b + x - y)$$
$$= [c^2 + (a - b)^2] (2a + 2b + c) - 4c (a - b)^2 - 2 (a - b) (a^2 - b^2)$$
$$= c^3 + 2c^2 (a + b) - 3c (a - b)^2,$$

leading to the result given.

2. Let A, B, C be the three given points, Ω, Ω' the points at infinity on the asymptotes and O the centre. Apply Pascal's Theorem to the hexagon $ABC\Omega\Omega\Omega'$. The points of intersection of opposite sides are

$$L (A\Omega', C\Omega); \quad M (BC, \Omega\Omega'); \quad N (AB, \Omega\Omega).$$

The first of these is the point of intersection of lines through A and C in the given directions of $O\Omega'$, $O\Omega$; the second is at infinity on BC; the third is the intersection of AB with the asymptote $O\Omega$. These three points are therefore collinear. Hence the construction:—Draw lines through A and C parallel to the given directions of $O\Omega'$, $O\Omega$ respectively, meeting in L. Through L draw LN parallel to BC cutting AB in N. Then the line through N parallel to the given direction of $O\Omega$ is one asymptote, and the other may be constructed by a similar method.

3. We have $r^3 \,.\, C_r^2 = r\,(r \,.\, C_r)^2 = r \,.\, (n \,.\, {}^{n-1}C_{r-1})^2$

$$= n^2 \{(r-1) \,.\, {}^{n-1}C_{r-1}{}^2 + {}^{n-1}C_{r-1}{}^2\}$$

$$= n^2 \{(n-1) \,.\, {}^{n-2}C_{r-2} \,.\, {}^{n-1}C_{r-1} + {}^{n-1}C_{r-1}{}^2\}.$$

Now $\quad \Sigma\, {}^{n-2}C_{r-2} \,.\, {}^{n-1}C_{r-1} =$ absolute term in $x(1+x)^{n-2}\left(1+\dfrac{1}{x}\right)^{n-1}$

$$= \text{coefficient of } x^{n-2} \text{ in } (1+x)^{2n-3}$$

$$= \frac{(2n-3)!}{(n-2)!\,(n-1)!},$$

and $\quad \Sigma\, {}^{n-1}C_{r-1}{}^2 =$ absolute term in $(1+x)^{n-1}\left(1+\dfrac{1}{x}\right)^{n-1}$

$$= \text{coefficient of } x^{n-1} \text{ in } (1+x)^{2n-2}$$

$$= \frac{(2n-2)!}{\{(n-1)!\}^2}.$$

Hence the sum of the given series is

$$n^2\left[\frac{(2n-3)!}{\{(n-2)!\}^2} + \frac{(2n-2)!}{\{(n-1)!\}^2}\right]$$

$$= n^2 \,.\, \frac{(2n-3)!}{\{(n-1)!\}^2}\,[(n-1)^2 + (2n-2)].$$

4. Multiplying the rows by $1, x^{n-1}, x^{n-2}, \ldots, x^2, x$, and subtracting the first row from each of the others, we obtain a determinant in which the elements of the leading diagonal are

$$x^{n-1},\ x^{n-2}(x^n-1),\ x^{n-3}(x^n-1),\ \ldots,\ x^n-1,$$

and in which every element to the left of the leading diagonal is zero. The value of this determinant is therefore

$$x^{\overline{n-1}+\overline{n-2}+\ldots+1}\,(x^n-1)^{n-1}.$$

Hence that of the original determinant (which has been multiplied by factors x^{n-1}, x^{n-2}, ..., x, 1) is $(x^n-1)^{n-1}$. In the second case we use the same multipliers and add.

5. Let $\alpha, \beta, \gamma, \delta$ be the roots. Then, by squaring the array

$$\left.\begin{matrix} 1, & 1, & 1, & 1 \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2 \end{matrix}\right\},$$

we see that the value of the given determinant is

$$D = \Sigma (\beta-\gamma)^2 (\gamma-\alpha)^2 (\alpha-\beta)^2.$$

Now if $z = ax + b$, the biquadratic becomes

$$z^4 + 6Hz^2 + 4Gz + a^2I - 3H^2 = 0,$$

and assuming $z = \sqrt{p} + \sqrt{q} + \sqrt{r}$, then p, q, r are the roots of Euler's cubic

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0 \quad \text{............(i).}$$

Hence, taking the double signs with $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$ so that

$$a\alpha + b = \sqrt{p} - \sqrt{q} - \sqrt{r}, \qquad a\gamma + b = -\sqrt{p} - \sqrt{q} + \sqrt{r},$$
$$a\beta + b = -\sqrt{p} + \sqrt{q} - \sqrt{r}, \qquad a\delta + b = \sqrt{p} + \sqrt{q} + \sqrt{r},$$

we find

$$\begin{aligned} \tfrac{1}{64} a^6 D &= (\sqrt{p} - \sqrt{q})^2 (\sqrt{q} - \sqrt{r})^2 (\sqrt{r} - \sqrt{p})^2 \\ &\qquad + \Sigma (\sqrt{p} + \sqrt{q})^2 (\sqrt{r} + \sqrt{p})^2 (\sqrt{q} - \sqrt{r})^2 \\ &= 4\{(p+q)(q+r)(r+p) - 8pqr\} \\ &= 4(\Sigma p \,.\, \Sigma qr - 9pqr). \end{aligned}$$

Hence, from (i),

$$\tfrac{1}{64} a^6 D = 4\left[-3H\left(3H^2 - \frac{a^2I}{4}\right) - \frac{9G^2}{4}\right],$$

and remembering that $G^2 + 4H^3 \equiv a^2HI - a^3J$, we get

$$\tfrac{1}{64} a^6 D = -6a^2HI + 9a^3J,$$

$$\text{i.e. } D = -\frac{192}{a^4}(2HI - 3aJ).$$

6. Let E, F be the middle points of AB, CD, and H, K those of AC, BD. Then $EKFH$ is a parallelogram, and we have to find the angle between its diagonals. If this angle is ϕ, then

$$EF \,.\, HK \cos\phi = EH^2 - EK^2 = \tfrac{1}{4}(b^2 - d^2) \quad \ldots\ldots\ldots(\text{i}).$$

Now
$$EC^2 + ED^2 = 2EF^2 + \tfrac{1}{2}c^2,$$

$$d^2 + y^2 = 2ED^2 + \tfrac{1}{2}a^2, \quad b^2 + x^2 = 2EC^2 + \tfrac{1}{2}a^2,$$

from which
$$4EF^2 = b^2 + d^2 + x^2 + y^2 - (c^2 + a^2),$$

$$4HK^2 = a^2 + b^2 + c^2 + d^2 - (x^2 + y^2);$$

$$\therefore\ 16EF^2 \,.\, HK^2 = (b^2 + d^2)^2 - (x^2 + y^2 - a^2 - c^2)^2.$$

Hence, from (i), the result follows.

7. If r is the radius of one of the circles, then $r = \dfrac{c}{2}\sec\theta$, and the length of the common chord is

$$\frac{a\sqrt{4r^2 - a^2}}{r} = \frac{2a}{c\sec\theta}(c^2\sec^2\theta - a^2)^{\frac{1}{2}} = \frac{2a^2}{c}\left(\frac{c^2}{a^2} - \cos^2\theta\right)^{\frac{1}{2}},$$

and, by the question, this is $2a \operatorname{sech}\alpha\,(\cosh^2\alpha - \cos^2\theta)^{\frac{1}{2}}$.

Now

$$\cosh 2n\alpha - \cos 2n\theta = 2^{n-1} \,.\, \prod_{0}^{n-1}\left\{\cosh 2\alpha - \cos\left(2\theta + \frac{2r\pi}{n}\right)\right\};$$

$$\therefore\ \cosh^2 n\alpha - \cos^2 n\theta = 2^{2n-2} \,.\, \Pi\left\{\cosh^2\alpha - \cos^2\left(\theta + \frac{r\pi}{n}\right)\right\}.$$

But the product of the chords is

$$(2a\operatorname{sech}\alpha)^n \,.\, \Pi\left\{\cosh^2\alpha - \cos^2\left(\theta + \frac{r\pi}{n}\right)\right\}^{\frac{1}{2}},$$

which is therefore equal to

$$(2a\operatorname{sech}\alpha)^n \,.\, \frac{1}{2^{n-1}}(\cosh^2 n\alpha - \cos^2 n\theta)^{\frac{1}{2}}.$$

8. Any circle which cuts the three circles $s_1 = 0$, $s_2 = 0$, $s_3 = 0$ orthogonally will, from the form of the condition, cut every circle of the system

$$\lambda s_1 + \mu s_2 + \nu s_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i})$$

orthogonally.

Hence (i) represents a family of circles cutting a fixed circle orthogonally, this fixed circle being the orthogonal circle of s_1, s_2, s_3.

But if P be any point on (i) and PT_1, PT_2, PT_3 the tangents from P to s_1, s_2, and s_3, then, from (i),

$$\lambda \,.\, PT_1^2 + \mu \,.\, PT_2^2 + \nu \,.\, PT_3^2 = 0,$$

and therefore if A, B, C be the centres of the circles and a_1, a_2, a_3 their radii, we have

$$\lambda \,.\, PA^2 + \mu \,.\, PB^2 + \nu \,.\, PC^2 = \Sigma\lambda a_1^2,$$

and now making the three circles point-circles, the result follows.

9. The centre (x_0, y_0, z_0) is determined by

$$lx_0 = my_0 = nz_0 = \frac{1}{\Sigma l^{-1}} = k \text{ say.}$$

If we put $x - x_0 = \xi$, etc. the equation to the conic is

$$\Sigma l\,(\xi + x_0)^2 = 0, \text{ i.e. } \Sigma l\xi^2 + k = 0 \quad \ldots\ldots\ldots\ldots\text{(i)},$$

since $$\Sigma\xi = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

The distance between the two points (x, y, z), (x_0, y_0, z_0) is given by

$$\rho^2 = \Sigma bc \cos A \,.\, \xi^2,$$

and, to get the semi-axes, we have to make this a maximum or minimum subject to (i) and (ii). We have then

$$\Sigma bc \cos A \,.\, \xi d\xi = 0, \quad \Sigma l\xi d\xi = 0, \quad \Sigma d\xi = 0,$$

and using multipliers p, q such that

$$bc \cos A \,.\, \xi + p \,.\, l\xi + q = 0, \text{ etc.}$$

we get, on multiplying by ξ, η, ζ and adding,

$$\rho^2 - k \,.\, p = 0,$$

whence $$k \,.\, bc \cos A \,.\, \xi + \rho^2 l\xi + kq = 0, \text{ etc.}$$

$$\text{i.e. } \xi = -\frac{kq}{kbc \cos A + \rho^2 l},$$

and, by (ii), this gives

$$\Sigma\,(\rho^2 m + k \,.\, ca \cos B)(\rho^2 n + k \,.\, ab \cos C) = 0,$$

so that, if ρ_1, ρ_2 are the semi-axes, we have

$$\rho_1^2\rho_2^2 = \frac{k^2.\Sigma a^2bc \cos B \cos C}{\Sigma mn}.$$

The sum in the numerator is

$$abc \,.\, 2R \,.\, \Sigma \sin A \cos B \cos C = abc \,.\, 2R \sin A \sin B \sin C = 4\Delta^2.$$

Hence
$$\rho_1\rho_2 = \pm \frac{2\Delta k}{(\Sigma mn)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{\pi\rho_1\rho_2}{\Delta} = \pm \frac{2\pi lmn}{(\Sigma mn)^{\frac{3}{2}}}, \text{ since } k = \frac{lmn}{\Sigma mn}.$$

For the numerical ratio we choose the negative sign, since one of the three l, m, n must be negative.

10. If BC makes an angle ϕ with the horizontal, we have $4a \cos\theta + 4a \cos\phi + 2a = 4a$, where $2a$ is the length of each rod,

$$\text{i.e. } \cos\theta + \cos\phi = \tfrac{1}{2} \quad \text{......................(i).}$$

For a small symmetrical displacement, the equation of virtual work is

$$2W \,.\, \delta(a \sin\theta) + 2W \,.\, \delta(2a \sin\theta + a \sin\phi) + W \,.\, \delta(2a \sin\theta + 2a \sin\phi) = 0,$$

$$\text{i.e. } 2 \cos\theta \,.\, \delta\theta + \cos\phi \,.\, \delta\phi = 0.$$

But, from (i), $\quad \sin\theta \,.\, \delta\theta + \sin\phi \,.\, \delta\phi = 0.$

Hence $\tan\phi = \frac{1}{2}\tan\theta$, and if $\cos\theta = x$, then, from (i),

$$\tfrac{1}{2} - x = \frac{1}{\sqrt{1 + \tan^2\phi}} = \frac{2}{\sqrt{4 + \tan^2\theta}} = \frac{2x}{\sqrt{3x^2 + 1}}.$$

11. Let f be the upward acceleration of the pulley at any time, T the tension of the upper string, T' of the lower. Then the upward accelerations of the two masses are

$$\frac{T'}{m} - g \text{ and } \frac{T'}{m'} - g.$$

Hence since the lower string is inextensible, we have

$$T'\left(\frac{1}{m} + \frac{1}{m'}\right) - 2g = 2f.$$

Also, since the pulley is weightless, $2T' - T = 0$;

$$\therefore \quad T = 4(g+f) \cdot \frac{mm'}{m+m'}.$$

But if x is the extension, $T = \frac{\lambda x}{a}$, leading to

$$f = \frac{\lambda}{a} \cdot \frac{m+m'}{4mm'} \left(x - \frac{a}{\lambda} \cdot \frac{4mm'}{m+m'} \cdot g \right).$$

Hence the acceleration varies as the distance from a fixed point and the motion is simple harmonic, the constant being

$$\mu = \frac{\lambda}{4a} \cdot \frac{m+m'}{mm'}.$$

12. We have

$$ds^2 = dr^2 + r^2 d\theta^2$$
$$= (4a^2n^2 \sin^2 n\theta + 4a^2 \cos^2 n\theta)\, d\theta^2.$$

Hence the perimeter of a loop is

$$2\int_0^{\frac{\pi}{2n}} 2a\sqrt{n^2 \sin^2 n\theta + \cos^2 n\theta}\, d\theta$$
$$= \frac{4a}{n}\int_0^{\frac{\pi}{2}} \sqrt{n^2 \sin^2\phi + \cos^2\phi}\, d\phi \quad \text{(putting } n\theta = \phi\text{)},$$

which is also the perimeter of the ellipse given by the equations $x = a\cos\phi$, $y = \frac{a}{n}\sin\phi$, i.e. an ellipse of semi-axes a and $\frac{a}{n}$.

The area of the loop is

$$\int_0^{\frac{\pi}{2n}} r^2 d\theta = 4a^2 \int_0^{\frac{\pi}{2n}} \cos^2 n\theta\, d\theta$$
$$= \frac{4a^2}{n}\int_0^{\frac{\pi}{2}} \cos^2\phi\, d\phi$$
$$= \frac{\pi a^2}{n},$$

which is the area of the same ellipse.

XCVIII.

1. Let $ABCD$ be the quadrilateral, and let AB, CD touch another circle in P and Q. Join PQ, and let it meet AD and BC produced in R and S respectively. Then evidently

$$B\hat{P}S = D\hat{Q}R \text{ and } P\hat{B}S = Q\hat{D}R.$$

Hence the triangles BPS, QDR are similar; therefore $\frac{BP}{BS} = \frac{DQ}{DR}$. In like manner the triangles APR, CQS are similar, and

$$\frac{AP}{AR} = \frac{CQ}{CS}.$$

Further, in the triangles ARP, BSP the angles ARP, BSP are equal, and the angles APR, BPS supplementary. Hence

$$\frac{AP}{AR} = \frac{BP}{BS}.$$

Again, since $D\hat{R}Q = C\hat{S}Q$, a circle can be drawn to touch AD, BC at R and S, and, from what has been proved, it appears that the ratios of the tangents drawn from the four points A, B, C, D to this circle, and to the circle touching AB, CD, are equal. Hence A, B, C, D must lie on a circle coaxal with these two circles. In other words, the circle RS is coaxal with the circles $ABCD$ and PQ.

2. Let the circle and conic touch at B, and let A be their remaining intersection. Let the other common tangent touch the circle at E and the conic at F, and let the common tangents meet in T. Let AB produced meet EF in O. Then

$$OE^2 : OF^2 = OA \,.\, OB : OF^2 = d^2 : d'^2,$$

where d, d' are the diameters of the conic parallel to AB and EF. But since TB, AB make equal angles with the axis of the conic, the diameters parallel to them are equal; therefore

$$TB^2 : TF^2 = d^2 : d'^2.$$

Hence $\quad OE : OF = TB : TF = TE : TF,$

i.e. T and O divide EF harmonically.

3. Since a cyclic interchange of x, y, z does not alter the given expression, the factors, if they exist, must be of the form

$$ax + by + cz, \quad cx + ay + bz, \quad bx + cy + az.$$

(These factors will be distinct provided $a^3 \neq b^3 \neq c^3$, involving, as appears from the equations below, $\beta^3 \neq \gamma^3$.)

Taking then

$$a \,.\, \Sigma x^3 + \beta \,.\, \Sigma x^2 y + \gamma \,.\, \Sigma xy^2 + \delta \,.\, xyz$$
$$\equiv (ax + by + cz)(cx + ay + bz)(bx + cy + az) \ldots \text{(i)},$$

we have

$$abc = \alpha, \quad \Sigma a^2 b = \beta, \quad \Sigma ab^2 = \gamma, \quad \Sigma a^3 + 3abc = \delta;$$

involving $\beta^3 - \gamma^3 = \Pi (a^3 - b^3)$.

From these

$$\beta\gamma = \Sigma a^3 b^3 + \alpha\delta, \quad \beta^2 = \Sigma a^4 b^2 + 2\alpha\gamma, \quad \gamma^2 = \Sigma a^2 b^4 + 2\alpha\beta.$$

But, putting $x = a^2$, etc. in (i), we have

$$\alpha \,.\, \Sigma a^6 + \beta \,.\, \Sigma a^4 b^2 + \gamma \,.\, \Sigma a^2 b^4 + \delta \,.\, a^2 b^2 c^2 = \Sigma a^3 \,.\, \Sigma ab^2 \,.\, \Sigma a^2 b,$$

i.e. $\alpha\left[(\delta - 3\alpha)^2 - 2(\beta\gamma - \alpha\delta)\right] + \beta(\beta^2 - 2\alpha\gamma) + \gamma(\gamma^2 - 2\alpha\beta) + \delta\alpha^2$
$$= (\delta - 3\alpha)\beta\gamma,$$

leading to the given condition.

4. The series

$$\frac{x^3}{1\,.\,3} + \frac{x^5}{3\,.\,5} + \frac{x^7}{5\,.\,7} + \ldots \qquad (x < 1)$$

$$= \tfrac{1}{2}\left[(1 - \tfrac{1}{3})x^3 + (\tfrac{1}{3} - \tfrac{1}{5})x^5 + (\tfrac{1}{5} - \tfrac{1}{7})x^7 + \ldots\right]$$

$$= \tfrac{1}{2}(x^3 + \tfrac{1}{3}x^5 + \tfrac{1}{5}x^7 + \ldots) - \tfrac{1}{2}(\tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots)$$

$$= \tfrac{1}{4}x^2 \log\frac{1+x}{1-x} - \tfrac{1}{2}\left(\tfrac{1}{2}\log\frac{1+x}{1-x} - x\right).$$

Putting $x = \frac{1}{2}$, this becomes

$$\tfrac{1}{16}\log 3 - \tfrac{1}{2}(\tfrac{1}{2}\log 3 - \tfrac{1}{2}) = \tfrac{1}{4} - \tfrac{3}{16}\log 3.$$

5. Let

$$\frac{1}{a+bxy+cy^2} \equiv \frac{1}{a(1-\alpha y)(1-\beta y)} = \frac{1}{a(\alpha-\beta)}\left[\frac{\alpha}{1-\alpha y} - \frac{\beta}{1-\beta y}\right].$$

Then, expanding $(1-\alpha y)^{-1}$ and $(1-\beta y)^{-1}$, we see that

$$P_n = \frac{\alpha^{n+1}-\beta^{n+1}}{a(\alpha-\beta)}.$$

Hence, if $P_n = 0$, we must have

$$\left(\frac{\alpha}{\beta}\right)^{n+1} = 1 \qquad (\alpha \neq \beta)$$

$$= \cos 2r\pi + i \sin 2r\pi,$$

r being any integer.

$$\therefore \frac{\alpha}{\beta} = \cos\frac{2r\pi}{n+1} + i\sin\frac{2r\pi}{n+1} \qquad (r \neq 0)$$

$$= (\cos a + i \sin a)^2, \quad \left(\text{where } a = \frac{r\pi}{n+1}\right)$$

$$= \frac{\cos a + i \sin a}{\cos a - i \sin a},$$

$$\therefore \frac{\alpha+\beta}{\alpha-\beta} = \frac{1}{i}\cot a,$$

$$\therefore \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2 = -\cot^2 a, \text{ whence } \frac{(\alpha+\beta)^2}{4\alpha\beta} = \cos^2 a.$$

But $$\alpha+\beta = -\frac{bx}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Hence $$\frac{b^2x^2}{4ac} = \cos^2 a,$$

$$\text{i.e. } x = \pm\frac{2\sqrt{ac}}{b}\cos\frac{r\pi}{n+1}, \qquad (r = 1, 2, \ldots).$$

Thus, since a and c have the same sign, these values of x are all real. If n is even, the roots are equal and opposite in pairs, the values of r ranging from 1 to $\frac{n}{2}$. If n is odd, one root $\left(\text{for } r = \frac{n+1}{2}\right)$ is zero, and the other roots are equal and opposite in pairs, the values of r ranging from 1 to $\frac{n-1}{2}$.

6. The rational form of the first equation is

$$2\Sigma(b^2-b'^2)(c^2-c'^2)-\Sigma(a^2-a'^2)^2=0,$$

and since $$2\Sigma b^2c^2-\Sigma a^4=16\Delta^2,$$

this is $$8\Delta^2+8\Delta'^2-\Sigma(b^2c'^2+b'^2c^2-a^2a'^2)=0 \quad\ldots\ldots\ldots\text{(i)}.$$

But the rational form of the second equation is

$$2\Sigma(b^2\Delta'^2-b'^2\Delta^2)(c^2\Delta'^2-c'^2\Delta^2)-\Sigma(a^2\Delta'^2-a'^2\Delta^2)^2=0,$$

i.e. $16\Delta^2.\Delta'^4+16\Delta'^2.\Delta^4-2\Delta^2\Delta'^2.\Sigma(b^2c'^2+b'^2c^2-a^2a'^2)=0,$

and this is the same as (i). Hence the existence of one of the first irrational relations involves the existence of one of the second.

7. If $$f(x)=x+\frac{x^4}{4}+\frac{x^7}{7}+\ldots$$

and the given series be denoted by S, then

$$S=\tfrac{1}{2}\{f(x)-f(-x)\}.$$

Now from the logarithmic series, we get

$$f(x)=-\tfrac{1}{3}[\log(1-x)+\omega^2\log(1-\omega x)+\omega\log(1-\omega^2x)].$$

Hence $$S=\frac{1}{6}\left\{\log\frac{1+x}{1-x}+\omega^2\log\frac{1+\omega x}{1-\omega x}+\omega\log\frac{1+\omega^2x}{1-\omega^2x}\right\},$$

and since $$\omega=\frac{-1+\sqrt{3}\,i}{2},\quad \omega^2=\frac{-1-\sqrt{3}\,i}{2},$$

the sum of the last two terms in the bracket is

$$-\frac{1}{2}\log\frac{(1+\omega x)(1+\omega^2x)}{(1-\omega x)(1-\omega^2x)}-\frac{\sqrt{3}\,i}{2}\log\frac{(1+\omega x)(1-\omega^2x)}{(1-\omega x)(1+\omega^2x)}$$

$$=-\frac{1}{2}\log\frac{1-x+x^2}{1+x+x^2}-\frac{\sqrt{3}\,i}{2}\log\frac{1-x^2+\sqrt{3}\,ix}{1-x^2-\sqrt{3}\,ix},$$

and the value of the latter logarithm is

$$2i\tan^{-1}\frac{\sqrt{3}x}{1-x^2}.$$

Hence $$S=\frac{1}{6}\left(\log\frac{1+x}{1-x}+\frac{1}{2}\log\frac{1+x+x^2}{1-x+x^2}+\sqrt{3}\tan^{-1}\frac{\sqrt{3}x}{1-x^2}\right).$$

8. The tangent at α is $\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$, or, in Cartesians,
$$(e + \cos\alpha)\,x + y\sin\alpha = l,$$
and the chord of curvature is therefore of the form
$$(e + \cos\alpha)\,x - y\sin\alpha = k,$$
or
$$\frac{k}{r} = e\cos\theta + \cos(\theta + \alpha),$$
and since this passes through α, we have
$$k\,.\,\frac{1 + e\cos\alpha}{l} = e\cos\alpha + \cos 2\alpha,$$
$$\text{i.e. } k = l\,.\,\frac{e\cos\alpha + \cos 2\alpha}{1 + e\cos\alpha}.$$
The circle of curvature is now of the form
$$(1 - e^2)\,x^2 + y^2 + 2elx - l^2$$
$$+ \lambda\left[(e + \cos\alpha)\,x + y\sin\alpha - l\right]\left[(e + \cos\alpha)\,x - y\sin\alpha - k\right] = 0 \ldots\text{(i)},$$
and the condition for a circle is
$$1 - e^2 + \lambda\,(e + \cos\alpha)^2 = 1 - \lambda\sin^2\alpha\,;$$
$$\therefore\ \lambda = \frac{e^2}{e^2 + 2e\cos\alpha + 1}.$$

The equation (i) in polars is
$$1 - \left(\frac{l}{r} - e\cos\theta\right)^2 + \lambda\left[e\cos\theta + \cos(\theta - \alpha) - \frac{l}{r}\right]$$
$$\times\left[e\cos\theta + \cos(\theta + \alpha) - \frac{k}{r}\right] = 0,$$
and substituting for λ and k, we obtain the result as given.

9. The equation to a rectangular hyperbola of the system is
$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0, \text{ with the condition } \Sigma u = 0 \ldots\ldots\text{(i)}.$$
The tangential equation is $\Sigma\frac{l^2}{u} = 0$, so that if (f, g, h), (f', g', h') are the foci, we must have
$$\frac{l^2}{u} + \frac{m^2}{v} + \frac{n^2}{w} \equiv (lf + mg + nh)(lf' + mg' + nh') - \lambda\,(\Sigma l^2 - 2\Sigma mn\cos A).$$

Hence $\frac{1}{u} = ff' - \lambda$, etc., so that (i) becomes

$$\Sigma\,(ff' - \lambda)^{-1} = 0 \quad\text{.......................(ii).}$$

Further $$gh' + g'h + 2\lambda \cos A = 0,$$

$$\text{i.e. } \frac{g'}{g} + \frac{h'}{h} = -\frac{2\lambda \cos A}{gh}, \text{ etc.,}$$

whence $$\frac{f'}{f} = -\frac{\lambda}{fgh}(-f \cos A + g \cos B + h \cos C), \text{ etc.};$$

$$\therefore\ ff' - \lambda = -\frac{\lambda}{fgh}\{f^2(-f \cos A + g \cos B + h \cos C) + fgh\}.$$

Hence, from (ii), replacing f, g, h by α, β, γ, the required locus is

$$\Sigma\,\{\alpha^2(-\alpha \cos A + \beta \cos B + \gamma \cos C) + \alpha\beta\gamma\}^{-1} = 0.$$

10. If the masses are at P and Q, and if C is the vertex, and the tangents at P, Q make angles ψ, ψ' with the horizontal, we have

$$T = mg \sin \psi = m'g \sin \psi'.$$

Also $$\text{arc } PQ = CP + CQ,$$

$$\text{i.e. } l = 4a \sin \psi + 4a \sin \psi';$$

$$\therefore\ \frac{4a \sin \psi}{m'} = \frac{4a \sin \psi'}{m} = \frac{l}{m + m'},$$

$$\text{i.e. } 4a \sin \psi = \frac{m'}{m + m'} \,.\, l.$$

11. Let A be the upper vertex, and when the particle is at P let the velocity be v. Suppose ϕ is the eccentric angle at P, and let the normal at P make an angle θ with the vertical. Then if R is the pressure at P, we have

$$\frac{mv^2}{\rho} = R + mg \cos \theta, \quad v^2 = u^2 - 2g(a + a \cos \phi);$$

$$\therefore\ R = m \,.\, \frac{u^2 - 2ga(1 + \cos \phi)}{\rho} - mg \cos \theta.$$

Now $$\rho = \frac{b'^3}{ab} \qquad \cos \theta = \frac{b \cos \phi}{b'}$$

$$\therefore \quad \rho \cos \theta = \frac{b'^2}{a} \cos \phi = a\,(1 - e^2 \cos^2 \phi) \cos \phi\,;$$

$$\therefore \quad R = \frac{m}{\rho}\,[u^2 - ga\,(2 + 3 \cos \phi - e^2 \cos^3 \phi)],$$

and R will be positive throughout if it is positive at the highest point, i.e. when $\phi = 0$, the condition for which is that

$$u^2 - ga\,(5 - e^2)$$

should be positive.

12. Let ϕ be the angle between the radius vector and the tangent, so that $\sin \phi = \dfrac{p}{r}$. Then the integral in question is

$$\int \frac{\sin \phi\, ds}{r^2}.$$

But $\sin \phi$ is also equal to $r\dfrac{d\theta}{ds}$, so that the integral is simply $\displaystyle\int \frac{d\theta}{r}$ taken round the ellipse, and since $\dfrac{l}{r} = 1 + e \cos \theta$, the integral is

$$\frac{1}{l} \int_0^{2\pi} (1 + e \cos \theta)\, d\theta = \frac{2\pi}{l}.$$

XCIX.

1. Suppose the four lines ABC, $AB'C'$, $BC'A'$, $CA'B'$ all touched by a circle whose centre is O, and invert with respect to this circle. Let a be the inverse of A, etc. Then the inverses of these lines, viz. the circles

$$Oabc, \quad Oab'c', \quad Obc'a', \quad Oca'b',$$

are four equal circles, and it is evident that in this case the circles

$$a'b'c', \quad a'bc, \quad b'ca, \quad c'ab \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

are also equal.

Now it is easy to shew that the circles

$$A'B'C', \quad A'BC, \quad B'CA, \quad C'AB \quad \ldots\ldots\ldots\ldots\text{(ii)}$$

meet in a point, say P, and therefore their inverses must also meet in a point. Hence the circles (i) are four equal circles meeting in a point, and therefore they are all touched by another circle having its centre at that point. Hence the circles (ii) are all touched by a fifth circle, and also by a sixth, viz. the point-circle P. Hence six of their centres of similitude lie on the radical axis of this fifth circle and the point P.

2. Reciprocating from the common focus, we get three circles touching in pairs at L, M, N, and the directrices become their centres A, B, C. Hence for the three points in question we get the lines AL, BM, CN. But if a, b, c be the radii, then $BL = b$, $CL = c$, etc. Hence

$$\frac{BL}{LC} = \frac{b}{c}, \quad \frac{CM}{MA} = \frac{c}{a}, \quad \frac{AN}{NB} = \frac{a}{b},$$

$$\text{i.e. } \frac{BL}{LC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1;$$

therefore the lines AL, BM, CN are concurrent. Hence the three points of which they are the reciprocals are collinear.

3. The expression may be written in the form

$$I = \frac{1}{a}\left[(ax + hy + gz)^2 + (ab - h^2)\,y^2 + 2\,(af - gh)\,yz + (ac - g^2)\,z^2\right],$$

and therefore if a is positive, I will be always positive, provided

$$(ab - h^2)\,y^2 + 2\,(af - gh)\,yz + (ac - g^2)\,z^2 \quad \text{.........(i)}$$

is positive.

Now

$$Py^2 + 2Qyz + Rz^2 \equiv \frac{1}{P}\{(Py + Qz)^2 + (PR - Q^2)\,z^2\},$$

and is therefore positive, provided P and $PR - Q^2$ are positive.

Hence the expression (i) will be always positive provided

$$ab - h^2 \;\text{ and }\; (ab - h^2)(ac - g^2) - (af - gh)^2$$

are positive. But the latter expression is

$$a\,(abc + 2fgh - af^2 - bg^2 - ch^2).$$

Hence the conditions are equivalent to those stated.

4. Let $\frac{1}{3}(n^2-1)=p^2$, so that $n^2-3p^2=1$. All solutions of this equation are derivable from the convergents to $\sqrt{3}$, and a general expression can be given for all of them in terms of the first one. In this case the first solution is obviously $n=2$, $p=1$. But if we put

$$x+y\sqrt{3}=(2+\sqrt{3})^m, \quad x-y\sqrt{3}=(2-\sqrt{3})^m,$$

then $x^2-3y^2=1$, so that x, y is also a solution. We have thus

$$2x=(2+\sqrt{3})^m+(2-\sqrt{3})^m,$$

i.e. n is included in the formula

$$\tfrac{1}{2}[(2+\sqrt{3})^m+(2-\sqrt{3})^m]=2^m+3\,.\,{}^mC_2\,.\,2^{m-2}+3^2\,.\,{}^mC_4\,.\,2^{m-4}+\ldots.$$

5. Under the given conditions, the equation

$$(ax^3+3bx^2+3cx+d)+\lambda(a'x^3+\ldots+d')=0 \quad \ldots\ldots\text{(i)}$$

can only have equal roots for two values of λ, for the roots must be either α, α, β or α, β, β, where α and β are the common roots of the given equations. Now the condition that the equation (i) has equal roots may be expressed by equating the determinant in question to zero. This can easily be seen as follows:—If the equation

$$Ax^3+3Bx^2+3Cx+D=0$$

has equal roots, it has a common root with

$$Ax^2+2Bx+C=0,$$

and since

$$Ax^3+3Bx^2+3Cx+D \equiv x(Ax^2+2Bx+C)+Bx^2+2Cx+D,$$

this implies that the quadratics

$$Ax^2+2Bx+C=0, \quad Bx^2+2Cx+D=0$$

have a common root. If x be this root, then

$$\begin{aligned} Ax^3+2Bx^2+\ \ Cx \quad\quad &=0,\\ Ax^2+2Bx+C&=0,\\ Bx^3+2Cx^2+\ \ Dx \quad\quad &=0,\\ Bx^2+2Cx+D&=0.\end{aligned}$$

Hence, eliminating x^3, x^2, x, 1, we get

$$\begin{vmatrix} A, & 2B, & C, & 0 \\ 0, & A, & 2B, & C \\ B, & 2C, & D, & 0 \\ 0, & B, & 2C, & D \end{vmatrix} = 0,$$

leading, in the case of (i), to the equation given.

This equation is generally of the fourth degree in λ, but since in this case it only gives rise to two distinct values of λ, it must have two pairs of equal roots.

6. By the given equations, the equation

$$x \sin \theta + y \cos \theta + z \sin^2 \theta = 1$$

is satisfied by α, β and γ.

Putting $\tan \dfrac{\theta}{2} = t$, the equation becomes

$$x \cdot \frac{2t}{1+t^2} + y \cdot \frac{1-t^2}{1+t^2} + z \cdot \frac{4t^2}{(1+t^2)^2} = 1,$$

$$\text{i.e.}\quad (1+y)\, t^4 - 2xt^3 + (2-4z)\, t^2 - 2xt + 1 - y = 0 \ldots\ldots\text{(i)}.$$

If the fourth root is $\tan \frac{1}{2}\delta$, then

$$\Sigma \tan \tfrac{1}{2}\alpha = \Sigma \tan \tfrac{1}{2}\beta \tan \tfrac{1}{2}\gamma \tan \tfrac{1}{2}\delta = \frac{2x}{1+y}.$$

Hence $\quad \tan \dfrac{\alpha+\beta+\gamma+\delta}{2} = 0; \qquad \therefore\ \dfrac{\delta}{2} = n\pi - \dfrac{\alpha+\beta+\gamma}{2}.$

We have now, for all values of t, the left side of (i) equal to

$$(1+y)\,(t - \tan \tfrac{1}{2}\alpha)\,(t - \tan \tfrac{1}{2}\beta)\,(t - \tan \tfrac{1}{2}\gamma)\,(t - \tan \tfrac{1}{2}\delta).$$

Dividing by $(1+t^2)^2$, and restoring the value of t, this leads to

$$1 - (x \sin \theta + y \cos \theta + z \sin^2 \theta) = (1+y) \cdot \frac{\Pi \sin \frac{1}{2}(\theta - \alpha)}{\Pi \cos \frac{1}{2}\alpha}.$$

But from (i)

$$\frac{4z}{1+y} = 1 + \Pi \tan \tfrac{1}{2}\alpha - \Sigma \tan \tfrac{1}{2}\alpha \tan \tfrac{1}{2}\beta$$

$$= \frac{\Pi \cos \frac{1}{2}\alpha + \Pi \sin \frac{1}{2}\alpha - \Sigma \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \cos \frac{1}{2}\gamma \cos \frac{1}{2}\delta}{\Pi \cos \frac{1}{2}\alpha}.$$

The numerator is equal to

$$\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}(\beta+\gamma+\delta)-\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}(\beta+\gamma+\delta)$$
$$=\cos\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)=(-1)^n.$$

Hence $$\frac{4z}{1+y}=\frac{(-1)^n}{\Pi\cos\frac{1}{2}\alpha};$$

$\therefore\quad 1-(x\sin\theta+y\cos\theta+z\sin^2\theta)=(-1)^n\,.\,4z\,.\,\Pi\sin\tfrac{1}{2}(\theta-\alpha),$
and substituting for δ, the result follows.

7. We have

$$\cosh y-\cos\alpha=2\sin\tfrac{1}{2}(\alpha+iy)\sin\tfrac{1}{2}(\alpha-iy)$$
$$=\tfrac{1}{2}(\alpha^2+y^2)\,.\,\prod_{n=1}\left\{1-\frac{(\alpha+iy)^2}{4n^2\pi^2}\right\}\left\{1-\frac{(\alpha-iy)^2}{4n^2\pi^2}\right\}$$
$$=\tfrac{1}{2}(\alpha^2+y^2)\,.\,\Pi\left\{1-\frac{\alpha^2-y^2}{2n^2\pi^2}+\frac{(\alpha^2+y^2)^2}{16n^4\pi^4}\right\}.$$

If we now put $y=\pi$, $\alpha=\pi\sqrt{3}$, we get

$$\cosh\pi-\cos\pi\sqrt{3}=2\pi^2\,.\,\prod_{n=1}\left(1-\frac{1}{n^2}+\frac{1}{n^4}\right),$$

and putting $y=\alpha=\pi\sqrt{2}$, we get

$$\cosh\pi\sqrt{2}-\cos\pi\sqrt{2}=2\pi^2\,.\,\prod_{n=1}\left(1+\frac{1}{n^4}\right),$$

and since $$\frac{1-\dfrac{1}{n^2}+\dfrac{1}{n^4}}{1+\dfrac{1}{n^4}}=\frac{n^4-n^2+1}{n^4+1}=1-\frac{1}{n^2+n^{-2}},$$

these lead to the result given.

8. Let (X, Y) be the pole of $lx+my=1$, so that

$$aX+hY=l,\quad hX+bY=m\ \ldots\ldots\ldots\ldots\ldots(\text{i}).$$

The line $\dfrac{x-X}{\cos\phi}=\dfrac{y-Y}{\sin\phi}=r$ meets the conic where

$$a(r\cos\phi+X)^2+2h(r\cos\phi+X)(r\sin\phi+Y)+b(r\sin\phi+Y)^2=1,$$

or $$r^2(a\cos^2\phi+2h\cos\phi\sin\phi+b\sin^2\phi)$$
$$+2r(l\cos\phi+m\sin\phi)+S'=0\ldots(\text{ii}),$$

where $$\begin{aligned} S' &= aX^2 + 2hXY + bY^2 - 1 \\ &= X(aX + hY) + Y(hX + bY) - 1 \\ &= lX + mY - 1 \\ &= l \cdot \frac{bl - hm}{ab - h^2} + m \cdot \frac{am - hl}{ab - h^2} - 1, \quad \text{as given.} \end{aligned}$$

If the line is a tangent, we have, from (ii),

$$(l\cos\phi + m\sin\phi)^2 = S'(a\cos^2\phi + 2h\cos\phi\sin\phi + b\sin^2\phi),$$

i.e. $$(l + m\tan\phi)^2 = S'(a + 2h\tan\phi + b\tan^2\phi) \quad \text{.........(iii).}$$

Now, if θ is the required angle, then

$$\tan\phi = \frac{-\dfrac{l}{m} + \tan\theta}{1 + \dfrac{l}{m}\tan\theta} = \frac{m\tan\theta - l}{l\tan\theta + m},$$

and $$l + m\tan\phi = \frac{(l^2 + m^2)\tan\theta}{l\tan\theta + m}.$$

Hence, from (iii), we get

$$(l^2 + m^2)^2\tan^2\theta = S'\left[a(l\tan\theta + m)^2 + 2h(m\tan\theta - l)(l\tan\theta + m) + b(m\tan\theta - l)^2\right],$$

reducing to the equation given.

9. Referring to the common self-conjugate triangle, let the conics be

$$\Sigma l\alpha^2 = 0 \ (S), \quad \Sigma l'\alpha^2 = 0 \ (S').$$

Then the tangent at $(\alpha', \beta', \gamma')$ to S is $\Sigma l\alpha\alpha' = 0$, and the polar of this point for S' is $\Sigma l'\alpha\alpha' = 0$.

These lines intersect in the point

$$\frac{\alpha}{(mn' - m'n)\beta'\gamma'} = \ldots = \ldots,$$

whence $$\frac{\alpha'}{(mn' - m'n)\beta\gamma} = \ldots = \ldots .$$

Hence, since $(\alpha', \beta', \gamma')$ is on S, the locus of (α, β, γ) is

$$\Sigma l(mn' - m'n)^2\beta^2\gamma^2 = 0,$$

a quartic curve passing through the points of reference.

Also the equation $\Sigma L\beta^2\gamma^2 = 0$ may be written

$$L\beta^2\gamma^2 + \alpha^2(M\gamma^2 + N\beta^2) = 0,$$

shewing that the lines $N\beta^2 + M\gamma^2 = 0$ are tangents at the point of reference A, i.e. the locus has a double point at A, and similarly for B and C.

10. Let G be the combined C. of G., A the point of contact, O the centre of the hemisphere. Then for stability

$$AG < \frac{Rr}{R+r}.$$

Now let L be the C. of G. of the hemisphere (so that $OL = \frac{1}{2}r$), and M the position of the particle on the rim, so that G is on the line LM. Then AG is a maximum when OG is a minimum, i.e. when OG is perpendicular to LM.

Now

$$LM^2 = r^2 + \left(\frac{1}{2}r\right)^2; \quad \therefore \; LM = \frac{\sqrt{5}}{2}r; \quad \therefore \; \sin OLM = \frac{2}{\sqrt{5}};$$

therefore in this case $\qquad OG = \dfrac{r}{\sqrt{5}}.$

The condition is therefore

$$r - \frac{r}{\sqrt{5}} < \frac{Rr}{R+r},$$

$$\text{i.e. } \frac{R+r}{R} < \frac{\sqrt{5}}{\sqrt{5}-1},$$

$$\text{i.e. } \frac{R}{r} > \sqrt{5} - 1.$$

11. Let V be the velocity of the cylinder, v that of the particle relative to the curve.

Then the horizontal velocity of the particle in space is $v\cos\phi - V$ and the vertical velocity $v\sin\phi$.

Hence the energy equation is

$$\tfrac{1}{2}m\left[(v\cos\phi - V)^2 + v^2\sin^2\phi\right] + \tfrac{1}{2}MV^2 = 2r\sin^2\phi \,.\, mg$$

(where r is the radius of the generating circle), i.e.

$$mv^2 - 2mvV\cos\phi + (M+m)V^2 = 4mgr\sin^2\phi \;\ldots\ldots\text{(i)},$$

and the equation of momentum is

$$m(v\cos\phi - V) = MV, \quad \text{i.e. } V = \frac{mv\cos\phi}{M+m} \quad \text{.........(ii).}$$

Eliminating V, we get

$$v^2(M + m\sin^2\phi) = 4gr(M+m)\sin^2\phi.$$

But the particle leaves the curve when $\dfrac{v^2}{\rho} = g\cos\phi$, and since $\rho = 4r\cos\phi$, this gives

$$4gr\cos^2\phi\,(M + m\sin^2\phi) = 4gr(M+m)\sin^2\phi;$$

$$\therefore \quad M + m\sin^2\phi = (M+m)\tan^2\phi,$$

$$\text{i.e.} \quad (M+m)\tan^2\phi + M = (M+m)\tan^2\phi\,(1+\tan^2\phi),$$

or

$$\tan^4\phi = \frac{M}{M+m}.$$

12. Taking the tangent and normal at the point of contact as axes, the co-ordinates of a point near the origin are expressible in the form

$$x = s - \frac{s^3}{6\rho^2} + \dots, \quad y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\cdot\rho' + \dots \quad \text{......(i),}$$

and the equation to the rectangular hyperbola takes the form

$$ax^2 + 2hxy - ay^2 = 2y \text{.....................(ii).}$$

If there is four-point contact, then, on substituting the co-ordinates (i) in (ii), the terms in s^2 and s^3 must disappear. We thus find

$$a = \frac{1}{\rho}, \quad h = -\frac{\rho'}{3\rho} \quad \text{.....................(iii).}$$

Now when (ii) is reduced to the centre as origin, it becomes

$$ax^2 + 2hxy - ay^2 + \frac{a}{a^2+h^2} = 0.$$

If, on turning to the principal axes, this takes the form

$$\alpha(x^2 - y^2) + \frac{a}{a^2+h^2} = 0,$$

then $\alpha^2 = a^2 + h^2$, and the square of the semi-axis is

$$\frac{a}{\alpha(a^2+h^2)} = \frac{a}{(a^2+h^2)^{\frac{3}{2}}} = \frac{\rho^2}{(1+\frac{1}{9}\rho'^2)^{\frac{3}{2}}}$$

from (iii).

C.

1. Let OA', OB' meet BC, CA in α, β respectively, and let $\alpha\beta$ meet AB in γ'. Then the pairs $A\alpha$, $B\beta$, $C\gamma'$ are opposite vertices of a complete quadrilateral, and therefore $O(A\alpha, B\beta, C\gamma')$ is an involution.

But if OC' meets AB in γ, then by the question

$$O(A\alpha,\ B\beta,\ C\gamma) \text{ is an involution.}$$

Hence $C\gamma$, $C\gamma'$ must coincide, i.e. γ and γ' must coincide; therefore α, β, γ are collinear, and similarly for the other set.

2. Through P and Q draw lines PAB, QAC very near to PO and QO, forming the quadrilateral $OABC$. Then the conics touching the sides of this quadrilateral have their director circles coaxal, the common radical axis being that of the circles on OA, BC and PQ as diameters. Now in the limit these conics become conics touching OP and OQ at P and Q, while the circles on OA and BC as diameters become point-circles at O. Hence the radical axis of the system of director circles becomes the radical axis of O and the circle on PQ as diameter.

3. We have

$$\frac{x_1+x_2+x_3}{3} \not< \sqrt[3]{x_1x_2x_3},$$

$$\text{i.e.}\quad (x_1+x_2+x_3)^3 \not< 27\,x_1x_2x_3,$$

or

$$\Sigma x_1^3 + 3\Sigma x_1^2x_2 \not< 21\,x_1x_2x_3.$$

Writing down all the possible inequalities of this type for every set of three suffixes, and adding the results, x_1^3 will occur in ${}^{n-1}C_2$ inequalities, and $x_1^2x_2$ in ${}^{n-2}C_1$ inequalities.

We therefore have

$$ {}^{n-1}C_2\,.\,\Sigma x_1^3 + 3\,.\,{}^{n-2}C_1\,.\,\Sigma x_1^2x_2 \not< 21\Sigma x_1x_2x_3.$$

Now

$$\Sigma x_1\,.\,\Sigma x_1^2 = \Sigma x_1^3 + \Sigma x_1^2x_2.$$

Hence the expression on the left is

$$\frac{(n-1)(n-2)}{2}\,.\,\Sigma x_1^3 + 3(n-2)(\Sigma x_1\,.\,\Sigma x_1^2 - \Sigma x_1^3)$$
$$= 3(n-2)\,.\,\Sigma x_1\,.\,\Sigma x_1^2 + \tfrac{1}{2}(n-2)(n-7)\,.\,\Sigma x_1^3.$$

4. Let $$1, \alpha, \beta, \gamma, \ldots N-1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$
be the $\phi(N)$ numbers less than N, and prime to N.

Then, if we divide the product
$$P = 1 \cdot \alpha \cdot \beta \cdot \gamma \ldots (N-1)$$
by each of these $\phi(N)$ numbers, we shall obtain $\phi(N)$ quotients, all different and prime to N. Hence, if we divide each of these quotients by N, we shall obtain for remainders the $\phi(N)$ numbers of the sequence (i) in some order. To see this, we have only to shew (1) that the remainders are all prime to N; (2) that they are all different.

Now (1) is evident, since the dividends are all prime to N. To prove (2), we remark that the difference between two dividends is of the form
$$[1 \cdot \gamma \cdot \delta \ldots (N-1)](\beta-\alpha),$$
and if N divides this product, it must divide $\beta-\alpha$ (since the other factor is prime to N), and this is impossible.

We have now therefore
$$\frac{P}{1} \cdot \frac{P}{\alpha} \cdot \frac{P}{\beta} \cdot \frac{P}{\gamma} \cdots \frac{P}{N-1} = M(N) + P,$$
$$\text{i.e. } \frac{P^{\phi(N)}}{P} = M(N) + P, \quad \text{i.e. } P^{\phi(N)} = M(N) + P^2.$$

But by Euler's Theorem, $P^{\phi(N)} - 1 = M(N)$. Hence it follows that
$$P^2 - 1 = M(N).$$

5. Suppose the roots are $\alpha \pm i\alpha'$, $\beta \pm i\beta'$, $\gamma \pm i\gamma'$, the common modulus being p. Then the equation must take the form
$$(x^2 - 2\alpha x + p^2)(x^2 - 2\beta x + p^2)(x^2 - 2\gamma x + p^2) = 0.$$

Now suppose 2α, 2β, 2γ are the roots of
$$y^3 + \lambda y^2 + \mu y + \nu = 0,$$
so that, for any value of y,
$$y^3 + \lambda y^2 + \mu y + \nu \equiv (y - 2\alpha)(y - 2\beta)(y - 2\gamma).$$

In this identity put $y = \dfrac{x^2 + p^2}{x}$, and we get
$$(x^2 + p^2)^3 + \lambda x(x^2 + p^2)^2 + \mu x^2(x^2 + p^2) + \nu x^3$$
$$\equiv (x^2 + p^2 - 2\alpha x)(x^2 + p^2 - 2\beta x)(x^2 + p^2 - 2\gamma x).$$

Hence the equation obtained by equating the left-hand side to zero must coincide with the original sextic. We must therefore have

$$\lambda = a\ldots\text{(i)},\quad 3p^2+\mu=b\ldots\text{(ii)},\quad 2p^2\lambda+\nu=c\ldots\text{(iii)},$$
$$3p^4+\mu p^2=d\ldots\text{(iv)},\quad p^4\lambda=e\ldots\text{(v)},\quad p^6=f\ldots\text{(vi)}.$$

From (i), (v) and (vi), we get $e^3=a^3f^2$, and from (iv), (v) and (ii),

$$de=\lambda p^6(3p^2+\mu)=abf.$$

Also $\quad p^2=\dfrac{af}{e};\quad \therefore\ \mu=b-\dfrac{3af}{e},\quad \nu=c-\dfrac{2a^2f}{e}.$

6. Let x, y, z be the distances of any point P from the angular points of ABC, and let $B\hat{P}C=\alpha$, $C\hat{P}A=\beta$, $A\hat{P}B=\gamma$, so that $\alpha+\beta+\gamma=2\pi$, and therefore

$$1-\Sigma\cos^2\alpha+2\cos\alpha\cos\beta\cos\gamma=0.$$

But $\quad \cos\alpha=\dfrac{y^2+z^2-a^2}{2yz}$, etc.

Hence, substituting, we get

$$4x^2y^2z^2-\Sigma x^2(y^2+z^2-a^2)^2+\Pi(y^2+z^2-a^2)=0,$$

which may be reduced to the form

$$\Sigma a^2(x^2-y^2)(x^2-z^2)+\Sigma a^2x^2(b^2+c^2-a^2)=a^2b^2c^2.$$

If we now write $x^2=\rho^2+r_1{}^2$, etc. (P being now the centre of the orthogonal circle), this becomes

$$-\Sigma a^2(r_1{}^2-r_2{}^2)(r_1{}^2-r_3{}^2)+16\Delta^2\rho^2+\Sigma a^2r_1{}^2(b^2+c^2-a^2)=a^2b^2c^2,$$

which is equivalent to the equation in question.

7. Let the angle $P_rON=\alpha_r$. Then

$$(2r-1)^2=\frac{NP_r}{NP_1}=\frac{\cot\alpha_1}{\cot\alpha_r}.$$

Hence $\quad \prod_1^\infty \operatorname{cosec}^2\alpha_r=\prod_1^\infty\left\{1+\dfrac{\cot^2\alpha_1}{(2r-1)^4}\right\}\ldots\ldots\ldots\ldots\ldots\text{(i)}.$

Now $\quad \cosh u+\cos u=2\cos\dfrac{u+iu}{2}\cos\dfrac{u-iu}{2},$

and since $\quad \cos\theta=\prod_1^\infty\left\{1-\dfrac{2^2\theta^2}{(2r-1)^2\pi^2}\right\},$

and $(u+iu)^2=2iu^2$, this gives

$$\cosh u+\cos u=2\prod_1^\infty\left\{1+\frac{2iu^2}{(2r-1)^2\pi^2}\right\}.\prod_1^\infty\left\{1-\frac{2iu^2}{(2r-1)^2\pi^2}\right\}$$
$$=2\prod_1^\infty\left\{1+\frac{4u^4}{(2r-1)^4\pi^4}\right\}.$$

If we now put $$u=\sqrt{\frac{\cot a_1}{2}}.\pi,$$

we get twice the product on the right of (i), and the result follows.

8. Taking the lines in the form

$$x\cos a_r+y\sin a_r-p_r=0,\quad (r=1,\ 2,\ 3),$$

the equation to the circle must be of the form

$$\Sigma\lambda\,(x\cos a_2+y\sin a_2-p_2)\,(x\cos a_3+y\sin a_3-p_3)=0,$$

and the conditions for a circle are

$$\Sigma\lambda\cos a_2\cos a_3=\Sigma\lambda\sin a_2\sin a_3,$$
$$\text{i.e.}\quad \Sigma\lambda\cos(a_2+a_3)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

and $$\Sigma\lambda\sin(a_2+a_3)=0\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

From (i) and (ii) we obtain

$$\frac{\lambda}{\sin(a_2-a_3)}=\frac{\mu}{\sin(a_3-a_1)}=\frac{\nu}{\sin(a_1-a_2)},$$

so that the equation to the circle is

$$\Sigma\,(x\cos a_2+\ldots)\,(x\cos a_3+\ldots)\sin(a_2-a_3)=0.$$

In this the coefficient of x^2 is $\Sigma\cos a_2\cos a_3\sin(a_2-a_3)$,

$\ldots\ldots\ldots\ldots\ldots\ \ y^2\ldots\ \Sigma\sin a_2\sin a_3\sin(a_2-a_3)$.

These are equal, and therefore each is equal to

$$\tfrac{1}{2}\Sigma\cos(a_2-a_3)\sin(a_2-a_3)$$
$$=\tfrac{1}{4}.\Sigma\sin 2(a_2-a_3)=\Pi\sin(a_2-a_3)\ldots\ldots(1).$$

The coefficient of x is

$$-\Sigma\,(p_2\cos a_3+p_3\cos a_2)\sin(a_2-a_3)$$
$$=-\Sigma p_1\left[\cos a_3\sin(a_3-a_1)+\cos a_2\sin(a_1-a_2)\right]$$
$$=\Sigma p_1\cos(a_2+a_3-a_1)\sin(a_2-a_3)\ \ldots\ldots\ldots\ldots\ldots(2).$$

Similarly the coefficient of y is

$$-\Sigma p_1 \sin(\alpha_2+\alpha_3-\alpha_1)\sin(\alpha_2-\alpha_3) \quad \ldots\ldots\ldots\ldots(3),$$

and the absolute term is

$$\Sigma p_2 p_3 \sin(\alpha_2-\alpha_3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If we now take

$$\frac{\cos\alpha_r}{a_r}=\frac{\sin\alpha_r}{b_r}=\frac{p}{-c_r}=\frac{1}{\sqrt{a_r{}^2+b_r{}^2}}, \quad (r=1,\ 2,\ 3),$$

and substitute in the expressions (1), (2), (3) and (4), we obtain the equation given in question.

9. The equation to the circle of curvature, if the origin is a centre of curvature, is $x^2+y^2=\rho^2$, and the condition that

$$S+k(x^2+y^2-\rho^2)=0$$

should represent straight lines is obtained by substituting

$$a+k, \quad b+k, \quad c-k\rho^2$$

for a, b, c in $\Delta=0$.

The result is

$$\Delta+\Delta(a'+b'-\rho^2c')k-(\overline{a+b}\rho^2-c)k^2-\rho^2k^3=0,$$

and this equation must have three equal roots, since the three pairs of chords of intersection coincide.

Now if $ak^3+\beta k^2+\gamma k+\delta$ is a perfect cube, we must have

$$\beta=3\alpha^{\frac{2}{3}}\delta^{\frac{1}{3}}, \quad \gamma=3\alpha^{\frac{1}{3}}\delta^{\frac{2}{3}}.$$

Hence in this case

$$-(\overline{a+b}\rho^2-c)=3(-\rho^2)^{\frac{2}{3}}\Delta^{\frac{1}{3}},$$

$$\text{i.e.}\quad a+b-c\rho^{-2}-=3\Delta^{\frac{1}{3}}\rho^{-\frac{2}{3}},$$

and

$$\Delta(a'+b'-\rho^2c')=3(-\rho^2)^{\frac{1}{3}}\Delta^{\frac{2}{3}},$$

$$\text{i.e.}\quad a'+b'-c'\rho^2=-3\Delta^{-\frac{1}{3}}\rho^{\frac{2}{3}}.$$

10. If, in any position, each of the upper rods makes an angle θ with the vertical, the length of each of the horizontal rods is $\sqrt{2}\,a\sin\theta$. Let T be the thrust in one of the horizontal rods. Then, since the weights of the rods are equivalent to a

weight $12mg$ at the centre of the figure, the equation of virtual work for a symmetrical virtual displacement is

$$4T\,.\,\delta(\sqrt{2}a\sin\theta)+12mg\,.\,\delta(a\cos\theta)$$
$$+Mg\,.\,\delta(2a\cos\theta-c\operatorname{cosec}\theta)=0,$$

i.e. $4T\,.\,\sqrt{2}a\cos\theta-12mg\,a\sin\theta$

$$+Mg(-2a\sin\theta+c\operatorname{cosec}\theta\cot\theta)=0,$$

and putting $\theta=45°$, for the equilibrium position, we obtain the value of T in question.

11. The mass M_1 will move downwards and M_2 upwards till the tension of the string destroys the motion. Let x be the distance M_1 descends and M_2 ascends, then $2x$ is the stretched length of the string. Hence, by the principle of energy,

$M_1gx-M_2gx=$ work done in stretching the string from length l_0 to length $2x$

$=$ extension $\times$ mean tension

$$=\frac{E}{2l_0}(2x-l_0)^2.$$

This quadratic in x gives the maximum depth.

When the string is tight, let x now be the distance each half is stretched beyond the natural length at any instant, f the acceleration of M_1 downwards and M_2 upwards, T the tension of the inelastic string. Then

$$M_1f=M_1g-\frac{2Ex}{l_0}-T,$$

$$M_2f=T-M_2g-\frac{2Ex}{l_0};$$

$$\therefore\ (M_1+M_2)f=(M_1-M_2)g-\frac{4Ex}{l_0};$$

$$\therefore\ f=\frac{4E}{l_0(M_1+M_2)}\left\{\frac{l_0(M_1-M_2)}{4E}\,.\,g-x\right\}.$$

Hence $f=\dfrac{4E}{l_0(M_1+M_2)}\times$ distance from a fixed point, so that the motion is simple harmonic, the period being

$$\pi\,.\,\sqrt{\frac{l_0(M_1+M_2)}{E}}.$$

12. The normal at α is

$$\frac{U'}{U} \cdot u = U' \cos(\theta - \alpha) - U \sin(\theta - \alpha),$$

where U, U' are the values of u and $\frac{du}{d\theta}$ at the point α. For this curve

$$\frac{u'}{u} = a^2u^2 \sin 2\theta = \tan 2\theta.$$

Hence the normal is

$$u \tan 2\alpha = U\left[\tan 2\alpha \cos(\theta - \alpha) - \sin(\theta - \alpha)\right],$$

$$\text{i.e.} \quad u \sin 2\alpha = U \sin(3\alpha - \theta) \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

The perpendicular normals are those at the points $\pm\frac{\pi}{6} + \alpha$. Taking the former, its equation is

$$u \sin\left(\frac{\pi}{3} + 2\alpha\right) = U_1 \cos(3\alpha - \theta) \quad \ldots\ldots\ldots\ldots\text{(ii)},$$

where U_1 is the value of u at $\frac{\pi}{6} + \alpha$.

For the intersection of (i) and (ii), we have

$$u^2 \left\{\frac{\sin^2 2\alpha}{U^2} + \frac{\sin^2\left(\frac{\pi}{3} + 2\alpha\right)}{U_1^2}\right\} = 1;$$

$$\text{i.e.} \quad r^2 = a^2 \left\{\sin^2 2\alpha \cos 2\alpha + \sin^2\left(\frac{\pi}{3} + 2\alpha\right)\cos\left(\frac{\pi}{3} + 2\alpha\right)\right\}.$$

Now $$\sin^2\phi \cos\phi = \tfrac{1}{4}(\cos\phi - \cos 3\phi);$$

$$\therefore \quad r^2 = \tfrac{1}{4}a^2 \left\{\cos 2\alpha + \cos\left(\frac{\pi}{3} + 2\alpha\right)\right\}$$

$$= \tfrac{1}{2}a^2 \cos\left(\frac{\pi}{6} + 2\alpha\right)\cos\frac{\pi}{6}$$

$$= 2^{-2}\, 3^{\frac{1}{2}} a^2 \cos\left(\frac{\pi}{6} + 2\alpha\right),$$

and similarly for the other normal.

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